ELSEVIER

Theoretical Computer Science 284 (2002) 499–518

Theoretical Computer Science

www.elsevier.com/locate/tcs

The closure properties on real numbers under limits and computable operators

Xizhong Zheng

Theoretische Informatik, BTU Cottbus, 03044 Cottbus, Germany

Abstract

In effective analysis, various classes of real numbers are discussed. For example, the classes of computable, semi-computable, weakly computable, recursively approximable real numbers, etc. All these classes correspond to some kind of (weak) computability of the real numbers. In this paper we discuss mathematical closure properties of these classes under the limit, effective limit and computable function. Among others, we show that the class of weakly computable real numbers is not closed under effective limit and partial computable functions while the class of recursively approximable real numbers is closed under effective limit and partial computable functions.

Keywords: Weakly computable reals; Computable real functions; Closure properties

1. Introduction

In computable analysis, a real number x is called *computable*, if there is a computable sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers which converges to x effectively. That is, the sequence satisfies the condition that $|x_n - x| < 2^{-n}$, for any $n \in \mathbb{N}$. In this case, the real number x is not only approximable by some effective procedure, there is also an effective error-estimation in this approximation. In practice, it happens very often that some real values can be effectively approximated, but an effective error-estimation is not always available. To characterize this kind of real numbers, the concept of recursively approximable real numbers is introduced. Namely, a real number x is *recursively approximable* (r.a., in short) if there is a computable sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers which converges to x. It is first noted by Ernst Specker in [15] that there is a

E-mail address: zheng@informatik.tu-cottbus.de (X. Zheng).

0304-3975/02/$ - see front matter
PII: S0304-3975(01)00107-4

recursively approximable real number which is not computable by encoding the halting problem into the binary expansion of a recursively approximable real numbers.

The class $\mathbf{C}_e$ of computable real numbers and the class $\mathbf{C}_{ra}$ of recursively approximable real numbers shares a lot of mathematical properties. For example, both $\mathbf{C}_e$ and $\mathbf{C}_{ra}$ are closed under the arithmetical operations and hence they are algebraic fields. Furthermore, these two classes are closed under the computable real functions, namely, if x is a computable (r.a.) real number and f is a computable real function in the sense of, say, Grzegorczyk [6], then $f(x)$ is also computable (resp. r.a.).

The classes of real numbers between $\mathbf{C}_e$ and $\mathbf{C}_{ra}$ are also widely discussed (see e.g. [3, 4, 11, 12, 18, 20]). Among others, the class of the so-called recursively enumerable real numbers might be the first widely discussed such class. A real number x is called *recursive enumerable* if its left Dedekind cut is an r.e. set of rational numbers, or equivalently, there is an increasing computable sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers which converges to x. We prefer to call such real numbers *left computable* because it can be approximated from “left” side on the real line and it is also very naturally related to the left topology $\tau_< := \{(a;\infty)\colon\ a\in\mathbb{R}\}$ of the real numbers by the admissible representation of Weihrauch [16]. Symmetrically, a real number x is called *right computable* if it is a limit of some decreasing computable sequence of rational numbers. Left and right computable real numbers are called *semi-computable*. Soare [11, 12] discusses widely the recursion-theoretical properties of the left Dedekind cuts of the left computable real numbers. Ceĭtin [4] shows that there is an r.a. real number which is not semi-computable. Another very interesting result, shown by a series works of Chaitin [5], Solovay [14], Calude et al. [3] and Slaman [10], says that a real number x is r.e. random if and only if it is an Ω-number of Chaitin which is the halting probability of an universal self-delimiting Turing machine. We omit the details about these notions here and refer the interested readers to a nice survey paper of Calude [2].

Although the class of left computable real numbers has a lot of nice properties, it is not symmetrical in the sense that the real number $-x$ is right computable but usually not left computable for a left computable real number x. Furthermore, even the class of semi-computable real numbers is not closed under the arithmetical operations as shown by Weihrauch and Zheng [18]. Namely, there are left computable real numbers y and z such that $y-z$ is neither left nor right computable. As the arithmetical closure of semi-computable real numbers, Weihrauch and Zheng [18] introduces a new class of weakly computable real numbers. A real number x is called *weakly computable* if there are two left computable real numbers y and z such that $x=y-z$. It is shown in [18] that x is weakly computable if and only if there is a computable sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers which converges to x weakly effectively, i.e. $\lim_{n\to\infty} x_n = x$ and $\sum_{n=0}^{\infty} |x_n - x_{n+1}|$ is finite. By means of this characterization, it is also shown in [18] that the class of weakly computable real numbers is an algebraic field and is strictly between the classes of semi-computable and r.a. real numbers. In this paper we will discuss other closure properties of weakly computable real numbers for limits, effective limits and computable real functions. We show that weakly computable real numbers are not closed under the effective limits and partial computable real functions. For other classes mentioned above, we also carry out the corresponding discussions.

At the end of this section, let us explain some notions. For any set $A \subseteq \mathbb{N}$, denote by $x_A := \sum_{n \in A} 2^{-n}$ the real number whose binary expansion corresponds to set A. For any $k \in \mathbb{N}$, we define $kA := \{kn\colon n \in A\}$. For any function $f:\mathbb{N} \to \mathbb{N}$, a set A is called f-r.e. if there is a computable sequence $(A_n)_{n\in\mathbb{N}}$ of finite subsets of $\mathbb{N}$ such that $A = \bigcup_{i\in\mathbb{N}} \bigcap_{j \geqslant i} A_j$ and $|\{s\colon n \in A_s \Delta A_{s+1}\}| < f(n)$ for all $n \in \mathbb{N}$, where $A\Delta B := (A\backslash B) \cup (B\backslash A)$. If $f(n) := k$ is a constant function, then f-r.e. sets are also called k-r.e. A is called ω-r.e. iff there is a recursive function f such that A is f-r.e. A (possibly) partial function f from A to B is always denoted by $f :\subseteq A \to B$, while corresponding total function is denoted by $f : A \to B$. For any subset $A \subseteq \mathbb{N}$ and any $n \in \mathbb{N}$, two kinds of the restrictions of A are used in this paper, namely $A \upharpoonright n := \{x \in A\colon x < n\}$ and $A \downharpoonleft n := \{x \in A\colon x > n\}$.

2. Computabilities of real numbers

In this section we give at first the formal definition of various computabilities of real numbers and then recall some important properties about these notions. We assume that the reader familiar with the computability about subsets of the natural numbers $\mathbb{N}$ and number-theoretical functions. A sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers is called computable means that there are recursive functions $a, b, c : \mathbb{N} \to \mathbb{N}$ such that $x_n = (a(n) - b(n))/(c(n) + 1)$. Obviously, as a finite object, any rational number $r \in \mathbb{Q}$ is computable and we can also effectively determine whether $r_1 < r_2$ or $r_1 = r_2$ for any $r_1, r_2 \in \mathbb{Q}$. For real number, we summarize the computability notions as follows.

Definition 2.1. For any real number $x \in \mathbb{R}$,

1. x is *computable* iff there is a computable sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers such that $x = \lim_{n\to\infty} x_n$ and $\forall n\ (|x_n - x_{n+1}| < 2^{-(n+1)})$. In this case, the sequence $(x_n)_{n\in\mathbb{N}}$ is called *fast convergent* and it converges to x *effectively*. The class of computable real numbers is denoted by $\mathbf{C}_e$.
2. x is *left* (*right*) *computable* iff there is an increasing (decreasing) computable sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers such that $x = \lim_{n\to\infty} x_n$. The classes of left and right computable real numbers are denoted by $\mathbf{C}_{lc}$ and $\mathbf{C}_{rc}$, respectively. Left and right computable real numbers are all called *semi-computable*. The class of all semi-computable real numbers is denoted by $\mathbf{C}_{sc}$.
3. x is *weakly computable* (*w.c.* in short) iff there is a computable sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers such that $x = \lim_{n\to\infty} x_n$ and $\sum_{n=0}^{\infty} |x_n - x_{n+1}|$ is finite. The class of all w.c. real numbers is denoted by $\mathbf{C}_{wc}$.
4. x is *recursively approximable* (*r.a.*, in short) iff there is a computable sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers such that $x = \lim_{n\to\infty} x_n$. The class of all r.a. real numbers is denoted by $\mathbf{C}_{ra}$.

