DESIGN OF A $d$-CONNECTED DIGRAPH WITH A MINIMUM NUMBER OF EDGES AND A QUASIMINIMAL DIAMETER

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Two fundamental considerations in the design of a communications network are reliability and maximum transmission delay, which can be respectively measured by the connectivity and diameter of a graph representing the network. A $d$-connected digraph (directed graph) is constructed, which has a minimum number of edges and the diameter is at most one larger than the lower bound for any $d$ and any number of nodes $n>d^3$. It improves upon previous designs, which achieve maximum connectivity with a diameter twice as large as the lower bound.

1. Introduction

A communications network or a multiprocessor network is conveniently modeled by a graph $G=(V,E)$, in which the set of nodes $V$ corresponds to processors or switching elements, and the set of edges $E$ corresponds to communication links [1,10]. Overall reliability and maximum transmission delay are two fundamental considerations in the design of such networks [1,4,6,8,10]. Based on such a model, the overall reliability can be measured by the connectivity of the graph and the maximum transmission delay can be measured by the diameter. The connectivity $\kappa$ of a graph $G$ corresponds to the minimum number of nodes whose break-down disrupts communication between a pair of nodes. The diameter $D$ corresponds to the maximum over the lengths of the shortest paths between any pair of nodes.

The following problem is proposed by Schumacher [8]: Given $n$ and $d$, construct a graph $G=(V,E)$ with $|V|=n$ which has the following properties:

- $G$ has connectivity $d$,
- $|E|$ is minimal,
- $G$ has minimal diameter.
First, minimizing the number of edges, we get $d$-regular graphs where the number of connections for every node is exactly $d$. Among these we look for graphs with connectivity $d$ and a small diameter. We call $d$-regular graphs with connectivity $d$ \textit{maximally connected $d$-regular graphs}. (Minimizing the diameter first would lead us to the complete graph $K_n$.) Schumacher [8] presents a nearly optimal solution of this problem for undirected graphs. He gives an algorithm of constructing a maximally connected $d$-regular undirected graph with a diameter twice as large as the lower bound.

Designing such graphs for any number of nodes $n$ has a much stronger justification if the graphs represent local or metropolitan area packet communications networks than if they represent processor interconnection networks [9]. In a local or metropolitan area network, use of unidirectional links is more desirable than bidirectional links, because it can reduce the number of required transmitters and receivers, and it is compatible with current optical fiber transmission technology. Therefore, the above problem is more important for digraphs (directed graphs) representing networks with unidirectional links.

For $d$-regular digraphs $G$ with $n$ nodes, the lower bound of the diameter is given as

$$D \geq \lceil \log_d(n(d-1)+1) \rceil - 1,$$

where $1 < d$ and $\lceil x \rceil$ denotes the minimum integer not less than $x$ [5]. Sengupta, Joshi and Bandyopadhyay [9] present a method of constructing a maximally connected $d$-regular digraph $G$ with a diameter $D(G) \leq 2\lceil \log_d n \rceil + 1$ for any number of nodes $n$ and $d$. From (1), the diameter of this digraph is about twice as large as the lower bound.

On the other hand, a \textit{de Bruijn digraph} $G_{B}(d^{m}, d)$ [2] is proposed as a minimum diameter digraph, which has connectivity $d-1$ (one less than the upper bound for the given numbers of nodes and edges) and can only be constructed when the number of nodes $n$ is a power of $d$ [5, 7]. An extension of the de Bruijn digraph, which can be constructed for any number of nodes, is independently proposed by Imaic and Itoh [5], and Reddy, Pradhan and Kuhl [7]. Such a digraph is called a \textit{generalized de Bruijn digraph} $G_{B}(n, d)$ [3], and its connectivity is shown to be $d-1$ (one less than the upper bound) [6] and the diameter is shown to be \textit{quasiminimal (defined as at most one larger than the lower bound)} [5].

