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Isomorphism classes of the hypergroups of type U on the right of size five

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ABSTRACT

By means of a blend of theoretical arguments and computer algebra techniques, we prove that the number of isomorphism classes of hypergroups of type U on the right of order five, having a scalar (bilateral) identity, is 14 751. In this way, we complete the classification of hypergroups of type U on the right of order five, started in our preceding papers [M. De Salvo, D. Freni, G. Lo Faro, A new family of hypergroups and hypergroups of type U on the right of size five, Far East J. Math. Sci. 26(2) (2007) 393–418; M. De Salvo, D. Freni, G. Lo Faro, A new family of hypergroups and hypergroups of type U on the right of size five Part two, Mathematicki Vesnik 60 (2008) 23–45; M. De Salvo, D. Freni, G. Lo Faro, On the hypergroups of type U on the right of size five, with scalar identity (submitted for publication)]. In particular, we obtain that the number of isomorphism classes of such hypergroups is 14 865.

1. Introduction

Hypergroups of type U on the right were introduced in [1] to analyze properties of quotient hypergroups H/h of a hypergroup H with respect to a subhypergroup $h \subset H$ ultraclosed on the right. The class of hypergroups of type U on the right is rather wide, see e.g., [2–4], and includes that of hypergroups of type C on the right, see [5,6], and, in particular, that of cogroups [7–10] and that of quotient hypergroups G/g of a group G with respect to a non-normal subgroup $g \subset G$ (D-hypergroups) [9–12]. Papers [2,4] are indeed quite relevant in this context; the former introduces the concept of homology hypergroup within the category of hypergroups of type U on the right, and some results already known in the field of homology of abelian groups are extended to non-commutative groups, while in the latter the analysis of relationships existing between hypergroups of type U and hypergroups of double cosets is furthered.

Recently, in the paper [13] the authors proved the existence of finite semi-hypergroups of type U on the right and used this result to construct hypergroups of this kind with a stable part which is not a subhypergroup. Moreover, in [14] they proved that there exists a unique proper semihypergroup of this kind having order 6 and that the least order for a hypergroup of type U on the right to have a stable part which is not a subhypergroup is 9. Furthermore, in [13] it is shown that all semihypergroups of type U on the right having size not larger than 5 are hypergroups. Since in [15] the multiplicative tables of hypergroups of type U on the right of size ≤ 4 have been found, in the papers [16–18] the authors have begun the classification of hypergroups of type U on the right of size 5. This classification is determined according to the possible cases of the family $P_\varepsilon = \{\varepsilon \circ x | x \in H\}$, where ε is the right scalar identity of a hypergroup H of type U on the right. In particular,

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if $H = \{1, 2, 3, 4, 5\}$ and $\varepsilon = 1$, the possible cases for the family P_ε are the following:

- $1 \circ 1 = \{1\}$; $1 \circ 2 = 1 \circ 3 = 1 \circ 4 = 1 \circ 5 = \{2, 3, 4, 5\}$;
- $1 \circ 1 = \{1\}$; $1 \circ 2 = 1 \circ 3 = \{2, 3, 4, 5\}$; $1 \circ 4 = 1 \circ 5 = \{4, 5\}$;
- $1 \circ 1 = \{1\}$; $1 \circ 2 = 1 \circ 3 = 1 \circ 4 = \{2, 3, 4\}$; $1 \circ 5 = \{5\}$;
- $1 \circ 1 = \{1\}$; $1 \circ 2 = 1 \circ 3 = \{2, 3\}$; $1 \circ 4 = \{4\}$; $1 \circ 5 = \{5\}$;
- $1 \circ 1 = \{1\}$; $1 \circ 2 = 1 \circ 3 = \{2, 3\}$; $1 \circ 4 = 1 \circ 5 = \{4, 5\}$;
- $1 \circ x = \{x\}$, for all $x \in H$ (that is, $\varepsilon = 1$ is a scalar identity).

The first five cases are analyzed in [16,17]. In those papers it is shown that, up to isomorphisms, there exist 114 hypergroups of type U on the right of size 5 in which the right scalar identity is not also a left scalar identity. In [18] it is shown that the hypergroups of type U on the right of size 5 with bilateral scalar identity are isomorphic to the group $\mathbb{Z}_5$, if there exists a hyperproduct $x \circ y$ of size 1, with x, y not equal to the identity. If this is not the case, then all hyperproducts $x \circ y$, with x, y different from the identity, always contain the identity and have size greater than or equal to 3, see Theorem 2.1.