As shown in [18], the relationship among these classes looks like the following:

$$\mathbf{C}_e = \mathbf{C}_{lc} \cap \mathbf{C}_{rc} \subsetneq \begin{smallmatrix} \mathbf{C}_{lc} \\ \mathbf{C}_{rc} \end{smallmatrix} \subsetneq \mathbf{C}_{sc} = \mathbf{C}_{lc} \cup \mathbf{C}_{rc} \subsetneq \mathbf{C}_{wc} \subsetneq \mathbf{C}_{ra}.$$

Note that in the above definition, we define various version of computabilities of real numbers in a similar way. Namely, a real number x is of some version of computability iff there is a computable sequence of rational numbers which satisfies some special property and converges to x. For example, if $P_{lc}[(x_n)]$ means that $(x_n)_{n\in\mathbb{N}}$ is increasing, then $x \in \mathbf{C}_{lc}$ iff there is a computable sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers such that $P_{lc}[(x_n)]$ and $\lim_{n\to\infty} x_n = x$. The properties of sequences corresponding to the above definition can be summarized as follows: for any sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers,

$$E[(x_n)_{n\in\mathbb{N}}] \iff \forall n\ (|x_n - x_{n+1}| < 2^{-(n+1)}),$$

$$LC[(x_n)_{n\in\mathbb{N}}] \iff \forall n\ (x_n \leqslant x_{n+1}),$$

$$RC[(x_n)_{n\in\mathbb{N}}] \iff \forall n\ (x_n \geqslant x_{n+1}),$$

$$SC[(x_n)_{n\in\mathbb{N}}] \iff \forall n\ (x_n \geqslant x_{n+1}) \text{ or } \forall n\ (x_n \leqslant x_{n+1}),$$

$$WC[(x_n)_{n\in\mathbb{N}}] \iff \exists k \left(\sum_{n=0}^{\infty} |x_n - x_{n+1}| \leqslant k\right),$$

and $RA[(x_n)]$ is satisfied for all sequence $(x_n)_{n\in\mathbb{N}}$.

In general, for any reasonable property on sequences, we can define a corresponding class of real numbers which have some kind of (weak) computability. This can even be extended to the case of sequences of real numbers as follows.

Definition 2.2. Let P be any property about the sequences of real numbers. Then

(1) A real number x is called *P-computable* iff there is a computable sequence $(x_n)_{n\in\mathbb{N}}$ of rational numbers which satisfies the property P and converges to x. The class of all P-computable real numbers is denoted by $\mathbf{C}_P$.

(2) A sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is called *P-computable*, or it is a computable sequence of $\mathbf{C}_P$ iff there is a computable double sequence $(r_{nm})_{nm\in\mathbb{N}}$ such that the sequences $(r_{nm})_{m\in\mathbb{N}}$ satisfy P and $\lim_{m\to\infty} r_{nm} = x_n$ for all $n \in \mathbb{N}$.

(3) The class $\mathbf{C}_P$ defined in (1) is called *closed under limits*, iff for any computable sequences $(x_n)_{n\in\mathbb{N}}$ of $\mathbf{C}_P$, the limits $x := \lim_{n\to\infty} x_n$ is also in $\mathbf{C}_P$ whenever $(x_n)_{n\in\mathbb{N}}$ satisfies P and converges.

(4) The class $\mathbf{C}_P$ defined in (1) is called *closed under effective limits*, iff for any fast convergent computable sequences $(x_n)_{n\in\mathbb{N}}$ of $\mathbf{C}_P$, the limits $x := \lim_{n\to\infty} x_n$ is also in $\mathbf{C}_P$.

For example, $(x_n)_{n\in\mathbb{N}}$ is a computable sequence of $\mathbf{C}_{wc}$ (or it is a *weakly computable* sequence of real numbers) if there is a computable sequence $(r_{ij})_{i,j\in\mathbb{N}}$ of rational numbers such that, for any $i \in \mathbb{N}$, the sequence $(r_{ij})_{j\in\mathbb{N}}$ converges to x_i weakly effectively. And $(x_n)_{n\in\mathbb{N}}$ is a computable sequence of $\mathbf{C}_e$ iff there is a computable sequence $(r_{ij})_{i,j\in\mathbb{N}}$ of rational numbers such that, for any $i\in\mathbb{N}$, the sequence $(r_{ij})_{j\in\mathbb{N}}$ satisfies the condition E and converges to x_i. It is easy to see, by the definition of E, that the sequence $(r_{ij})_{j\in\mathbb{N}}$ converges effectively in i and j in the sense of [9] to the sequence $(x_i)_{i\in\mathbb{N}}$. Therefore, the notion of computable sequence of $\mathbf{C}_e$ is just same as that of computable sequence of real numbers in the standard sense.

On the closureness of $\mathbf{C}_P$ under limits, we consider only such computable sequences of $\mathbf{C}_P$ which are convergent and satisfy the condition P. Therefore, e.g., $\mathbf{C}_e$ is closed under limits means that all computable sequence of $\mathbf{C}_e$ converges to a computable real numbers, if it converges effectively (i.e. satisfies the condition E). And $\mathbf{C}_{lc}$ is closed under limits means that any (bounded) increasing computable sequence of $\mathbf{C}_{lc}$ converges to a left computable real number, etc.

Now we introduce the notion of computable real function. There are a lot of approaches to define the computability of real functions (see [6, 8, 17]). Here we use Grzegorczyk–Ko–Weihrauch's approach and define computable real function in terms of "Type-two Turing Machine" (TTM, in short) of Weihrauch.

Let Σ be any alphabet. Σ^* and Σ^∞ are sets of all finite strings and infinite sequences on Σ, respectively. Roughly, TTM M extends the classical Turing machine in such a way that it can be input and also can output infinite sequences as well as finite strings. For any $p\in\Sigma^*\cup\Sigma^\infty$, $M(p)$ outputs a finite string q, if $M(p)$ writes q in output tape and halt in finite steps similar to the case of classical Turing machine. On the other hand, $M(p)$ outputs an infinite sequence q means that $M(p)$ will compute forever and keep writing q on the output tape. We omit the formal details about TTM here and refer the interested readers to [16, 17]. We will also omit the details about the encoding of rational numbers by Σ^* and take directly the sequences of rational numbers as inputs and outputs to TTMs.

Definition 2.3. A real function $f:\subseteq\mathbb{R}\to\mathbb{R}$ is called *computable* if there is a TTM M such that, for any $x\in\mathrm{dom}(f)$ and any sequence $(u_n)_{n\in\mathbb{N}}$ of rational numbers which converges effectively to x, $M((u_n)_{n\in\mathbb{N}})$ outputs a sequence $(v_n)_{n\in\mathbb{N}}$ of rational numbers which converges to $f(x)$ effectively.

Note that, in this definition we do not add any restriction on the domain of computable real function. Hence a computable real function can have any type of domain, because $f\upharpoonright A$ is a computable function with domain A for any set $A\subseteq\mathrm{dom}(f)$, whenever f is computable. Furthermore, for a total function $f:[0,1]\to\mathbb{R}$, f is computable iff f is sequentially computable and effectively uniformly continuous (see [9]).

Definition 2.4. For any subset $\mathbf{C}\subseteq\mathbb{R}$,

1. $\mathbf{C}$ is *closed under computable operators*, iff, for any $x\in\mathbf{C}$ and any total computable real function $f:\mathbb{R}\to\mathbb{R}$, $f(x)\in\mathbf{C}$.
2. $\mathbf{C}$ is *closed under partial computable operators*, iff, for any $x\in\mathbf{C}$ and any partial computable real function $f:\subseteq\mathbb{R}\to\mathbb{R}$ with $x\in\mathrm{dom}(f)$, $f(x)\in\mathbf{C}$.

Following proposition follows immediately from the definition.

Proposition 2.5. 1. $x_A \in \mathbf{C}_e \Leftrightarrow A$ *is recursive.*

2. $x_A \in \mathbf{C}_{ra} \Leftrightarrow A$ *is a* Δ_2^0*-set, or equivalently,* $A \leqslant_T \emptyset'$.

3. $\mathbf{C}_e$ *and* $\mathbf{C}_{ra}$ *are closed under arithmetical operations* $+$, $-$, $\times$ *and* $\div$, *hence they are algebraic fields.*

4. $\mathbf{C}_e$ *are closed under limits and computable real functions.*

5. $\mathbf{C}_{lc}$ *and* $\mathbf{C}_{rc}$ *are closed under addition.*

Some other non-trivial closure properties are shown in [18] and [19].