This paper presents a method of modifying the generalized de Bruijn digraph $G_{B}(n, d)$ to be maximally connected, by clarifying the properties of minimum cutsets in $G_{B}(n, d)$. By this method, for any $n > d^{3}$ and $d \geq 3$, we can construct a maximally connected $d$-regular digraph with a quasiminimal diameter. Since Reddy, Pradhan and Kuhl [7] give a method of constructing maximally connected 2-regular digraphs $D_{n}$ with quasiminimal diameter for any $n$, we can consider the problem is settled for $d \geq 2$ and any $n > d^{3}$. 
2. Definitions and properties of generalized de Bruijn digraphs

This section summarizes properties of generalized de Bruijn digraphs after defining several digraph terms used in this paper.

2.1. Definitions

Let \( G = (V, E) \) be a digraph where \( V \) is a set of nodes and \( E \) is a set of (directed) edges (i.e., ordered pairs of nodes). An edge \((u, v) \in E\), where \( u = v \), is called a self loop. For a node \( v \), the outdegree (indegree) is the number of nodes which are adjacent from (to) node \( v \). The maximum degree of a digraph \( G \) is the maximum outdegree and indegree of every node. A digraph \( G \) is called a \( d \)-regular digraph if the out- and indegrees of every node are equal to \( d \).

If \((u, v) \in E\), then \( u \) is a predecessor of \( v \); similarly, \( v \) is a successor of \( u \). A walk in \( G \) is an alternating sequence of nodes and edges, say \( v_0, e_1, v_1, \ldots, v_{j-1}, e_j, v_j, \ldots, e_k, v_k \) where \( e_i = (v_{i-1}, v_i) \). The distance from node \( u \) to node \( v \), denoted by \( \text{dis}(u, v) \), is the number of the edges contained in a shortest walk from \( u \) to \( v \). For a node-subset \( \mathcal{W} \subseteq V \) and a node \( v \) in \( V - \mathcal{W} \), \( \text{dis}(v, \mathcal{W}) \) denotes \( \min\{\text{dis}(v, u) \mid u \in \mathcal{W}\} \), while \( \text{dis}(\mathcal{W}, v) \) denotes \( \min\{\text{dis}(u, v) \mid u \in \mathcal{W}\} \). The diameter of \( G \), \( D(G) \), is the maximum distance from any node to any other node. A digraph \( G \) is said to be strongly connected if there is a walk from \( u \) to \( v \) and vice versa for every pair of distinct nodes, \( u \) and \( v \). The connectivity of \( G \), \( \kappa(G) \), is defined as the minimum number of nodes whose removal results in a trivial or not strongly connected digraph.

For a digraph \( G = (V, E) \) and a node-subset \( \mathcal{V}' \subseteq V \), \( S(\mathcal{V}') \) is defined as the set of the successors of \( \mathcal{V}' \), and \( P(\mathcal{V}') \) is defined as the set of the predecessors of \( \mathcal{V}' \). Namely, \( S(\mathcal{V}') = \{u \mid v \in \mathcal{V}' \text{ and } (u, v) \in E\} \) and \( P(\mathcal{V}') = \{u \mid v \in \mathcal{V}' \text{ and } (u, v) \in E\} \).

For a node set \( \mathcal{V}' \subseteq V \),

\[
S'(\mathcal{V}') \overset{\text{def}}{=} S(S^{-1}(\mathcal{V}')), \quad P'(\mathcal{V}') \overset{\text{def}}{=} P(P^{-1}(\mathcal{V}')).
\]

In other words, \( S'(\mathcal{V}') \) is the set of nodes to which there is a \( t \)-length walk from some node \( v \) in \( \mathcal{V}' \), while \( P'(\mathcal{V}') \) is the set of nodes from which there is a \( t \)-length walk to some node \( u \) in \( \mathcal{V}' \).