In the present paper, we exploit the aforementioned properties in order to complete the classification of those hypergroups. To achieve this result, we use a blend of theoretical arguments and computational techniques: Firstly, we partition the set of all hypergroups of our concern into pairwise disjoint, non-isomorphic families and subfamilies; by the analysis of each of them, we obtain some incomplete hyperproduct tables, as more informative as possible; finally, we use computer algebra tools to generate all hypergroups belonging to each case, and count their isomorphism classes. We stress that, owing to the huge number of hypergroups of type U on the right having size five, a brute force approach, letting the computer to construct all these hypergroups and count their isomorphism classes, without any preceding analysis, would require an extremely long time. On the other hand, the computer analysis of the largest subfamilies took a few hours on a laptop PC.

In summary, we found that the number of hypergroups of type U on the right having size five is fairly large; indeed, apart from isomorphisms, they are 14 865, of which 14 751 having a scalar (bilateral) identity. Obviously, we cannot list all their hyperproduct tables within this paper; rather, we limit ourselves to state their properties and expose the theoretical and computational tools that we used for this computation.

The plan of this paper is the following: After introducing in Section 2 some basic definitions, notations and main theorems, in Section 3 we analyze some properties of the hypergroups of type U on the right having size five and scalar bilateral identity, that are not isomorphic to the group $\mathbb{Z}_5$. In Section 4 we determine five pairwise distinct hypergroup families, that will be used in Section 5 to compute the number of isomorphism classes of those hypergroups. Finally, we describe in the Appendix the two algorithms that we used to enumerate the hypergroups within each isomorphism class. The first of them is applied to the partially specified tables presented in Section 3, and computes all hypergroups belonging to a given class. The second takes as input the lists produced by the former and computes the number of pairwise non-isomorphic hypergroups. In this way, we found that there exist 14 751 distinct isomorphism classes of hypergroups of type U on the right of order five, with bilateral scalar identity, see Theorem 5.1. By considering the results obtained in our preceding papers [16,17], we conclude that there exist 14 865 isomorphism classes of hypergroups of type U on the right of order five, see Theorem 5.2.

2. Basic definitions and results

A *semi-hypergroup* is a non-empty set H with a hyperproduct $\circ$, that is, a possibly multi-valued associative product. A *hypergroup* is a semi-hypergroup H such that $x \circ H = H \circ x = H$ (this condition is called *reproducibility*). If a hypergroup H contains an element ε with the property that, for all x in H, one has $x \in x \circ \varepsilon$ (resp., $x \in \varepsilon \circ x$), then we say that ε is a *right identity* (resp., *left identity*) of H. If $x \circ \varepsilon = \{x\}$ (resp., $\varepsilon \circ x = \{x\}$), for all x in H, then ε is a *right scalar identity* (resp., *left scalar identity*). The element ε is said to be an *identity* (resp., *scalar identity or bilateral scalar identity*), if it is both right and left identity (resp., right and left scalar identity).

A hypergroup H is said to be of *type U on the right* if it fulfils the following axioms:

U_1: H has a right scalar identity ε;
U_2: For all $x, y \in H$, $x \in x \circ y \Rightarrow y = \varepsilon$.

We refer to [13,14,19] for other basic concepts and definitions in hypergroup theory. Moreover, we recall from [18] the following theorem:

Theorem 2.1. *Let H be a hypergroup of type U on the right of size 5, with ε as scalar identity, such that H is not isomorphic to the group $\mathbb{Z}_5$. Then, for every $x, y \in H - \{\varepsilon\}$, we have $\varepsilon \in x \circ y$ and $|x \circ y| \geq 3$.*

3. Properties of hyperproducts

In this section, we prove some relevant properties of hyperproducts of hypergroup of type U on the right having order five. Based on these properties, we will construct partial hyperproduct tables that, with the help of the algorithms described in the Appendix, allow us to enumerate the isomorphism classes of our interest, in reasonable time.

Proposition 3.1. *Let $H = \{\varepsilon, x, y, z, t\}$ be a hypergroup of type U on the right, with ε as scalar identity. If x, y, z fulfil $x \circ x = \{\varepsilon, y, z\}$, then:*

1. $z \circ x \supseteq \{\varepsilon, y, t\}$;
2. $z \in y \circ y$.