Theorem 2.6 (Weihrauch and Zheng [18, 19]). 1. $\mathbf{C}_{sc}$ *is not closed under addition, i.e. there are left computable* y *and right computable* z *such that* $y+z$ *is neither left nor right computable.*

2. $\mathbf{C}_{wc}$ *is closed under arithmetical operations. In fact* $\mathbf{C}_{wc}$ *is just the closure of* $\mathbf{C}_{sc}$ *under the arithmetical operations.*

It is not very surprising that the classes $\mathbf{C}_{lc}$ and $\mathbf{C}_{rc}$ are not closed under "subtraction" and, in general, under computable real functions, because they are not symmetrical. On the other hand, the class $\mathbf{C}_{wc}$ is symmetrical and closed under arithmetical operations. So it is quite natural to ask whether it is also closed under limits and computable real functions. In the following we will give the negative answers to both questions. To do that we need the following observations about weakly computable real numbers.

Theorem 2.7 (Ambos-Spies, Weihrauch and Zheng [1]). 1. *If* $A, B \subseteq \mathbb{N}$ *are incomparable under Turing reduction, then* $x_{A\oplus\bar{B}}$ *is not semi-computable.*

2. *For any set* $A \subseteq \mathbb{N}$, *if* x_{2A} *is weakly computable, then* A *is* f*-r.e. for* $f(n) := 2^{3n}$, *hence* A *is* ω*-r.e.*

Theorem 2.8. *There is a non-*ω*-r.e.* Δ_2^0*-set* A *such that* x_A *is weakly computable.*

The proof of Theorem 2.8 needs a finite priority construction which is included in Section 5.

3. Closure property under limits

In this section, we will discuss the closure properties of several classes of real numbers under limits. Remember that $\mathbf{C}_P$ is closed under limits means that every computable sequence of $\mathbf{C}_P$ converges to an element of $\mathbf{C}_P$, if it converges and satisfies the condition P. We first consider the classes of left and right computable real numbers.

Theorem 3.1. *The classes of left and right computable real numbers are closed under limits, respectively.*

Proof. We prove the case of left computable real numbers. For right computable real numbers the proof is similar. By Definition 2.2(3), it suffices to show that any (bounded) increasing computable sequence of $\mathbf{C}_{lc}$ converges to a left computable real number.

Let $(x_n)_{n\in\mathbb{N}}$ be an increasing computable sequence of $\mathbf{C}_{lc}$ which converges to x. By Definition 2.2(2), there is a computable double sequence $(r_{nm})_{n,m\in\mathbb{N}}$ of rational numbers such that, for any $n\in\mathbb{N}$, $(r_{nm})_{m\in\mathbb{N}}$ is an increasing sequence which converges to x_n. Define a computable sequence $(y_n)_{n\in\mathbb{N}}$ by $y_n := \max\{r_{ij}\colon i,j \leqslant n\}$. Obviously, $(y_n)_{n\in\mathbb{N}}$ is non-decreasing and bounded above by x. We claim that $y := \lim_{n\to\infty} y_n = x$, hence x is left computable. Otherwise, if $y<x$, then there is an n_1 such that $y<x_{n_1}$ since $\lim_{n\to\infty} x_n = x$. Because of $\lim_{n\to\infty} r_{n_1 n} = x_{n_1}$, there is furthermore an n_2 such that $y<r_{n_1 n_2}$. Let $N := \max\{n_1, n_2\}$. Then $y<r_{n_1 n_2}<y_N$. This contradicts the fact that $(y_n)_{n\in\mathbb{N}}$ is non-decreasing and $\lim_{n\to\infty} y_n = y$. $\square$

Next theorem shows that the situation is different for semi-computable real numbers.

Theorem 3.2. *The class $\mathbf{C}_{sc}$ of semi-computable real numbers is not closed under limits.*

Proof. Define, for any $n,s\in\mathbb{N}$, at first the following sets:

$$A := \{e \in \mathbb{N}\colon\ \varphi_e \text{ is total}\};$$

$$A_n := \{e \in \mathbb{N}\colon\ (\forall x \leqslant n)\varphi_e(x)\downarrow\};$$

$$A_{n,s} := \{e \in \mathbb{N}\colon\ (\forall x \leqslant n)\varphi_{e,s}(x)\downarrow\}.$$

Since $A_{n,s} \subseteq A_{n,s+1}$, $(x_{A_{n,s}})_{n,s\in\mathbb{N}}$ is obviously a computable sequence of rational numbers such that, for any $n \in \mathbb{N}$, $(x_{A_{n,s}})_{s\in\mathbb{N}}$ is non-decreasing and converges to x_{A_n}. That is, $(x_{A_n})_{n\in\mathbb{N}}$ is a computable sequence of $\mathbf{C}_{lc}$, hence it is a computable sequence of $\mathbf{C}_{sc}$ which satisfies the monotonic condition *SC*. But its limit x_A is not semi-computable. In fact x_A is even not r.a. by Proposition 2.5. $\square$

Note that in the above proof, as a computable sequence of $\mathbf{C}_{lc}$, $(x_{A_n})_{n\in\mathbb{N}}$ is also a computable sequence of $\mathbf{C}_{wc}$ and $\mathbf{C}_{ra}$. It obviously satisfies the conditions *WC* and *RA* also. Then the next corollary follows immediately.

Corollary 3.3. *The classes $\mathbf{C}_{wc}$ and $\mathbf{C}_{ra}$ are not closed under the limit.*

Now we discuss the closure property under the effective limits. We will show that the class of semi-computable real numbers and also the class of r.a. real numbers are closed under effective limits and the class of weakly computable real numbers is not closed under effective limits.

Theorem 3.4. *The class $\mathbf{C}_{sc}$ is closed under the effective limits.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a computable sequence of $\mathbf{C}_{sc}$ which satisfies the condition that $\forall n\ (|x_n - x_{n+1}| < 2^{-(n+1)})$ and converges to x. We shall show that $x \in \mathbf{C}_{sc}$.

By Definition 2.2(2), there is a computable sequence $(r_{ij})_{i,j\in\mathbb{N}}$ of rational numbers such that, for any $n\in\mathbb{N}$, $(r_{nj})_{j\in\mathbb{N}}$ satisfies the condition SC and is monotonic and it converges to x_n. For any n, we can effectively determine whether x_n is left or right computable by simply comparing, say, r_{n0} and r_{n1}. Therefore, the sequence $(x_n)_{n\in\mathbb{N}}$ can be split effectively into two computable subsequences $(x_{n_i})_{i\in\mathbb{N}}$ and $(x_{m_i})_{i\in\mathbb{N}}$ of left and right computable real numbers, respectively. At least one of them is infinite. Suppose w.l.o.g. that $(x_{n_i})_{i\in\mathbb{N}}$ is an infinite sequence. Obviously, it is also a fast convergent computable sequence, i.e. $|x_{n_i} - x_{n_{i+1}}| < 2^{-i}$. Define a new sequence $(y_n)_{n\in\mathbb{N}}$ by $y_i := x_{n_i} - 2^{-(i-1)}$. Since

$$
\begin{aligned}
y_{i+1} &= x_{n_{i+1}} - 2^{-i} \\
&= (x_{n_{i+1}} - x_{n_i} + 2^{-i}) + (x_{n_i} - 2^{-(i-1)}) \\
&\geqslant x_{n_i} - 2^{-(i-1)} = y_i,
\end{aligned}
$$

$(y_i)_{i\in\mathbb{N}}$ is an increasing sequence. Let $r'_{ij} := r_{n_ij} - 2^{-(i-1)}$. Then $(r'_{ij})_{i,j\in\mathbb{N}}$ is a computable sequence of rational numbers such that, for any i, $(r'_{ij})_{j\in\mathbb{N}}$ is increasing and converges to y_i. Namely, $(y_i)_{i\in\mathbb{N}}$ is an increasing computable sequence of $\mathbf{C}_{lc}$. By Theorem 3.1, its limit $\lim_{i\to\infty} y_i = \lim_{i\to\infty} x_{n_i} = \lim_{i\to\infty} x_i = x$ is also left computable, i.e., $x \in \mathbf{C}_{lc} \subseteq \mathbf{C}_{sc}$. $\square$

Theorem 3.5. *The class $\mathbf{C}_{wc}$ is not closed under effective limits.*

Proof. Suppose, by Theorem 2.8, that A is a non-ω-r.e. Δ^0_2-set such that x_A is weakly computable. Then x_{2A} is not weakly computable by Theorem 2.7. Let $(A_s)_{s\in\mathbb{N}}$ be a recursive approximation of A such that $(x_{A_s})_{s\in\mathbb{N}}$ converges to x_A weakly effectively, i.e., $\sum_{s=0}^{\infty} |x_{A_s} - x_{A_{s+1}}| \leqslant C$ for some $C \in \mathbb{N}$.