In the digraph shown in Fig. 1,

\[
S(0) = \{0,1,2\}, \quad P(0) = \{0,4,8\},
\]

\[
S^2(0) = \{0,1,2,3,4,5,6,7,8\}, \quad P^2(0) = \{0,1,2,4,5,6,8,9,10\}.
\]

2.2. Properties of generalized de Bruijn digraphs

In the generalized de Bruijn digraph \( G_B(n,d) = (V, E) \),

\[
V = \{0, 1, \ldots, n - 1\},
\]

\[
E = \{(u, v) \mid v = u \cdot d + a \text{ (mod } n\}, \quad a = 0, 1, \ldots, d - 1\}.
\]
Figure 1 shows a digraph $G_B(12,3)$ with diameter 3 and connectivity 2.

The diameter and connectivity of $G_B(n,d)$ have been shown in [5, 6, 7].

**Property 2.1** [5, 7]. Let $D(G_B(n,d))$ be $D$. Then $D = \lceil \log_d n \rceil$. Namely, $d^{D-1} < n \leq d^D$. From (1), this means that the diameter of $G_B(n,d)$ is quasiminimal.

**Property 2.2** [6]. If $n > d^3$ (i.e., the diameter $D \geq 4$), then $\kappa(G_B(n,d)) = d - 1$.

It is not so hard to see the following properties of $G_B(n,d)$ and the proofs can be found in [3, 6, 7].

**Property 2.3** [6]. For any node $v$ in $G_B(n,d) = (V,E)$, if $t < D(G_B)$,

$$|S'(v)| = |P'(v)| = d', \quad S'(v)/d'$$

where $V'/q$ means that the labels of all nodes in node-subset $V'$ take $q$ consecutive values mod $n$.

**Property 2.4** [6]. Let $v$ be a node of $G_B(n,d)$. If $V' \subseteq S'^{-1}(v)$ and $t < D(G_B)$, then $|S(V')| = d \cdot |V'|$, and if $V' \subseteq P'^{-1}(v)$ and $t < D(G_B)$, then $|P(V')| = d \cdot |V'|$.

**Property 2.5** [3, 7]. Let $\gcd(n,d-1) = g$, where $\gcd(p,q)$ is the greatest common divisor of $p$ and $q$. Then $d + g - 1$ nodes of $G_B(n,d)$ have a self loop.
3. Construction method

This section presents a method of constructing a maximally connected d-regular digraph with a quasiminimal diameter for any number of nodes \( n > d^3 \), by modifying generalized de Bruijn digraphs.

Let the set of nodes with self loops in \( G_B(n, d) \) be denoted by \( V_s = \{ u_1, \ldots, u_d, \ldots, u_s \} \), where \( s = d + \gcd(n, d - 1) - 1 \). Theorem 3.1 shows that the following method of modifying self loops in \( G_B(n, d) \) gives a maximally connected d-regular digraph with a quasiminimal diameter.

**Construction method of \( G_B^*(n,d) \).** Remove \( s \) self loops and add \( s \) edges, \((u_1, u_2), (u_2, u_3), \ldots, (u_{s-1}, u_s), (u_s, u_1)\), that will connect the nodes originally with self loops into a cycle of length \( s \).

Figure 2 shows \( G_B^*(12,3) \) with diameter 3 and connectivity 3.

**Theorem 3.1.** If \( n > d^3 \) and \( d \geq 3 \), then \( \kappa(G_B^*(n,d)) = d \) and \( D(G_B^*(n,d)) \leq \lfloor \log_d n \rfloor \).

4. Proof of Theorem 3.1

This section proves Theorem 3.1, by clarifying the properties of minimum cut-sets in \( G_B(n,d) \). Since it is clear that \( D(G_B^*(n,d)) \leq D(G_B(n,d)) = \lceil \log_d n \rceil \), it is enough to prove that \( \kappa(G_B^*(n,d)) = d \).
Let $T \subseteq V$ be an arbitrary minimum cut-set of $G_B(n, d) = (V, E)$, and two disjoint nonempty sets $Y$ and $Y'$ be a partition of $V - T$ such that $G_B(n, d) - T$ contains no edges from $Y$ to $Y'$. For $T$, $Y$ and $Y'$,

$K \overset{\text{def}}{=} \max_{y \in Y} \text{dis}(y, T), \quad K' \overset{\text{def}}{=} \max_{y \in Y'} \text{dis}(T, y).