Proof. 1. By Theorem 2.1, we know that $\varepsilon \in z \circ x$, hence we have to prove that $\{y, t\} \subset z \circ x$. Suppose that $y \notin z \circ x$. By $(x \circ x) \circ x = x \circ (x \circ x)$, we obtain $x \circ y \cup x \circ z \subseteq y \circ x \cup z \circ x$. By Theorem 2.1, since $y \notin y \circ x \cup z \circ x$, it follows $x \circ y = x \circ z = \{\varepsilon, z, t\}$. Now, from $(z \circ x) \circ x = \{\varepsilon, x, t\} \circ x = \{x\} \cup \{\varepsilon, y, z\} \cup (t \circ x)$, we obtain $t \notin z \circ (x \circ x) = z \circ \{\varepsilon, y, z\} = \{z\} \cup (z \circ y) \cup (z \circ z)$ and so $z \circ y = z \circ z = \{\varepsilon, x, y\}$. This is a contradiction because $z \circ (z \circ z) = H$ while $y \notin (z \circ z) \circ z$. Then $y \in z \circ x$.

Suppose now that $t \notin z \circ x$. By Theorem 2.1 and reproducibility, we have $z \circ x = \{\varepsilon, x, y\}$ and $t \in y \circ x$. Moreover, since $(y \circ x) \circ x \subseteq (H - \{y\}) \circ x = \{x\} \cup \{\varepsilon, y, z\} \cup \{\varepsilon, x, y\} \cup (t \circ x)$, we obtain $t \notin y \circ (x \circ x) = y \circ \{\varepsilon, y, z\} = \{y\} \cup (y \circ y) \cup (y \circ z)$ and so $y \circ y = y \circ z = \{\varepsilon, x, z\}$. We have again a contradiction because $y \circ (z \circ x) = H$, while $t \notin (y \circ z) \circ x$. Then $t \in z \circ x$.
2. Suppose $z \notin y \circ y$. By Theorem 2.1, we have $y \circ y = \{\varepsilon, x, t\}$. By 1, it follows $x \circ y \supseteq \{\varepsilon, z, t\}$ and $t \circ y \supseteq \{\varepsilon, x, z\}$. From $(y \circ y) \circ x = \{\varepsilon, x, t\} \circ x = \{x\} \cup \{\varepsilon, y, z\} \cup (t \circ x)$, we obtain $t \notin y \circ (y \circ x)$ and so $x \notin y \circ x$ (otherwise, by $x \circ x = \{\varepsilon, y, z\}$ and 1, we have $t \in y \circ x \subseteq y \circ (y \circ x)$, a contradiction). Then $y \circ x = \{\varepsilon, z, t\}$. Furthermore, since $y \circ (y \circ x) = y \circ \{\varepsilon, z, t\} = \{y\} \cup (y \circ z) \cup (y \circ t)$, we have also $y \circ z = y \circ t = \{\varepsilon, x, z\}$. Analogously, by $(x \circ x) \circ y = x \circ (x \circ y)$, we obtain $z \notin (x \circ x) \circ y = x \circ (x \circ y)$ and so $y \notin x \circ y$ (otherwise, by $y \circ y = \{\varepsilon, x, t\}$ and 1, we have $z \in x \circ y \subseteq x \circ (x \circ y) = (x \circ x) \circ y$, a contradiction). Therefore $x \circ y = \{\varepsilon, z, t\}$ and, since $z \notin (x \circ x) \circ y = x \circ (x \circ y) = x \circ \{\varepsilon, z, t\} = \{x\} \cup (x \circ z) \cup (x \circ t)$, it results $x \circ z = x \circ t = \{\varepsilon, y, t\}$. Now, from $x \circ (x \circ z) = H$, we obtain $H = (x \circ x) \circ z = \{\varepsilon, y, z\} \circ z = \{z\} \cup \{\varepsilon, x, z\} \cup (z \circ z)$ and so $\{\varepsilon, y, t\} \subseteq z \circ z$. Finally, we arrive at the contradiction $x \circ (z \circ z) \supseteq x \circ \{\varepsilon, y, t\} = H$, while $(x \circ z) \circ z = \{\varepsilon, y, t\} \circ z = \{z\} \cup \{\varepsilon, x, z\} \cup (t \circ z) = H - \{t\}$. Thus $z \in y \circ y$. □

The purpose of the forthcoming propositions is to determine some useful properties of hypergroups of type U on the right having size five and scalar identity 1, whose diagonal products are $a \circ a = H - \{a\}$, for all $a \in H - \{\varepsilon\}$.

Proposition 3.2. *Let $H = \{\varepsilon, x, y, z, t\}$ be a hypergroup of type U on the right of size 5, with ε as scalar identity and $a \circ a = H - \{a\}$, for all $a \in H - \{\varepsilon\}$. If $x \circ y = \{\varepsilon, y, z\}$, then we have:*

1. $t \in x \circ z$;
2. $y \in z \circ x$ *and* $z \in y \circ x$;
3. $t \in z \circ y$;
4. $z \in y \circ t$ *and* $y \in z \circ t$.