For any $n, s \in \mathbb{N}$, define

$$
\begin{aligned}
B_{n,s} &:= 2(A_s \upharpoonright (n+1)) \cup (A_s \downharpoonleft 2n), \\
B_n &:= 2(A \upharpoonright (n+1)) \cup (A \downharpoonleft 2n).
\end{aligned}
$$

It is easy to see that $(B_{n,s})_{n,s\in\mathbb{N}}$ is a computable sequence of finite subsets of $\mathbb{N}$, hence $(x_{B_{n,s}})_{n,s\in\mathbb{N}}$ is a computable sequence of rational numbers.

Since $\lim_{s\to\infty} A_s = A$, there is an $N(n)$, for any $n \in \mathbb{N}$ such that, for any $s \geqslant N(n)$, $A_s \upharpoonright (n+1) = A \upharpoonright (n+1)$. Let $C_1 = \sum_{s=0}^{N(n)} |x_{B_{n,s}} - x_{B_{n,s+1}}|$. Then

$$\begin{aligned}\sum_{s=0}^{\infty} |x_{B_{n,s}} - x_{B_{n,s+1}}| &= \sum_{s=0}^{N(n)} |x_{B_{n,s}} - x_{B_{n,s+1}}| + \sum_{s>N(n)}^{\infty} |x_{B_{n,s}} - x_{B_{n,s+1}}| \\ &= C_1 + \sum_{s>N(n)}^{\infty} |x_{A_{n,s}} - x_{A_{n,s+1}}| \\ &< C_1 + C.\end{aligned}$$

On the other hand, it is easy to see that $\lim_{s\to\infty} x_{B_{n,s}} = x_{B_n}$. Therefore, the sequence $(x_{B_{n,s}})_{s\in\mathbb{N}}$ converges to x_{B_n} weakly effectively. Hence $(x_{B_n})_{n\in\mathbb{N}}$ is a weakly computable sequence of real numbers. By the definition of B_n, $B_n \Delta 2A \subseteq \{2n+1, 2n+2, \ldots\}$. It follows that $|x_{B_n} - x_{2A}| \leqslant 2^{-2n} \leqslant 2^{-n}$. This means that $(x_{B_n})_{n\in\mathbb{N}}$ converges to x_{2A} effectively and this ends the proof of the theorem. $\square$

Theorem 3.6. *The class* $\mathbf{C}_{ra}$ *is closed under effective limits.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be any computable sequence of $\mathbf{C}_{ra}$ which converges effectively to x. Assume w.l.o.g. that it satisfies, for all $n \in \mathbb{N}$, the condition $|x_n - x_{n+1}| < 2^{-(n+2)}$. By Definition 2.2(2), there is a computable sequence $(r_{ij})_{i,j\in\mathbb{N}}$ of rational numbers such that, for any $n \in \mathbb{N}$, $\lim_{s\to\infty} r_{ns} = x_n$. We shall show that $x \in \mathbf{C}_{ra}$.

It suffices to construct a computable sequence $(u_s)_{s\in\mathbb{N}}$ of rational numbers such that $\lim_{s\to\infty} u_s = x$. The sequence $(u_s)_{s\in\mathbb{N}}$ will be constructed as a subsequence of $(r_{ij})_{i,j\in\mathbb{N}}$. For any fixed n, only finite many r_{ni} $(i \in \mathbb{N})$ can be chosen. To this end, we choose some element r_{ni} to be an element of $(u_s)_{s\in\mathbb{N}}$ only if either no other $r_{ni'}$, for any $i' \in \mathbb{N}$, is already been chosen or r_{ni} is "far from" the last chosen element $r_{ni'}$, i.e., $|r_{ni} - r_{ni'}| \geqslant 2^{-(n+1)}$. In the following construction, $i(s)$ denotes the index of last constructed element of $(u_s)_{s\in\mathbb{N}}$ till stage s. If there is still no $j \in \mathbb{N}$ such that r_{ij} is chosen to $(u_s)_{s\in\mathbb{N}}$ till stage s, then we define $t(i,s) = -1$. Otherwise, $t(i,s) = k$, if we have chosen some r_{ij} most recently as u_k till stage s.

The construction of $(u_s)_{s\in\mathbb{N}}$:

Stage $s = 0$: Define $u_0 := r_{00}$, $t(0,0) := 0$ and $i(0) := 0$.

Stage $s+1$: Given $i(s)$, $u_0, \ldots, u_{i(s)}$ and $t(j,s)$ for all $j \leqslant s$. If there is some $j \leqslant s$ satisfying

$$j \neq 0 \Rightarrow |u_{t(j-1,s)} - r_{js}| < 2^{-(j-1)} \tag{1}$$

such that either

$$t(j,s) = -1 \tag{2}$$

or

$$t(j,s) \neq -1 \ \& \ |u_{t(j,s)} - r_{js}| \geqslant 2^{-(j+1)}, \tag{3}$$

then choose j_0 as minimal such j and define

$$(*) \quad \begin{cases} i(s+1) := i(s)+1, \\ u_{i(s+1)} := r_{j_0 s}, \\ t(j,s+1) := \begin{cases} t(j,s) & \text{if } 0 \leqslant j < j_0, \\ i(s)+1 & \text{if } j = j_0, \\ -1 & \text{if } j_0 < j \leqslant s+1. \end{cases} \end{cases}$$

Otherwise, if no such j exists, then define, $i(s+1) := i(s)$, $t(s+1,s+1) := -1$ and $t(j,s+1) := t(j,s)$ for all $j \leqslant s$.

To show that this construction succeeds, we prove at first the following claims.

Claim 1. *For any $j \in \mathbb{N}$, the limit $t(j) := \lim_{s\to\infty} t(j,s)$ exists and satisfies the conditions that $t(j) \neq -1$ and $|u_{t(j)} - x_j| \leqslant 2^{-(j+1)}$.*

Proof. Assume by induction hypothesis that the claim holds for any $i < j$. We consider now the case of j. Choose a minimal s_0 such that $|r_{js} - x_j| < 2^{-(j+1)}$, $t(i,s) = t(i,s_0) = t(i) \neq -1$ for all $i < j$ and $s \geqslant s_0$, and hence also $|u_{t(j-1,s_0)} - x_{j-1}| = |u_{t(j-1)} - x_{j-1}| \leqslant 2^{-j}$.

If $t(j,s_0) = -1$, then condition (2) is satisfied. Furthermore, $|u_{t(j-1,s_0)} - r_{js_0}| \leqslant |u_{t(j-1,s_0)} - x_{j-1}| + |x_{j-1} - x_j| + |x_j - r_{js_0}| < 2^{-(j-1)}$. That is, condition (1) is satisfied too. Therefore, at stage s_0+1, we define $t(j,s_0+1) := i(s_0)+1$ and $u_{t(j,s_0+1)} := r_{js_0}$ by $(*)$. After stage s_0+1, conditions (3) and (2) will never be satisfied again for j_0. This means that $t(j,s) = t(j,s_0+1)$ for any $s \geqslant s_0+1$, hence $t(j) = \lim_{s\to\infty} t(j,s) = t(j,s_0+1)$ and $|u_{t(j)} - x_j| \leqslant 2^{-(j+1)}$.

Suppose now that $t(j,s_0) \neq -1$. From $(*)$, it is easy to see that $t(j,s) \neq -1$ for all $s \geqslant s_0$, hence (2) will never be satisfied after stage s_0.

If the value of $t(j,s)$ is never changed after s_0, then $\lim_{s\to\infty} t(j,s) = t(j,s_0)$. We claim that $|u_{t(j)} - x_j| \leqslant 2^{-(j+1)}$ holds too. Otherwise, if $|u_{t(j)} - x_j| > 2^{-(j+1)}$, then there is an $s' > s_0$ such that $|u_{t(j,s')} - r_{js'}| = |u_{t(j)} - r_{js'}| > 2^{-(j+1)}$ and $|x_j - r_{js'}| < 2^{-(j+1)}$. This implies that $|u_{t(j-1,s')} - r_{js'}| \leqslant |u_{t(j-1)} - x_{j-1}| + |x_{j-1} - x_j| + |x_j - r_{js'}| \leqslant 2^{-(j-1)}$. That is, (1) and (3) are satisfied, hence $t(j,s)$ will be redefined at stage s'. A contradiction.

Suppose that $t(j,s)$ is redefined at stages: $s_1 < s_2 < s_3 < \cdots$ after stage s_0. Then conditions (1) and (3) must be satisfied at these stages. From (3) and the definition $(*)$, it follows that $|u_{t(j,s_i)} - u_{t(j,s_{i+1})}| \geqslant 2^{-(j+1)}$, hence $|r_{j,s_i} - r_{j,s_{i+1}}| \geqslant 2^{-(j+1)}$. Since the sequence $(r_{j,s})_{s\in\mathbb{N}}$ converges, there are at most finitely many such s_i's. Let s_{i_0} be the last such s_i. Then $\lim_{s\to\infty} t(j,s) = t(j,s_{i_0})$. Similar to the above proof, we can show also that $|u_{t(j,s_{i_0})} - x_j| < 2^{-(j+1)}$. $\square$

Claim 2. $\lim_{s\to\infty} i(s) = +\infty$ *and* $\mathrm{rang}(i) = \mathbb{N}$.

