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On Sendov's Conjecture for Roots near the Unit Circle

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1. INTRODUCTION

Sendov's conjecture asserts that, if a polynomial (with complex coefficients) has all its roots in the unit disk, then within one unit of each of its roots lies a root of its derivative. First advanced in 1962, this conjecture has given rise to over 30 papers (for references see [1, 3]), but has been verified in general only for polynomials of degree less than 6. In this paper, we verify Sendov's conjecture for roots which are sufficiently close to the unit circle.

Let n be an integer greater than 1, and let β be a complex number of modulus at most 1. Define $S(n, \beta)$ to be the set of complex polynomials of degree n with all roots in the unit disk and at least one root at β. For any polynomial P of degree at least 2, let $|P|_\beta$ be the distance between β and the closest root of P'. In this notation, Sendov's conjecture becomes

Conjecture 1. If $P \in S(n, \beta)$, then $|P|_\beta \leqslant 1$.

To date, the only known examples of polynomials $P \in S(n, \beta)$ with $|P|_\beta = 1$ occur when $|\beta| = 1$. As such, obvious values of β to check for counterexamples to Sendov's conjecture are those on or near the unit circle.

We will assume (by rotation) that $0 \leqslant \beta \leqslant 1$. In [2], Rubinstein verified Sendov's conjecture for roots on the unit circle, by proving

THEOREM 2. *If $P \in S(n, 1)$, then $|P|_1 \leqslant 1$, with equality only if $P(z) = c(z^n - 1)$.*

In this paper, we verify Sendov's conjecture for roots sufficiently close to the unit circle, as a consequence of

THEOREM 3. *There are constants $K_n > 0$ so that, if β is sufficiently close to 1 and $P \in S(n+1, \beta)$, then $|P|_\beta \leqslant 1 - K_n(1-\beta)$. Furthermore, one can choose the K_n so that $\lim_{n\to\infty} K_n = 1/3$.*

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Note that, while the first statement of Theorem 3 is plausible if one believes Sendov's conjecture, the fact that the limit of the K_n does not approach 0 is quite startling. Indeed, Theorem 3 hints at the existence of a result which is even stronger than Sendov's conjecture.

For $P \in S(2, \beta)$, it can be easily seen that $|P|_\beta \leqslant (1+\beta)/2$, and so we may choose $K_1 = 1/2$. Thus, it will suffice to prove Theorem 3 for $n \geqslant 2$.

By choosing $K_n \leqslant 1$ we will take care of the case when $|P|_\beta \leqslant \beta$, for then $|P|_\beta \leqslant 1-(1-\beta) \leqslant 1-K_n(1-\beta)$. Define $T(n, \beta)$ to be the set of polynomials $P \in S(n, \beta)$ such that P' is monic and $|P|_\beta > \beta$. It will thus suffice to prove Theorem 3 for those polynomials $P \in T(n+1, \beta)$.

2. Approximating the Roots of P

We will prove Theorem 3 by showing that, if the roots of P are in the unit disk, then the roots of P' cannot be too far from β. To accomplish this, we will approximate the roots of P in terms of the coefficients of P', making use of

Lemma 4. *If $P'(w_0) \neq 0$, then there is a root z of P so that $|z-w_0| \leqslant n\,|P(w_0)/P'(w_0)|$.*

Proof. If $P(w_0)=0$, then take $z=w_0$. Otherwise, suppose that the roots of P are $z_1, ..., z_n$. Then

$$\left|\frac{P'(w_0)}{P(w_0)}\right| = \left|\sum_{i=1}^{n} \frac{1}{w_0 - z_i}\right| \leqslant \sum_{i=1}^{n} \frac{1}{|w_0 - z_i|},$$

so for some i, we have $|w_0 - z_i| \leqslant n\,|P(w_0)/P'(w_0)|$. ∎

If b_ε is a complex number depending on ε, we define $b_\varepsilon = \mathcal{O}(\varepsilon^m)$ to mean that there is a constant C (which depends only on the integer n from Theorem 3) such that for all sufficiently small $\varepsilon > 0$, we have $|b_\varepsilon| < C\varepsilon^m$. We define $b_\varepsilon = d_\varepsilon + \mathcal{O}(\varepsilon^m)$ to mean that $b_\varepsilon - d_\varepsilon = \mathcal{O}(\varepsilon^m)$.