Remark that $K \geq 1$, $K' \geq 1$ and $K + K' \leq D(G_B(n, d))$.

The following lemma is useful for proving Theorem 3.1.

**Lemma 4.1.** If $n > d^3$, $T \cap V_s = \emptyset$, $Y \cap V_s \neq \emptyset$ and $Y' \cap V_s \neq \emptyset$ for any $T$, $Y$ and $Y'$, then $\kappa(G_B(n, d)) = d$.

**Proof.** Any cut-set of $G_B(n, d)$ is also a cut-set of $G_B$. Any minimum cut-set $T$ of $G_B$ does not cut $G_B$ from the precondition. Thus, there is no cut-set of $G_B$ whose cardinality is $|T|$ or less, which implies the validation of this lemma, because $|T| = d - 1$. □

Hence, it is enough to prove that $G_B(n, d)$ satisfies the precondition of Lemma 4.1. First, we will prove in Lemma 4.2 that $G_B(n, d)$ does not have a minimum cut-set $T$ such that $K \neq 1$ and $K' \neq 1$. Next, we will prove in Lemma 4.3 that $G_B(n, d)$ satisfies the precondition of Lemma 4.1 in the case of $K = 1$ or $K' = 1$.

**Lemma 4.2.** For $d \geq 3$, $G_B(n, d)$ does not have a minimum cut-set $T$ such that $K \neq 1$ and $K' \neq 1$.

**Proof.** First, we will prepare the following inequalities (3) and (4). For each $t_j \in T$, let $Y_i(t_j) = \{y \in Y \mid \text{dis}(y, t_j) = i\}$ $(1 \leq i \leq K)$ and $Y'_i(t_j) = \{y' \in Y' \mid \text{dis}(t_j, y') = i\}$ $(1 \leq i \leq K')$. Since $G_B$ has maximum degree $d$, it is clear that $|Y_i(t_j)| \leq d^i$ and $|Y'_i(t_j)| \leq d^i$. Since every $y \in Y$ is contained in some $Y_i(t_j)$,

$$|Y| \leq \sum_{i=1}^{K} \sum_{t_j} |Y_i(t_j)| \leq \sum_{i=1}^{K} |T| \cdot d^i = |T| \cdot d^K - d - 1. \quad (3)$$

Let $D$ be $D(G_B)$. In a similar way, since $K + K' \leq D$, $Y'$ can be estimated as

$$|Y'| \leq \sum_{i=1}^{K'} \sum_{t_j} |Y'_i(t_j)| \leq \sum_{i=1}^{K'} |T| \cdot d^i \leq \sum_{i=1}^{D - K} |T| \cdot d^i = |T| \cdot d^{D - K} - d - 1. \quad (4)$$

Assume that $K \neq 1$ and $K' \neq 1$. We will derive a contradiction. Since $K + K' \leq D$, it is enough to consider that $D \geq 4$ (i.e., $n > d^3$) and the following three cases: (1) $K = 2$, (2) $K' = 2$, and (3) $3 \leq K \leq D - 3$.