Proof. It follows from Claim 1 immediately, because $i(s)$ increases by 1 whenever $t(i,s)$ is redefined from -1 to a new value for any i. $\square$

Claim 3. $\forall j \forall s\, (s \geqslant \max\{t(i){:}\ i \leqslant j\} \Rightarrow (|u_{t(j)} - u_s| \leqslant 2^{-j})$.

Proof. For any $j \in \mathbb{N}$, choose by Claim 1 a minimal s_0 such that $t(i,s) = t(i,s_0) = t(i)$ for all $s \geqslant s_0$ and $i \leqslant j$. By the minimality of s_0, it follows that $t(j,s_0) = t(j) = \max\{t(i){:}\ i \leqslant j\}$. If $t > \max\{t(i){:}\ i \leqslant j\} = t(j)$, by Claim 2 and the choice of s_0, there are $k > 0$ and $s' > s_0$ such that $t = i(s') = t(j+k, s')$. By the construction, there are $s_1 < s_2 < \cdots < s_k = s'$ such that, from condition (1) and the definition $(*)$, for any $0 < i \leqslant k$,

$$|u_{t(j+i-1,s_i)} - u_{t(j+i,s_i)}| < 2^{-(j+i-1)}.$$

Therefore

$$\begin{aligned} |u_{t(j)} - u_t| &= |u_{t(j,s_0)} - u_{t(j+k,s_k)}| \\ &\leqslant \sum_{i=1}^{i=k} |u_{t(j+i-1,s_i)} - u_{t(j+i,s_i)}| \\ &< \sum_{i=1}^{i=k} 2^{-(j+i-1)} < 2^{-(j-1)}. \qquad \square \end{aligned}$$

Now it is clear that the sequence $(u_s)_{s \in \mathbb{N}}$ constructed above is a computable infinite sequence of rational numbers by Claims 2. Furthermore, by Claims 1 and 3, this sequence converges to x. This completes the proof of theorem. $\square$

4. Closure property under computable operators

In this section we will discuss the closure property under computable operators. The following proposition about left and right computable real numbers is immediate by the fact that the real function f defined by $f(x) = -x$ is computable.

Proposition 4.1. *The classes* $\mathbf{C}_{lc}$ *and* $\mathbf{C}_{rc}$ *are not closed under the computable operators, hence also not closed under partial computable operators.*

To discuss the closure property under partial computable operators for other classes, we will apply the following observation which essentially belongs to Ko [7].

Theorem 4.2. *For any sets* $A, B \subseteq \mathbb{N}$, $A \leqslant_T B$ *iff there is a (partial) computable real function* $f :\subseteq \mathbb{R} \to \mathbb{R}$ *such that* $f(x_B) = x_A$.

Proof. (idea) "⇒" Suppose that $A \leqslant_T B$. If x_B is rational, then the set B, and hence set A, is recursive, i.e. the real number x_A is computable. So the computable constant function $f(x) := x_A$ maps obviously x_B to x_A. Otherwise suppose that x_B is not rational. It is not difficult to construct a Turing machine M such that, if p is a fast convergent Cauchy sequence converging to the irrational number x_B, then $M(p)$ computes at first the characteristic function χ_B of set B, then using the reduction of $A \leqslant_T B$, computes the characteristic function χ_A of A from which M can easily compute and output a fast convergent Cauchy sequence q converging to x_A. This Turing machine M computes a (partial) computable real function f which maps x_B to x_A.

"$\Leftarrow$" If x_A is rational, then A is finite or co-finite, hence it is recursive and $A \leqslant_T B$ for any set B. Suppose that x_A is not rational and there is a computable real function $f :\subseteq \mathbb{R} \to \mathbb{R}$ such that $f(x_B) = x_A$. Let M be a Turing machine which computes f.

From the characteristic function χ_B, we can construct a fast convergent Cauchy sequence $p := (u_n)_{n \in \mathbb{N}}$ by $u_n := x_{B \upharpoonright n}$. Then $M(p)$ outputs a fast convergent Cauchy sequence $(v_n)_{n \in \mathbb{N}}$ which converges to x_A. Since x_A is irrational, we can compute the characteristic function χ_A of A from the sequence $(v_n)_{n \in \mathbb{N}}$ effectively. This procedure shows the reduction of $A \leqslant_T B$. $\square$

From this result, it is easy to show that a lot of classes of real numbers are not closed under the partial computable operators.

Theorem 4.3. *The classes $\mathbf{C}_{sc}$ and $\mathbf{C}_{wc}$ are not closed under the partial computable operators. The class $\mathbf{C}_{ra}$ is closed under partial computable operators.*

Proof. 1. For class $\mathbf{C}_{sc}$. By Muchnik–Friedberg Theorem (see [13]), there are two r.e. sets A and B such that they are incomparable under Turing reduction. Then $x_{A \oplus \bar{B}}$ is not semi-computable by Theorem 2.7. On the other hand, $x_{A \oplus B}$ is left computable since $A \oplus B$ is r.e. Obviously, we have the reduction that $A \oplus \bar{B} \leqslant_T A \oplus B$. By Theorem 4.2, there is a computable real function f such that $f(x_{A \oplus B}) = x_{A \oplus \bar{B}}$. Therefore, $\mathbf{C}_{sc}$ is not closed under partial computable operators.

2. For class $\mathbf{C}_{wc}$. By Theorem 2.8, there is a non-ω-r.e. set A such that x_A is weakly computable. On the other hand, x_{2A} is not weakly computable by Theorem 2.7 since $2A$ is obviously not ω-r.e. Because $2A \leqslant_T A$, by Theorem 4.2, there is a computable real function f such that $f(x_A) = x_{2A}$. That is, $\mathbf{C}_{wc}$ is not closed under the partial computable operators.

3. For class $\mathbf{C}_{ra}$, it follows immediately from the fact that a real number x_A is r.a. iff A is a Δ_2-set and the class of all Δ_2-sets is closed under the Turing reduction, i.e. if $A \leqslant_T B$ and B is Δ_2-set, then A is also Δ_2-set. $\square$

It is shown in Theorem 2.6 that the class $\mathbf{C}_{sc}$ is not closed under addition. Hence it is not closed under the polynomial functions with several arguments. Namely, if $p(x, \ldots, x_n)$ is a polynomial (with even rational coefficients) and $a_1, \ldots, a_n$ are semi-computable real numbers, then $p(a_1, \ldots, a_n)$ is not necessary semi-computable. Next lemma shows that $\mathbf{C}_{sc}$ is closed under rational polynomials with one argument.

Lemma 4.4. *If $p(x)$ is a rational polynomial and a is a semi-computable real number, then $p(a)$ is also semi-computable.*

Proof. Note that, for any polynomial p and any real number x, there are always rational numbers a, b such that p is monotonic on both intervals $[a, x]$ and $[x, b]$. If all coefficients of p are rational numbers and x is, say, left computable, then there is an increasing computable sequence $(r_n)_{n\in\mathbb{N}}$ of rational numbers such that $\lim_{n\to\infty} r_n = x$. Fix an N large enough so that $r_n \in [a, x]$ for all $n \geqslant N$. Then $(p(r_{N+n}))_{n\in\mathbb{N}}$ is also a monotonic computable sequence of rational numbers which converges to $p(x)$. Therefore $p(x)$ is semi-computable. □

Unfortunately, Lemma 4.4 cannot be extended to the case of the sequence. Namely, if $(p_n)_{n\in\mathbb{N}}$ is a computable sequence of rational polynomials and x is a semi-computable real number, then the sequence $(p_n(x))_{n\in\mathbb{N}}$ is not necessarily a computable sequence of semi-computable real numbers in the sense of Definition 2.2, although every $p_n(x)$ is a semi-computable real number. Hence the closureness of semi-computable real number class under total computable operators cannot be directly followed from Lemma 4.4, Theorem 3.4 and the effective Weistrass theorem (cf. [9]) and it is not yet clear.