Take any polynomial $P \in T(n+1, \beta)$ and write

$$P'(z) = \prod_{j=1}^{n} (z - \zeta_j) = \sum_{k=0}^{n} a_k z^k.$$

If β is close to 1, then the roots of P' must be close to 0. (This follows from the observation that any sequence of polynomials $P_k \in T(n+1, \beta_k)$ with β_k tending to 1 has by compactness a subsequence which converges to a polynomial $P \in S(n+1, 1)$ with $|P|_1 \geqslant 1$. By Theorem 2 this polynomial must be of the form $P(z) = c(z^{n+1}-1)$, so the roots of the P'_k tend to the roots of P', which are all 0.) Thus, given any $\varepsilon > 0$, we may choose β sufficiently close to 1 so that each $|\zeta_j| \leqslant \varepsilon$. Since each a_{n-m} is (plus or minus) the mth elementary symmetric function of $\zeta_1, ..., \zeta_n$, we have $|a_{n-m}| \leqslant \binom{n}{m} \varepsilon^m$ and so $a_{n-m} = \mathcal{O}(\varepsilon^m)$.

Our only requirement on β is that it be sufficiently close to 1, so (taking β larger if necessary) we may assume without loss of generality that $\beta \geqslant 1 - \varepsilon^2$. Then $\beta = 1 + \mathcal{O}(\varepsilon^2)$, and thus $\beta^k = [1 + (\beta - 1)]^k = 1 + \binom{k}{1}(\beta - 1) + \cdots = 1 + \mathcal{O}(\varepsilon^2)$.

By choice of β, the roots of P are close to the roots of $z^{n+1} - 1$. Using Lemma 4, we provide an $\mathcal{O}(\varepsilon^2)$ approximation to the roots of P in terms of the coefficients of P' via

LEMMA 5. *For every root z_0 of $z^{n+1} - 1 = 0$, there is a root z_ε of P so that $z_\varepsilon = z_0 + a_{n-1}(z_0 - 1)/n + \mathcal{O}(\varepsilon^2)$.*

Proof. Recall that $\beta^k = 1 + \mathcal{O}(\varepsilon^2)$, that $a_{n-1} = \mathcal{O}(\varepsilon^1)$, and that for $m > 1$ we have $a_{n-m} = \mathcal{O}(\varepsilon^2)$. Let $w_0 = z_0 + a_{n-1}(z_0 - 1)/n$, and note that by the binomial theorem, we have

$$w_0^k = z_0^k + k z_0^{k-1} a_{n-1}(z_0 - 1)/n + \mathcal{O}(\varepsilon^2).$$

Then

$$\begin{aligned} P(w_0) &= \int_\beta^{w_0} P'(w)\, dw \\ &= \sum_{k=0}^{n} \frac{a_k (w_0^{k+1} - \beta^{k+1})}{k+1} \\ &= \frac{w_0^{n+1} - \beta^{n+1}}{n+1} + \frac{a_{n-1}(w_0^n - \beta^n)}{n} + \mathcal{O}(\varepsilon^2) \\ &= \frac{(z_0^{n+1} + (n+1)\, z_0^n a_{n-1}(z_0 - 1)/n) - 1}{n+1} \\ &\quad + \frac{a_{n-1}(z_0^n - 1)}{n} + \mathcal{O}(\varepsilon^2) \\ &= \frac{a_{n-1} z_0^n (z_0 - 1)}{n} + \frac{a_{n-1}(z_0^n - 1)}{n} + \mathcal{O}(\varepsilon^2) \\ &= \mathcal{O}(\varepsilon^2). \end{aligned}$$