When $D \geq 4$, from Property 2.2, $|T| = \kappa(G_B) = d - 1$. For any $V' \subseteq Y$, since $S(V') \subseteq Y \cup T$,
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\[ |S(V') \cap Y| = |S(V')| - |S(V') \cap T| \geq |S(V')| - |T| = |S(V')| - (d-1). \]  
\hspace{1cm} (5)

**Case 1**: $K=2$. For $T$, $Y$ and $Y'$, let $a = \min_{v \in Y_1} |T(v)|$, where $T(v) = \{ t \in T \mid \text{dis}(v, t) = 2 \}$ and $Y_2 = \{ y \in Y \mid \text{dis}(y, T) = 2 \}$. Thus $1 \leq a \leq |T|$. Let $w$ be a node in $Y_2$ such that $|T(w)| = a$. Since $w \in Y_2$ and there are no edges from $Y$ to $Y'$, $S^2(w) \subseteq T \cup Y$. Then, from Property 2.3 and $|S^2(w) \cap T| = |T(w)| = a$,

\[ |S^2(w) \cap Y| = |S^2(w)| - |S^2(w) \cap T| = d^2 - a. \]

From (5), Property 2.4, and $|S^2(w) \cap Y| = d^2 - a$,

\[ |S^3(w) \cap Y| \geq |S(S^2(w) \cap Y) \cap Y| \geq |S(S^2(w) \cap Y)| - (d-1)^2 \geq (d^2 - a)d - (d-1) = d^3 - (a+1)d + 1. \]  
\hspace{1cm} (6)

When $D \geq 5$, also

\[ |S^4(w) \cap Y| \geq |S(S^3(w) \cap Y) \cap Y| \geq |S(S^3(w) \cap Y)| - (d-1)^3 \geq (d^3 - (a+1)d + 1)^2 = d^4 - (a+1)d^2 + 1. \]

Since $a \leq d-1$,

\[ |Y| \geq |S^4(w) \cap Y| \geq d^4 - (a+1)d^2 + 1 \geq d^4 - d^3 + 1. \]

On the other hand, by substituting $K=2$ and $|T| = d-1$ into (3), we get $|Y| \leq (d-1)(d+d^2) = d^3 - d$; this contradicts $|Y| \geq d^4 - d^3 + 1$. Hence $D \leq 4$.

Hereafter, we will prove the remaining case, i.e., the case of $D=4$. Since $K + K' \leq D = 4$ and $K' \neq 1$, we get $K' = 2$. From the definition of $a$, every $v \in Y_2$ is contained in at least $\text{"a"}$ sets of $Y_2(t_j)$. Then

\[ |Y_2| \leq \sum_{t_j} |Y_2(t_j)| \leq |T| \frac{d^2}{a}. \]

Thus,

\[ |Y| \leq \sum_{t_j} |Y_1(t_j)| + |Y_2| \leq |T| \cdot d + |T| \frac{d^2}{a} = (d-1) \left( d + \frac{d^2}{a} \right). \]  
\hspace{1cm} (7)

Recall that $w$ is a node in $Y_2$ such that $|T(w)| = a$. Since dis$(w, y') \leq D = 4$ for every $y' \in Y'$, dis$(w, T(w)) = 2$, and dis$(w, T - T(w)) \geq 2$, it is valid that dis$(T(w), y') \leq D - 2 = 2$ or dis$(T - T(w), y') < D - 2 = 2$. In other words, dis$(T, y') < 2$ or dis$(T(w), y') = 2$ for every $y' \in Y'$. Thus,

\[ |Y'| \leq \sum_{t_j \in T} |Y_1'(t_j)| + \sum_{t_j \in T(w)} |Y_2'(t_j)| \leq |T| \cdot d + a \cdot d^2. \]  
\hspace{1cm} (8)

From (6) and (7), \((d-1)(d+d^2/a) \geq |Y| \geq d^3 - (a+1)d^2 + 1\); it is concluded that $a = 1$. Then $|Y| \geq d^3 - 2d + 1$. In a similar way, we can derive $|Y'| \geq d^3 - 2d + 1$. Thus,
\[ n = |T| + |Y| + |Y'| \geq d - 1 + 2(d^3 - 2d + 1) = 2d^3 - 3d + 1. \quad (9) \]

On the other hand, from (7), (8), \(|T| = d - 1\), and \(a = 1\), we get

\[ n = |T| + |Y| + |Y'| \leq d - 1 + (d - 1)(d + d^2) + (d - 1)d + d^2 = d^3 + 2d^2 - d - 1; \]

since \(d \geq 3\), this contradicts (9).