It remains also open whether the class $\mathbf{C}_{wc}$ is closed under (total) computable operators. We guess it is not. One possible approach is to define a computable real function which maps some weakly computable real number x_A for a non-ω-r.e. set A to a not weakly computable real number x_{2A}. Using the idea in the proof of Theorem 4.2, it is not difficult to show that there is a computable partial real function $f :\subseteq \mathbb{R} \to \mathbb{R}$ such that $f(x_A) = x_{2A}$ for any irrational x_A. Unfortunately, such function cannot be extended to a total computable real function as shown by the next result.

Theorem 4.5. 1. *Let $f :\subseteq \mathbb{N} \to \mathbb{N}$ be a function such that $f(x_A) = x_{2A}$ for any irrational x_A. If x_A is a rational number, then there is a sequence $(x_n)_{n\in\mathbb{N}}$ of irrational numbers such that $\lim_{n\to\infty} x_n = x_A$ and $\lim_{n\to\infty} f(x_n) = x_{2A}$.*

2. *The function $f : [0; 2] \to \mathbb{R}$ defined by $f(x_A) := x_{2A}$ for any $A \subseteq \mathbb{N}$ is not continuous at any rational points, hence it is not computable.*

Proof. 1. Suppose that function $f :\subseteq \mathbb{N} \to \mathbb{N}$ satisfies $f(x_A) = x_{2A}$ for any irrational x_A. Let x_A be rational, hence A is a finite set. We define a sequence $(x_n)_{n\in\mathbb{N}}$ of irrational numbers by $x_n := x_A + \sqrt{2}\cdot 2^{-(n+1)}$. Let n_0 be the maximal element of A. Define a set A_n by $x_{A_n} = \sqrt{2}\cdot 2^{-(n+1)}$ for any $n \in \mathbb{N}$. Then for any $n > n_0$, $x_{A_n} < 2^{-n} \leqslant 2^{-n_0}$. This implies that A_n contains only the elements which are bigger than n. If $n \geqslant n_0$, then $A \cap A_n = \emptyset$ and $f(x_n) = f(x_A + \sqrt{2}\cdot 2^{-(n+1)}) = f(x_A + x_{A_n}) = f(x_{A\cup A_n}) = x_{2(A\cup A_n)} = x_{(2A)\cup(2A_n)} = x_{2A} + x_{2A_n}$. Since $\lim_{n\to\infty} x_{A_n} = 0$, it is easy to see that $\lim_{n\to\infty} x_{2A_n} = 0$ too. So we conclude that $\lim_{n\to\infty} f(x_n) = x_{2A}$.

2. Suppose that $f : [0; 2] \to \mathbb{R}$ satisfies $f(x_A) = x_{2A}$ for any $A \subseteq \mathbb{N}$. For any rational x_A, A is finite. Let n_0 be the maximal element of A and $A' := A\setminus\{n_0\}$ and define,

for all $n \in \mathbb{N}$, a finite set A_n by

$$A_n := A' \cup \{n_0 + 1, n_0 + 2, \ldots, n_0 + n\}. \tag{4}$$

Then it is easy to see that $\lim_{n\to\infty} x_{A_n} = x_A$. On the other hand, we have

$$\begin{aligned} f(x_{A_n}) = x_{2A_n} &= x_{2A'} + \sum_{i=1}^{n} 2^{-2(n_0+i)} \\ &= x_{2A} - 2^{-2n_0} + 2^{-2n_0}(1 - 2^{-2n})/3. \end{aligned}$$

This implies that $\lim_{n\to\infty} f(x_{A_n}) = x_{2A} - 2^{-2n_0+1}/3 \neq x_{2A}$. $\square$

In summary, the known closure properties of several classes of real numbers under limits, effective limits, partial computable operators and computable operators are listed in the following table:

	Arithmetic operations	Limits	Effective limits	Computable operators	Partial computable operators
$\mathbf{C}_e$	Yes	Yes	Yes	Yes	Yes
$\mathbf{C}_{lc}$	No	Yes	Yes	No	No
$\mathbf{C}_{rc}$	No	Yes	Yes	No	No
$\mathbf{C}_{sc}$	No	No	Yes	?	No
$\mathbf{C}_{wc}$	Yes	No	No	?	No
$\mathbf{C}_{ra}$	Yes	No	Yes	Yes	Yes

5. Proof of Theorem 2.8

In the last section of this paper, we give a complete proof of Theorem 2.8. Our proof uses the following technical lemma whose proof is straightforward and omitted here.

Lemma 5.1. *Let $A, B, C \subset \mathbb{N}$ be finite sets such that $x_A = x_B - x_C$ and n, m and y be any natural numbers. If $n, m \in B \backslash C$, $n < y < m$ and $(B\upharpoonright(n;m))\backslash\{y\} = (C\upharpoonright(n;m))\backslash\{y\}$, then $n \notin A \Leftrightarrow y \in C\backslash B$.*

Proof of Theorem 2.8. Let $\{(V_i[s])_{s\in\mathbb{N}} : i \in \mathbb{N}\}$ enumerate effectively and uniformly all computable sequences of finite subsets of $\mathbb{N}$. Define $V_i := \lim_{s\to\infty} V_i[s]$ if the limit exists. Then, a set V is ω-r.e. iff there are $i, j \in \mathbb{N}$ such $V = V_i$ and φ_j is a total function which bounds the enumeration $(V_i[s])_{s\in\mathbb{N}}$, namely,

$$|\{s \in \mathbb{N}:\ y \in V_i[s] \Delta V_i[s+1]\}| < \varphi_j(y) \tag{5}$$

for any $y \in \mathbb{N}$. In this case, we say that the pair (i, j) *satisfies ω-condition*. As an approximation, we say that a pair (i, j) *satisfies ω-condition on x at stage s*, if the following conditions hold:

$$\forall y \leqslant x\ (\varphi_j(y)[s]\downarrow) \tag{6}$$

and

$$\forall y \leqslant x\ (|\{t < s\colon\ y \in V_i[t] \Delta V_i[t+1]\}| < \varphi_j(y)[s]). \tag{7}$$

It is not difficult to show that

Claim A. *A pair (i, j) satisfies ω-condition iff for any x there is an s such that (i, j) satisfies ω-condition on x at any stages $t \geqslant s$.*

For the proof of theorem, it suffices to construct two r.e. sets B, C and a set A which satisfy, for all $i, j \in \mathbb{N}$, the following requirements:

P: $x_A = x_B - x_C$,

$R_{\langle i,j\rangle}$: (i, j) satisfies ω-condition $\Rightarrow A \neq V_i$.

We will construct effectively sets A, B and C in stages. At the same time, we define also a sequence $(x_e[s])_{s\in\mathbb{N}}$ of witnesses, two sequences $(y_e[s])_{s\in\mathbb{N}}$ and $(z_e[s])_{s\in\mathbb{N}}$ of supplementary elements and a sequence $(a_e[s])_{s\in\mathbb{N}}$ of states for any requirement R_e. We will choose these sequences so that $x_e[s] < y_e[s] < z_e[s]$ and $a_e[s] \in \{0, 1, 2\}$ for any e, s. Suppose that $e = \langle i, j\rangle$. Then $x_e[s]$ is a possible witness for $V_i \neq A$. We will change $A(x_e)[s]$ by putting $y_e[s]$ into B or C if it is necessary and possible. The $z_e[s]$ serves as a "firewall" preserving the actions for R_e being disturbed by the actions for other requirements with lower priority. At any stages s, the requirement R_e is in one of three states: "inactive", "active" or "satisfied" which are denoted by $a_e[s] = 0$, 1 and 2, respectively. Roughly, a requirement R_e is in the state "inactive" means that we have done nothing for R_e after the current witness x_e having been appointed to R_e. R_e is in "active" means that R_e receives some actions for current witness x_e and is waiting for possible further action. R_e is in "satisfied" state means that the premise of R_e seems false and we need not do anything for R_e any more.