Further, $P'(w_0) = \sum_{k=0}^{n} a_k w_0^k = w_0^n + \mathcal{O}(\varepsilon^1) = z_0^n + \mathcal{O}(\varepsilon^1)$. Thus for sufficiently small ε (depending only on n), we have $P'(w_0) \neq 0$ and $P(w_0)/P'(w_0) = \mathcal{O}(\varepsilon^2)$, so by Lemma 4, there is a root z_ε of P such that $z_\varepsilon = w_0 + \mathcal{O}(\varepsilon^2)$. ∎

If b_ε is a real number depending on ε, we define $b_\varepsilon < \mathcal{O}(\varepsilon^m)$ to mean that there is a constant C (which may be either positive or negative, and which depends only on the integer n from Theorem 3) such that for all sufficiently small $\varepsilon > 0$, we have $b_\varepsilon < C\varepsilon^m$. Define $b_\varepsilon > \mathcal{O}(\varepsilon^m)$ similarly.

We now estimate the coefficients of P' with

PROPOSITION 6. (1) $a_{n-1} = \mathcal{O}(\varepsilon^2)$.
(2) $\Re(a_{n-2}) = (1/2) \sum_{j=1}^{n} |\zeta_j|^2 + \mathcal{O}(\varepsilon^4)$.

Proof. Since $P \in T(n+1, \beta)$ we know that $|P|_\beta > \beta$, so each $|\beta - \zeta_j| > \beta$. This implies that $\beta^2 - 2\beta\Re(\zeta_j) + |\zeta_j|^2 > \beta^2$, and hence that $\Re(\zeta_j) < |\zeta_j|^2/(2\beta)$. Since by choice of β each $|\zeta_j| \leqslant \varepsilon$, we have that $\Re(\zeta_j) < \mathcal{O}(\varepsilon^2)$. Since $a_{n-1} = -\sum_{j=1}^{n} \zeta_j$, it follows that $\Re(a_{n-1}) > \mathcal{O}(\varepsilon^2)$.

Let $z_1, ..., z_{n+1}$ be the roots of $z^{n+1} - 1 = 0$. We know by Lemma 5 that for each z_i there is a root z_ε of P so that

$$z_\varepsilon = z_i + a_{n-1}(z_i - 1)/n + \mathcal{O}(\varepsilon^2).$$

Since z_ε/z_i is in the unit disk, we have $\Re(z_\varepsilon/z_i) \leqslant 1$ and so

$$\Re[1 + a_{n-1}(1 - \overline{z_i})/n] < 1 + \mathcal{O}(\varepsilon^2).$$

Solving this inequality for $\Re(\overline{z_i} a_{n-1})$, we get $\Re(\overline{z_i} a_{n-1}) > \Re(a_{n-1}) + \mathcal{O}(\varepsilon^2)$. We have shown that $\Re(a_{n-1}) > \mathcal{O}(\varepsilon^2)$, so $\Re(\overline{z_i} a_{n-1}) > \mathcal{O}(\varepsilon^2)$. This implies that there is a constant $C > 0$ so that $\Re(\overline{z_i} a_{n-1}) > -C\varepsilon^2$ for $i = 1, ..., n+1$. Thus, a_{n-1} is located in a regular $(n+1)$-gon, each of whose sides is at a distance of $C\varepsilon^2$ from the origin, and so $a_{n-1} = \mathcal{O}(\varepsilon^2)$, which verifies the first statement of Proposition 6.