**Case 2:** \(K' = 2\). When \(D \geq 5\), we can derive a contradiction in the same manner as in Case 1, by replacing \(Y\), \(a\), \(T(v)\), \(Y_2\) with \(Y'\), \(a' = \min_{y \in Y_2} |T'(v)|\), \(T'(v) = \{t \in T \mid \text{dis}(t, u) = 2\}\), \(Y_2' = \{y \in Y' \mid \text{dis}(T', y) = 2\}\) and \(S'(\cdot)\) with \(P'(\cdot)\). When \(D \leq 4\), since \(K + K' \leq D\) and \(K \neq 1\), we get \(K = 2\). This was considered in Case 1.

**Case 3:** \(3 \leq K \leq D - 3\). From inequalities (3), (4) and \(|T| = d - 1\), we get \(n = |T| + |Y| + |Y'| \leq d - 1 + d(d^K - 1) + d(d^2 - K - 1)\). Since the right-hand side of this inequality is maximized under \(3 \leq K \leq D - 3\) by letting \(K = 3\) or \(K = D - 3\), we get \(n \leq d^{D-2} + d^4 - d - 1\). This contradicts \(d^{D-1} < n\).

Therefore, Lemma 4.2 holds. \(\square\)

Next, we will prove that \(G_B(n, d)\) satisfies the precondition of Lemma 4.1 in the case of \(K = 1\) or \(K' = 1\).

**Lemma 4.3.** If \(n > d^3\) (i.e., \(D(G_B(n, d)) \geq 4\)), \(d \geq 3\), and \(K = 1\) or \(K' = 1\), then \(T \cap V_s = \emptyset\), \(Y \cap V_s \neq \emptyset\) and \(Y' \cap V_s \neq \emptyset\).

**Proof.** From Property 2.2, \(|T| = \kappa(G_B) = d - 1\). Let \(T = \{t_1, t_2, \ldots, t_{d-1}\}\).

**Case 1:** \(K = 1\). First, we will show that \(|S(y) \cap Y| \leq 1\) for any \(y \in Y\). Assume \(|S(y) \cap Y| \geq 2\). Since \(D \geq 4\), from (5), Property 2.4, and this assumption,

\[ |S^2(y) \cap Y| \geq |S(S(y) \cap Y) \cap Y| \geq |S(S(y) \cap Y)| - (d - 1) = d\ |S(y) \cap Y| - (d - 1) \geq 2d - (d - 1) = d + 1, \]

\[ |S^3(y) \cap Y| \geq |S(S^2(y) \cap Y) \cap Y| \geq |S(S^2(y) \cap Y)| - (d - 1) \geq (d + 1)d - (d - 1) = d^2 + 1. \]

Thus, \(|Y| \geq |S^3(y) \cap Y| \geq d^2 + 1\). On the other hand, by substituting \(K = 1\) and \(|T| = d - 1\) into inequality (3), we get \(|Y| \leq (d - 1)d\). These are a contradiction. Consequently, \(|S(y) \cap Y| \leq 1\).

From this, \(|T| = d - 1\) and \(|S(y)| = d\), we get \(|S(y) \cap T| = |T| = d - 1\) and \(|S(y) \cap Y| = 1\). In other words, \(S(y)\) can be represented as

\[ S(y) = \{x, t_1, \ldots, t_{d-1}\} \quad (x \in Y) \quad \text{for any } y \in Y. \quad (10) \]