To meet the requirement R_e for $e = \langle i, j\rangle$, we have to change $A(x_e)$ from 0 to 1 or vice versa. Of course, it is only necessary if (i, j) seems satisfying ω-condition. More precisely, we change $A(x_e)$ at some stage s only if (i, j) satisfies ω-condition on x_e at stage s. If (i, j) satisfies ω-condition on x_e at some stage s_1 and it does not at a later stage s_2, then (i, j) will not satisfy ω-condition on x_e at any stage after s_2 any more. In this case we stop doing anything for R_e. If (i, j) do satisfy ω-condition, then $V_i(x_e)$ can change at most $\varphi_j(x_e)$ times. We reserve exclusively an interval $(m_0; z_e]$ with some proper $m_0 > x_e$ and $z_e = m_0 + \varphi_j(x_e) + 1$ for requirement R_e, so that we have enough chances to change the value of $A(x_e)$ by putting some element from this interval into B or C according to Lemma 5.1. We define $A[s]$, $B[s]$ and $C[s]$ so that they satisfy always $x_{A[s]} = x_{B[s]} - x_{C[s]}$. To put x_e into A at the first time, we simply put x_e into B. Since x_e cannot be taken out of B, we take x_e out of A by putting an element $y_e := x_e + 1$ into C. Later, if we want to put x_e into A again, we need only put y_e into B. In this case we need a new supplementary $y_e := y_e + 1$ which is ready to be put into C to force x_e out of A. To guarantee this procedure works and is not disturbed by the actions for lower priority requirements, we put z_e into B too. Then, by Lemma 5.1, this procedure does really work and can be repeated at most $2\varphi_j(x_e)$ times. Then, either we have enough chances to make $A(x_e)$ different from $V_i(x_e)$, if (i,j) satisfies ω-condition, or we can show at some stage that (i,j) does not satisfy ω-condition, if (i,j) does not satisfy ω-condition indeed. In both cases, R_e can be satisfied by this strategy.

The construction:

Stage $s = 0$: Define $A[0] = B[0] = C[0] = \emptyset$, $x_e[0] = 3e$, $y_e[0] := 3e+1$, $z_e[0] = 3e+2$ for all $e \in \mathbb{N}$. Set all requirement R_e into the state of "inactive" by defining $a_e[0] := 0$. We say that all requirement is *initialized* at this stage.

Stage $s+1$: Given $A[s]$, $B[s]$, $C[s]$ with $x_{A[s]} = x_{B[s]} - x_{C[s]}$, $x_e[s]$, $y_e[s]$, $z_e[s]$ and $a_e[s]$. A requirement R_e $(e = \langle i,j \rangle)$ *requires attention* if the following conditions hold:
(R1) R_e is not in the state of "satisfied", i.e. $a_e[s] \neq 2$;
(R2) $V_i(x_e)[s] = A(x_e)[s]$; and
(R3) (i,j) satisfies ω-condition on $x_e[s]$ at stage s.
If there is no requirement which requires attention, then go directly to next stage. All parameters remain unchanged. Otherwise, choose the minimal $e = \langle i,j \rangle$ such that R_e requires attention. Consider the following two cases:

Case 1: $a_e[s] = 0$, i.e. R_e is in the state of "inactive". This means that R_e does not receive attention yet for the current witness $x_e[s]$, hence $x_e[s]$ is neither in $B[s]$ nor $C[s]$ (In fact, $x_e[s]$ is also not in $A[s]$). Of course it is possible that some elements bigger than $x_e[s]$ have been put into $B[s]$ or $C[s]$ by actions for $R_{e'}$ with $e' > e$. Let $m_0 = \max(B[s] \cup C[s] \cup \{x_e[s]+1\})$ and redefine

$$x_e[s+1] = x_e[s],$$

$$y_e[s+1] = m_0 + 1,$$

$$z_e[s+1] = m_0 + 1 + \varphi_j(x_e)[s]. \tag{8}$$

Then put all natural numbers from the interval $(x_e[s], m]$ both into B and C and put $x_e[s]$ and $z_e[s+1]$ into B. (Notice that $x_e[s]$ and $z_e[s+1]$ will never be put into C by the action for R_e!)

We set now R_e into the state of "active" by defining $a_e[s+1] = 1$.

In this case, all requirements R_t with lower priority $(t > e)$ should be "initialized" by the following actions:
(I1) Define $a_t[s+1] := 0$;
(I2) Define $x_t[s+1] := z_e[s+1] + 3t + 1$, $y_t[s+1] := z_e[s+1] + 3t + 2$ and $z_t[s+1] := z_e[s+1] + 3t + 3$.

Case 2: $a_e[s]=1$, i.e. R_e is in the state of “active”. This means that R_e received attention for the current witness $x_e[s]$ before stage $s+1$. Hence $x_e[s]$ and $z_e[s]$ are already been put into $B[s]$.

Now if $y_e[s]\notin B[s]$, then simply put $y_e[s]$ into B.

Otherwise, suppose that $y_e[s]\in B[s]$. If $y_e[s]+1<z_e[s]$, then redefine $y_e[s+1]:=y_e[s]+1$ and put $y_e[s]$ into C. Otherwise, if $y_e[s]+1=z_e[s]$, then set R_e into the state of “satisfied” by defining $a_e[s+1]:=2$.

At last, define $A[s+1]$ as the unique finite set satisfying $x_{A[s+1]}=x_{B[s+1]}-x_{C[s+1]}$. All other parameters remain the same as in stage s. In both of these cases, we say that R_e *receives attention.*

End of the construction.

We show now that the construction suffices by proving the following sublemmas:

Sublemma 1. $B:=\lim_{s\to\infty} B[s]$ *and* $C:=\lim_{s\to\infty} C[s]$ *are r.e. sets.*

Proof. By the construction, we may put some elements into B or C at some stages and never take element out of B and C at any stage. This means that $B[s]\subseteq B[s+1]$ and $C[s]\subseteq C[s+1]$ hold for all s. Hence $(B[s])_{s\in\mathbb{N}}$ and $(C[s])_{s\in\mathbb{N}}$ are effective enumerations of B and C, respectively. That is $B=\lim_{s\to\infty} B[s]=\bigcup_{s\in\mathbb{N}} B[s]$ and $C=\lim_{s\to\infty} C[s]=\bigcup_{s\in\mathbb{N}} C[s]$. This implies that B and C are r.e. $\square$

Sublemma 2. *For any* $e\in\mathbb{N}$, *the requirement* R_e *requires and receives attentions at most finitely often.*

Proof. Assume by induction hypothesis that, for any $t<e=\langle i,j\rangle$, the requirement R_t requires and receives attentions at most finite often and is eventually satisfied. Choose a minimal s_0 such that no R_t $(t<e)$ requires and receives attention after stage s_0.

By the minimality of s_0, all requirement R_t for $t\geqslant e$ is initialized at stage s_0. Since R_e will never be initialized after stage s_0, $x_e[s]$ remains the same for all $s\geqslant s_0$. We denote $x_e:=x_e[s_0]$.

If R_e requires and hence receives no attention after stage s_0, then R_e requires and receives attentions at most finitely often.

Otherwise, suppose that R_e requires and receives attentions at stages $s_1<s_2<s_3<\cdots$ after stage s_0, respectively. Since R_e is initialized at stage s_0, R_e receives attention at stage s_1 according to Case 1. Namely we define $y_e[s_1]$, $z_e[s_1]$ according to (8) and put $x_e[s_1]$, $z_e[s_1]$ into B. Since R_e will never be initialized after stage s_1, $z_e[s]$ remains same for any $s\geqslant s_1$. We denote simply $z_e=z_e[s_1]$.

Note that x_e and z_e will never be put into C and $z_e<x_t[s], y_t[s], z_t[s]$ for any $t>e$ and $s\geqslant s_1$. It follows that $x_e, z_e\in(B\backslash C)[s]$ for all $s\geqslant s_1$. Applying Lemma 5.1 we can prove by induction on $k\geqslant 0$ that:

$$s_{2k+2}\leqslant s<s_{2k+3}\Rightarrow x_e\notin A[s]\ \&\ y_e[s]=y_e[s_1]+k, \tag{9}$$

$$s_{2k+3}\leqslant s<s_{2k+4}\Rightarrow x_e\in A[s]\ \&\ y_e[s]=y_e[s_1]+k \tag{10}$$

and furthermore

$$s_{2k+1} - 1 \leqslant s < s_{2k+2} - 1 \Rightarrow x_e \notin V_i[s], \tag{11}$$

$$s_{2k+2} - 1 \leqslant s < s_{2k+3} - 1 \Rightarrow x_e \in V_i[s]. \tag{12}$$

It follows that, if R_e receives two successive attentions at stages, say s_k and s_{k+1}, then y_e is increased by 1. On the other hand, y_e is always bounded by z_e according to the construction. Therefore, R_e can receive attentions at most finitely often. □

Sublemma 3. *$A := \lim_{s\to\infty} A[s]$ exists and satisfies that $x_A = x_B - x_C$.*

Proof. By Sublemma 1, $\lim_{s\to\infty}(x_{B[s]} - x_{C[s]}) = x_B - x_C$ exists. Since $x_{A[s]} = x_{B[s]} - x_{C[s]}$ for any $s \in \mathbb{N}$ hold, it suffices to show that $\lim_{s\to\infty} A[s]$ exists, i.e., for any x, $A(x)[s]$ changes at most finitely often.