A simple computation provides that $(a_{n-1})^2 = 2a_{n-2} + \sum_{j=1}^{n} \zeta_j^2$. We have just proved that $a_{n-1} = \mathcal{O}(\varepsilon^2)$, and so $a_{n-2} = -(1/2) \sum_{j=1}^{n} \zeta_j^2 + \mathcal{O}(\varepsilon^4)$. We have shown that each $\Re(\zeta_j) < \mathcal{O}(\varepsilon^2)$, and we know that $\Re(\zeta_1 + \cdots + \zeta_n) = -\Re(a_{n-1}) > \mathcal{O}(\varepsilon^2)$, so each $\Re(\zeta_j) > \mathcal{O}(\varepsilon^2)$ and hence each $\Re(\zeta_j) = \mathcal{O}(\varepsilon^2)$. This implies that $\Re(\zeta_j^2) = -|\zeta_j|^2 + \mathcal{O}(\varepsilon^4)$, so $\Re(a_{n-2}) = (1/2) \sum_{j=1}^{n} |\zeta_j|^2 + \mathcal{O}(\varepsilon^4)$. This completes the proof of Proposition 6. ∎

Having proved that $a_{n-1} = \mathcal{O}(\varepsilon^2)$, we are now able to provide an $\mathcal{O}(\varepsilon^3)$ approximation to the roots of P via

LEMMA 7. *For every root z_0 of $z^{n+1} - 1 = 0$, there is a root z_ε of P so that*

$$z_\varepsilon = \beta z_0 + \frac{a_{n-1}(z_0 - 1)}{n} + \frac{a_{n-2}(z_0 - \overline{z_0})}{n-1} + \mathcal{O}(\varepsilon^3).$$

Proof. Let

$$w_1 = \frac{a_{n-1}(z_0 - 1)}{n} + \frac{a_{n-2}(z_0 - \overline{z_0})}{n-1}$$

and let $w_0 = \beta z_0 + w_1$. Recall that $\beta^k = 1 + \mathcal{O}(\varepsilon^2)$, and note that $a_{n-m} = \mathcal{O}(\varepsilon^2)$ for $m \geq 1$, so $w_1 = \mathcal{O}(\varepsilon^2)$. Now for any positive integer k, we have

$$\begin{aligned} w_0^k - \beta^k &= (\beta z_0 + w_1)^k - \beta^k \\ &= \beta^k z_0^k + k\beta^{k-1} z_0^{k-1} w_1 - \beta^k + \mathcal{O}(\varepsilon^4) \\ &= \beta^k (z_0^k - 1) + k z_0^{k-1} w_1 + \mathcal{O}(\varepsilon^4), \end{aligned}$$

so $w_0^{n+1} - \beta^{n+1} = (n+1)\, z_0^n w_1 + \mathcal{O}(\varepsilon^4)$, and for $k \leq n$ we have $w_0^k - \beta^k = z_0^k - 1 + \mathcal{O}(\varepsilon^2)$. Then

$$\begin{aligned} P(w_0) &= \int_\beta^{w_0} P'(w)\, dw \\ &= \sum_{k=0}^{n} \frac{a_k (w_0^{k+1} - \beta^{k+1})}{k+1} \\ &= \frac{w_0^{n+1} - \beta^{n+1}}{n+1} + \frac{a_{n-1}(w_0^n - \beta^n)}{n} \\ &\quad + \frac{a_{n-2}(w_0^{n-1} - \beta^{n-1})}{n-1} + \mathcal{O}(\varepsilon^3) \\ &= z_0^n w_1 + \frac{a_{n-1}(z_0^n - 1)}{n} + \frac{a_{n-2}(z_0^{n-1} - 1)}{n-1} + \mathcal{O}(\varepsilon^3) \\ &= \mathcal{O}(\varepsilon^3) \qquad \text{(by choice of } w_1\text{)}. \end{aligned}$$

Now $P'(w_0) = \sum_{k=0}^{n} a_k w_0^k = w_0^n + \mathcal{O}(\varepsilon^2) = z_0^n + \mathcal{O}(\varepsilon^2)$. Thus for sufficiently small ε (depending only on n), we have $P'(w_0) \neq 0$ and $P(w_0)/P'(w_0) = \mathcal{O}(\varepsilon^3)$, so by Lemma 4, there is a root z_ε of P such that $z_\varepsilon = w_0 + \mathcal{O}(\varepsilon^3)$. ∎