Next, we will show that \(|Y \cap V_s| = 1\) or 2. For any node \(y \in Y \cap V_s\), from (10) and Property 2.3, \(\{y, t_1, \ldots, t_{d-1}\} /d\). Since it is clear that the candidate of \(y\) takes at most two values, we get \(|Y \cap V_s| \leq 2\). Further, we can show that \(Y \cap V_s \neq \emptyset\).
Assume $Y \cap V_s = \emptyset$. For a node $y_1 \in Y$, let $S(y_1) \cap Y$ be $\{y_2\}$, $S(y_2) \cap Y$ be $\{y_3\}$, and $S(y_3) \cap Y$ be $\{y_4\}$. Then $y_1 \neq y_2$, $y_2 \neq y_3$ and $y_3 \neq y_4$. From (10) and Property 2.3,

\[
\{y_2, t_1, \ldots, t_{d-1}\} \equiv d, \tag{11}
\]

\[
\{y_3, t_1, \ldots, t_{d-1}\} \equiv d, \tag{12}
\]

\[
\{y_4, t_1, \ldots, t_{d-1}\} \equiv d.
\]

Since at most two values of $u$ can satisfy $\{u, t_1, \ldots, t_{d-1}\} \equiv d$, we get $y_2 = y_4$. From (11) and (12), without loss of generality, let $y_2 = y_3 + d \pmod{n}$. Since $y_3 \in S(y_2)$ and $y_2 \in S(y_3)$, from (2),

\[
\begin{align*}
\{y_3 &\equiv y_3 \cdot d + a_1 \pmod{n} \quad (0 \leq a_1 < d), \\
y_3 &\equiv y_2 \cdot d + a_2 \pmod{n} \quad (0 \leq a_2 < d).
\end{align*}
\]

Substituting $y_2 = y_3 + d \pmod{n}$ in these equations, we get

\[
\begin{align*}
y_3 + d &\equiv y_3 \cdot d + a_1 \pmod{n}, \\
y_3 &\equiv (y_3 + d) + a_2 \pmod{n}.
\end{align*}
\]

Subtracting the first equation from the second, we get $d^2 + d + a_2 - a_1 \equiv 0 \pmod{n}$. Let $p$ be the left-hand side of this equation. Since $p \neq 0$, $n$ is equal to $|p|$ or a divisor of $|p|$. From this and $|p| \leq d^2 + 2d - 1$, we get $n \leq d^2 + 2d - 1$; this contradicts $n > d^3$. Hence, $Y \cap V_s \neq \emptyset$, and it is concluded that $|Y \cap V_s| = 1$ or $2$.

Further, we can show that $T \cap V_s = \emptyset$. Let $i$ be a node in $Y \cap V_s$ and assume there is a node $j \in T \cap V_s$. Then, there exist two paths of length 2 from $i$ to $j$, that is $i, i, j$ and $i, j, j$, contradicting Property 2.3.

Since $|Y \cap V_s| = 1$ or $2$, $T \cap V_s = \emptyset$, and $|V_s| = d - 1 + \gcd(d - 1, n) \geq 3$ for $d \geq 3$, we get $|Y' \cap V_s| \geq 1$. Consequently, this lemma holds for $K = 1$.

**Case 2:** $K = 1$. From Property 2.4, in a similar way to Case 1, $P(y)$ can be represented as

\[
P(y) = \{x, t_1, \ldots, t_{d-1}\} \quad (x \in Y') \quad \text{for any } y \in Y'.
\]

Then,

\[
S(t_j) \supseteq Y' \quad \text{for any } t_j \in T. \tag{13}
\]

First, we will show that $|Y' \cap V_s| \leq 2$. Assume that $|Y' \cap V_s| \geq 3$. Let $u, v$ and $w$ be three distinct nodes in $Y' \cap V_s$. From $u, v, w \in S(t_j)$, $u, v$ and $w$ are contained in the set of nodes with $d$ consecutive labels. Without loss of generality, let $u = v - s \pmod{n}$ and $w = v + t \pmod{n}$ ($s, t > 0$ and $s + t < d$). Since $v$ is contained in $V_s$, $v = v \cdot d + a \pmod{n}$ ($0 \leq a < d$). Thus, $u = u \cdot d + a - s \pmod{n}$ and $w = v \cdot d + a + t \pmod{n}$. From $s + t < d$, $s, t > 0$ and $0 \leq a < d$, it is valid that either $0 \leq a - s < d$ or $0 \leq a + t < d$. In other words, $v \in P(u)$ or $v \in P(w)$. Without loss of generality, let $v \in P(u)$. From this, $T \subset P(u)$ and $u \in P(u)$, we get $|P(u)| \geq d + 1$; this contradicts Property 2.3. Thus, $|Y' \cap V_s| \leq 2$. 