By the construction, $x_t[s] < y_t[s] < z_t[s] < x_{t+1}[s]$ holds for all $t, s \in \mathbb{N}$. This implies that $t \leqslant x_t[s]$ for any $t, s \in \mathbb{N}$. For any natural number n, by Sublemma 2, there is an s such that no requirement R_t $(t \leqslant n)$ requires and receives attentions after stage s. This means that no elements less than $x_n[s]$ will be put into or take out of B or C after stage s. That is, after stage s, only the elements bigger that $x_n[s]$ can enter or leave B or C. By a simple induction, we can show that $x_{(B\upharpoonright[x_t,z_t])[s]} - x_{(C\upharpoonright[x_t,z_t])[s]} \geqslant 0$ holds for any $t, s \in \mathbb{N}$. By the fact that $x_{A[s]} = x_{B[s]} - x_{C[s]}$, it follows from Lemma 5.1 that $A[s] \upharpoonright (x_n + 1)[s]$ will not change any more after stage s. Especially, this implies that the limit $\lim_{s\to\infty} A(n)[s]$ exists. Because n is arbitrary, it follows that $A := \lim_{s\to\infty} A[s]$ exists. □

Sublemma 4. *For any $e \in \mathbb{N}$, the requirement R_e is satisfied eventually.*

Proof. For any $e = \langle i, j\rangle$, there is, by Sublemma 2, a minimal s_0 such that no requirement R_t with $t < e$ requires and receives attentions after stage s_0. Then R_e is initialized at stage s_0 and $x_e[s] = x_e[s_0]$ holds for all $s \geqslant s_0$. Denote $x_e := x_e[s_0]$.

If R_e requires and receives no attentions after stage s_0, then R_e remains in the state of "inactive" after stage s_0, i.e. (R1) is always satisfied after stage s_0. It is also easy to see that $x_e \notin B[s]$ for any $s \geqslant s_0$. By a proof similar to that of Sublemma 3 we can show also that $x_e \notin A[s]$ for any $s \geqslant s_0$, hence $A(x_e) = \lim_{s\to\infty} A(x_e)[s] = 0$ by Sublemma 3. After stage s_0, conditions (R2) and (R3) cannot hold at the same time. If there is no s such that (R3) always holds after stage s, then x_e witnesses that (i, j) does not satisfy ω-condition by Claim A. Hence R_e is satisfied trivially. Suppose now that there is a $s' \geqslant s_0$ such that (R3) is satisfied after stage s'. We claim that $\lim_{s\to\infty} V_i(x_e)[s] \neq 0 = A(x_e)$, hence x_e witnesses the satisfaction of R_e. Assume by contradiction that $\lim_{s\to\infty} V_i(x_e)[s] = 0$. Then there is an $s'' > s'$ such that $V_i(x_e)[s''] = 0 = A(x_e)[s'']$. In this case all conditions (R1)–(R3) are satisfied at stage s''. R_e will require and also receive attention at stage s''. This contradicts the choice of s_0.

Suppose that R_e receives attentions at least once after stage s_0. Assume, by Sublemma 2, that $s_1 < s_2 < s_3 < \cdots < s_{k_0}$ are all stages bigger than s_0 at which R_e receives attention. Then, (9)–(12) hold for corresponding k. We consider the stage s_{k_0}. There are two possibilities:

Situation 1: R_e gets the state "satisfied" at stage s_{k_0}, i.e. $a_e[s_{k_0}] = 2$. By the construction, this means that $y_e[s_{k_0} - 1] + 1 = z_e[s_{k_0} - 1] = z_e[s_1] = y_e[s_1] + \varphi_j(x_e) + 1$. It follows from (9)–(12) that $V_i(x_e)[s]$ changes at least $2\varphi_j(x_e)$ times before stage s_{k_0}, hence

$$|\{t < s_{k_0}\colon\ x_e \in V_e[t] \Delta V_e[t+1]\}| > \varphi_j(x_e). \tag{13}$$

This implies that, for any $s \geqslant s_{k_0}$, the pair (i, j) does not satisfy ω-condition at stage s. Hence (i, j) does not satisfy ω-condition by Claim A. Therefore R_e is satisfied eventually because its premise is false.

Situation 2: R_e gets the state of "active" at stage s_{k_0}. Since R_e receives last attention at stage s_{k_0}, no element in the interval $[x_e[s_{k_0}]; z_e[s_{k_0}]]$ will enter or leave B or C after stage s_{k_0}. It is not difficult to see that $A(x_e) = A(x_e)[s_{k_0}]$. By the construction, $V_e(x_e)[s_{k_0} - 1] = A(x_e)[s_{k_0} - 1] \neq A(x_e)[s_{k_0}] = A(x_e)$. We claim that $V_i(x_e)[s] = V_e(x_e)[s_{k_0}]$ for any $s \geqslant s_{k_0}$. This implies that $V_i(x_e) = V_i(x_e)[s_{k_0}] \neq A(x_e)$, hence R_e is eventually satisfied. Otherwise if $V_i(x_e)[s'] \neq V_i(x_e)[s_{k_0}]$ for some minimal $s' \geqslant s_{k_0}$, then $V_i(x_e)[s'] = A(x_e)[s]$. Obviously, (i, j) satisfied ω-condition on x_e at stage s' too. Then (R1)–(R3) are all satisfied and R_e requires and receives attention at stage s'. This contradicts the choice of s_{k_0}.

So in both situations, R_e is satisfied eventually.

Now it is easy to see that the constructed set A satisfies the theorem. By Sublemmas 1 and 3, x_A is a weakly computable real number. On the other hand, if V is any ω-r.e. set, then there are i, j such that $V_i = V$ and φ_j is a total bounding function of the enumeration $(V_i[s])_{s \in \mathbb{N}}$. Then (i, j) satisfies ω-condition. By Sublemma 4, $A \neq V$. This concludes that A is not ω-r.e. and completes the proof of the theorem. $\square$

References

[1] K. Ambos-Spies, K. Weihranch, X. Zheng, Weakly computable real numbers, J. Complexity 6 (2000) 676–690.

[2] C. Calude, A characterization of c.e. random reals, CDMTCS Research Report Series 095, March 1999.

[3] C. Calude, P. Hertling, B. Khoussainov, Y. Wang, Recursive enumerable reals and Chaintin's Ω-number, STACS'98, 1998, pp. 596–606.

[4] G.S. Ceĭtin, A pseudofundamental sequence that is not equivalent to a monotone one, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 20 (1971) 263–271, 290 (in Russian).

[5] G.J. Chaitin, A theory of program size formally identical to information theory, J. ACM 22 (1975) 329–340.

[6] A. Grzegorczyk, On the definitions of recursive real continuous functions, Fund. Math. 44 (1957) 61–71.

[7] Ker-I Ko, Reducibilities of real numbers, Theoret. Comp. Sci. 31 (1984) 101–123.

[8] Ker-I Ko, Complexity Theory of Real Functions, Birkhäuser, Berlin, 1991.

[9] M. Pour-El, J. Richards, Computability in Analysis and Physics, Springer, Berlin, Heidelberg, 1989.

[10] T.A. Slaman, Randomness and recursive enumerability, preprint, 1999.

[11] R. Soare, Recursion theory and Dedekind cuts, Trans. Amer. Math. Soc. 140 (1969) 271–294.

[12] R. Soare, Cohesive sets and recursively enumerable Dedekind cuts, Pacific J. Math. 31 (1) (1969) 215–231.

[13] R. Soare, Recursively Enumerable Sets and Degrees, Springer, Berlin, Heidelberg, 1987.

[14] R. Solovay, Draft of a paper (or series of papers) on Chaitin's work … done for the most part during the period of Sept.–Dec. 1975, unpublished manuscript, IBM Thomas J. Watson Research Center, Yorktoen Heights, New York, May 1975, 215pp.

[15] E. Specker, Nicht konstruktive beweisbare Sätze der Analysis, J. Symbolic Logic 14 (1949) 145–158.

[16] K. Weihrauch, Computability, EATCS Monographs on Theoretical Computer Science, Vol. 9, Springer, Berlin, Heidelberg, 1987.

[17] K. Weihrauch, An Introduction to Computable Analysis, Springer, Berlin, Heidelberg, 2000.

[18] K. Weihrauch, X. Zheng, A finite hierarchy of the recursively enumerable real numbers, MFCS'98, Brno, Czech Republic, August 1998, pp. 798–806.

[19] K. Weihrauch, X. Zheng, Arithmetical hierarchy of real numbers, MFCS'99, Szklarska Poreba, Poland, September 1999, pp. 23–33.

[20] X. Zheng, Binary enumerability of real numbers, Computing and Combinatorics, Proc. COCOON'99, Tokyo, Japan, July 26–28, 1999, pp. 300–309.