3. Proof of Theorem 3

We now provide a bound for $|P|_\beta$ via

Proposition 8. *Suppose that $z_1, \ldots, z_m$ are roots of $z^{n+1} - 1 = 0$, not all 1, such that $\sum_{i=1}^m z_i$ and $\sum_{i=1}^m z_i^2$ are real, and such that $\sum_{i=1}^m z_i^2 \leq \sum_{i=1}^m z_i$. Choose any C with $(\sum_{i=1}^m z_i)/m < C < 1$. Then for every β sufficiently close to 1, we have $|P|_\beta < 1 - (1-\beta)\, C/(C-1)$.*

Proof. Note first that for each j we have

$$|P|_\beta^2 \leq |\zeta_j - \beta|^2 = |\zeta_j|^2 - 2\beta \Re(\zeta_j) + \beta^2,$$

and average these inequalities over all j to obtain

$$|P|_\beta^2 \leqslant \frac{1}{n} \sum_{j=1}^{n} |\zeta_j|^2 + \frac{2\beta}{n} \Re(a_{n-1}) + \beta^2. \qquad (*)$$

To estimate $|P|_\beta$, we will produce an $\mathcal{O}(\varepsilon^3)$ bound for $\Re(a_{n-1})$.

We know by Lemma 7 that for every z_i there is a root z_ε of P so that

$$z_\varepsilon = \beta z_i + \frac{a_{n-1}(z_i - 1)}{n} + \frac{a_{n-2}(z_i - \overline{z_i})}{n-1} + \mathcal{O}(\varepsilon^3).$$

Since z_ε / z_i is in the unit disk, we have $\Re(z_\varepsilon/z_i) \leqslant 1$, and so

$$\beta + \Re\left[\frac{a_{n-1}(1-\overline{z_i})}{n}\right] + \Re\left[\frac{a_{n-2}(1-\overline{z_i}^2)}{n-1}\right] < 1 + \mathcal{O}(\varepsilon^3).$$

Averaging these inequalities over $z_1, ..., z_m$, and recalling that by hypothesis $\sum_{i=1}^m z_i$ and $\sum_{i=1}^m z_i^2$ are real, we obtain

$$\beta + \frac{\Re(a_{n-1})}{n}\left[1 - \frac{1}{m}\sum_{i=1}^{m} z_i\right] + \frac{\Re(a_{n-2})}{n-1}\left[1 - \frac{1}{m}\sum_{i=1}^{m} z_i^2\right] < 1 + \mathcal{O}(\varepsilon^3).$$

Solving for $\Re(a_{n-1})$, and recalling that by Proposition 6 we have $\Re(a_{n-2}) = (1/2) \sum_{j=1}^n |\zeta_j|^2 + \mathcal{O}(\varepsilon^4)$, we get

$$\Re(a_{n-1}) < \frac{n(1-\beta)}{1 - (\sum_{i=1}^m z_i)/m} - \frac{n}{(n-1)}\left[\frac{1 - (\sum_{i=1}^m z_i^2)/m}{1 - (\sum_{i=1}^m z_i)/m}\right]\left[\frac{1}{2}\sum_{j=1}^{n} |\zeta_j|^2\right] + \mathcal{O}(\varepsilon^3).$$

Let $D = (\sum_{i=1}^m z_i)/m$. By hypothesis $\sum_{i=1}^m z_i^2 \leqslant \sum_{i=1}^m z_i$, so

$$\Re(a_{n-1}) < \frac{n(1-\beta)}{1-D} - \frac{n}{2(n-1)} \sum_{j=1}^{n} |\zeta_j|^2 + \mathcal{O}(\varepsilon^3).$$