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Next, we will show that $Y' \cap V_s \neq \emptyset$. Assume $Y' \cap V_s = \emptyset$. For a node $y_3 \in Y'$, let $P(y_3) \cap Y'$ be $\{y_2\}$, $P(y_2) \cap Y'$ be $\{y_1\}$, where $y_2 \neq y_1$ and $y_3 \neq y_2$ because $y_1$ and $y_2$ have no self loop. Then

$$y_2 \equiv y_1 \cdot d + a_1 \pmod{n} \quad (0 \leq a_1 < d), \quad (14)$$

$$y_3 \equiv y_2 \cdot d + a_2 \pmod{n} \quad (0 < a_2 < d). \quad (15)$$

Subtracting (14) from (15), we get

$$0 \equiv (y_2 - y_1)d + (a_2 - a_1) - (y_3 - y_2) \pmod{n}.$$

Denoting the right-hand side of this equation as $p$, we will prove $p \neq 0$. Since $y_1$, $y_2$ and $y_3$ are contained in the set of nodes with $d$ consecutive labels on mod $n$ from (13), we get $|y_2 - y_1| < d$, $|y_3 - y_2| < d$ and $|y_1 - y_3| < d$. If $y_2 - y_1 = 1$, from (14), $y_1 \equiv y_1 \cdot d + a_1 - 1 \pmod{n}$, where $0 \leq a_1 < d - 1$. Since $y_1 \notin V_s$, $a_1 - 1$ does not take a value from 0 to $d - 1$. Thus $a_1 = 0$. From this, $y_2 - y_1 = 1$, $a_2 \geq 0$ and $|y_1 - y_3| < d$, we get $p \neq 0$. In the case of $y_2 - y_1 = -1$, we can derive $p \neq 0$ in a similar way.

In the other cases, that is $|y_2 - y_1| \geq 2$, from $|a_2 - a_1| < d$ and $|y_3 - y_2| < d$, we get $p \neq 0$. Consequently, it is valid that $p \neq 0$. From this and $|p| \leq (d - 1)d + d - 1 = d^2 + d - 2$, we get $n \leq d^2 + d - 2$. This contradicts $n > d^3$. Hence, $Y' \cap V_s \neq \emptyset$.

From (13), it can be shown that $T \cap V_s = \emptyset$ in a similar way to Case 1. Since $|Y' \cap V_s| = 1$ or 2, $T \cap V_s = \emptyset$, and $|V_s| \geq 3$, we get $|Y \cap V_s| \geq 1$. Consequently, this lemma holds for $K_s = 1$. \square

5. Conclusion

This paper has considered the problem of constructing a $d$-connected digraph with a minimum number of edges and a quasiminimum diameter for any number of nodes $n$ in the range of $n > d^3$ and any $d > 2$. Since Reddy, Pradhan and Kuhl [7] presented a method of constructing maximally connected 2-regular digraphs $D_n$ with quasiminimal diameter for any $n$, we can consider the problem is settled for $d = 2$ and any $n > d^3$.

For $n \leq d^3$, we cannot obtain a unified method of constructing maximally connected $d$-regular digraphs only by replacing all self loops in $G_B(n, d)$ by a cycle, because the connectivity of some $G_B$, for example $G_B(d^2 - d, d)$, is less than $d - 1$. To settle this problem for any $n \leq d^3$, another method is required.

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