Substituting this bound for $\Re(a_{n-1})$ into inequality $(*)$, we obtain

$$|P|_\beta^2 \leqslant \sum_{j=1}^{n} |\zeta_j|^2 \left(\frac{1}{n} - \frac{\beta}{n-1}\right) + \frac{2\beta(1-\beta)}{1-D} + \beta^2 + \mathcal{O}(\varepsilon^3). \qquad (**)$$

Our only requirement on β is that it be sufficiently close to 1, so (taking β larger if necessary) we may assume without loss of generality that $1/n < \beta/(n-1)$.

The result (**) is true for all polynomials $P \in T(n+1, \beta)$, as long as ε is sufficiently small (depending only on n) and β is sufficiently close to 1 (requiring only that each $|\zeta_j| \leqslant \varepsilon$ and that $\beta \geqslant 1-\varepsilon^2$). Thus for any particular polynomial $P \in T(n+1, \beta)$, by taking ε smaller still, we may assume without loss of generality that either some $|\zeta_j| = \varepsilon$ or that $\beta = 1-\varepsilon^2$.

If some $|\zeta_j| = \varepsilon$, then

$$\sum_{j=1}^{n} |\zeta_j|^2 \left(\frac{1}{n} - \frac{\beta}{n-1}\right) + \mathcal{O}(\varepsilon^3) < \left(\frac{1}{n} - \frac{\beta}{n-1}\right) \varepsilon^2 + \mathcal{O}(\varepsilon^3)$$

$$< 0 \qquad \text{for sufficiently small } \varepsilon.$$

If instead we have that $\beta = 1-\varepsilon^2$, then

$$\sum_{j=1}^{n} |\zeta_j|^2 \left(\frac{1}{n} - \frac{\beta}{n-1}\right) + \mathcal{O}(\varepsilon^3) < \mathcal{O}(\varepsilon^3) = \mathcal{O}(1-\beta)^{3/2}.$$

In either case, inequality (**) implies that

$$|P|_\beta^2 < \beta^2 + \frac{2\beta(1-\beta)}{1-D} + \mathcal{O}(1-\beta)^{3/2},$$

and so (using the inequality $\sqrt{\beta^2 + 2x} < \beta + x/\beta$) we have

$$|P|_\beta < \beta + \frac{1-\beta}{1-D} + \mathcal{O}(1-\beta)^{3/2}.$$

By hypothesis $C > D$, so for β sufficiently close to 1 we have

$$\frac{1-\beta}{1-D} + \mathcal{O}(1-\beta)^{3/2} < \frac{1-\beta}{1-C}$$

so

$$|P|_\beta < \beta + \frac{1-\beta}{1-C} = 1 - (1-\beta)\frac{C}{C-1}. \quad \blacksquare$$

We now prove the first conclusion of Theorem 3. Let $z_1, ..., z_n$ be the roots of $z^{n+1} - 1 = 0$ which are not 1. Since $n \geqslant 2$ it follows that $\sum_{i=1}^{n} z_i^2 = \sum_{i=1}^{n} z_i = -1$, so we may choose $m = n$ and $C = -1/(n+1)$ in Proposition 8, and hence $K_n = C/(C-1) = 1/(n+2)$ in the first conclusion of Theorem 3.

To prove the second conclusion of Theorem 3, we first choose any C such that $-1/2 < C < 0$. For sufficiently large n, there will always be a root z_0 of $z^{n+1} - 1 = 0$ with $-1/2 < \Re(z_0) < C$. For such a root, $\Re(z_0^2) < -1/2$, so $z_0^2+\overline{z_0}^2<-1<z_0+\overline{z_0}$. Since $C>(z_0+\overline{z_0})/2$, we may by Proposition 8 with $m=2$ choose $K_n=C/(C-1)$. Let C tend to $-1/2$ to finish the proof of Theorem 3. ∎

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