Proof. 1. We have: $H = (x \circ x) \circ y \Rightarrow H = x \circ (x \circ y) = x \circ \{\varepsilon, y, z\} = \{x\} \cup (x \circ y) \cup (x \circ z) = \{x\} \cup \{\varepsilon, y, z\} \cup (x \circ z)$, hence $t \in x \circ z$.
2. We have that $y \in z \circ x$. Indeed, if $y \notin z \circ x$, by Theorem 2.1 and axiom U_2 we obtain $z \circ x = \{\varepsilon, x, t\}$ and $(z \circ x) \circ y = \{\varepsilon, x, t\} \circ y = \{y\} \cup x \circ y \cup t \circ y = \{\varepsilon, y, z\} \cup t \circ y$, whence $t \notin (z \circ x) \circ y = z \circ (x \circ y) = H$, a contradiction.

Now we prove that $z \in y \circ x$. If $x \notin y \circ x$, by Theorem 2.1 and axiom U_2 we have $y \circ x = \{\varepsilon, z, t\}$. Otherwise, if $x \in y \circ x$, then $x \circ (y \circ x) = H$ and so $H = (x \circ y) \circ x = \{\varepsilon, y, z\} \circ x = \{x\} \cup (y \circ x) \cup (z \circ x)$. Thus $z \in y \circ x$.
3. We have $H = y \circ (x \circ y) = (y \circ x) \circ y$. Hence, there exists $a \in y \circ x$ such that $t \in a \circ y$. By axiom U_2 and hypothesis $x \circ y = \{\varepsilon, y, z\}$ we obtain $a = z$, whence $t \in z \circ y$.
4. If $x \notin y \circ t$, then we have $y \circ t = \{\varepsilon, z, t\}$ and $z \in y \circ t$. Otherwise if $x \in y \circ t$, we have $x \circ (y \circ t) = H$ and so $H = (x \circ y) \circ t = \{\varepsilon, y, z\} \circ t = \{t\} \cup (y \circ t) \cup (z \circ t)$. Thus $z \in y \circ t$.

Now, if we suppose that $y \notin z \circ t$, then we have $z \circ t = \{\varepsilon, x, t\}$. We are led to $(z \circ t) \circ y = \{\varepsilon, x, t\} \circ y = \{y\} \cup (x \circ y) \cup (t \circ y) = \{\varepsilon, y, z\} \cup (t \circ y)$, whence $t \notin (z \circ t) \circ y = z \circ (t \circ y)$. Obviously, by the preceding point, we have $y \notin t \circ y$, whence $t \circ y = \{\varepsilon, x, z\}$. Finally, we obtain the contradiction $H = z \circ (t \circ y) = (z \circ t) \circ y = \{\varepsilon, x, t\} \circ y = \{y\} \cup (x \circ y) \cup (t \circ y) = H - \{t\}$. □

Proposition 3.3. *Let $H = \{\varepsilon, x, y, z, t\}$ be a hypergroup of type U on the right, with ε as scalar identity and $a \circ a = H - \{a\}$, for all $a \in H - \{\varepsilon\}$. If $x \circ y = \{\varepsilon, z, t\}$, then we have:*

1. $z \in t \circ y$ *and* $t \in z \circ y$;
2. $(x \circ z) \cup (x \circ t) = H - \{x\}$.

Proof. 1. From $H = x \circ (y \circ y)$ we have $H = (x \circ y) \circ y = \{\varepsilon, z, t\} \circ y = \{y\} \cup (z \circ y) \cup (t \circ y)$, and the claim follows.
2. Since $H = (x \circ x) \circ y$ we have $H = x \circ (x \circ y) = x \circ \{\varepsilon, z, t\} = \{x\} \cup (x \circ z) \cup (x \circ t)$ and so $(x \circ z) \cup (x \circ t) = H - \{x\}$. □

4. The main families

In this section we determine five large, pairwise distinct families of hypergroups that, in what follows, will allow us to complete the classification we are dealing with.

Taking aside the group $\mathbb{Z}_5$, let H be a hypergroup of type U on the right, having size five, with a scalar bilateral identity. For simplicity, we denote its elements by 1, 2, 3, 4, 5, and let 1 be the identity. Moreover, we denote by ν the number of diagonal hyperproducts $x \circ x$, where $x \in \{2, 3, 4, 5\}$, such that $x \circ x \neq H - \{x\}$. By Theorem 2.1, these hyperproducts contain at least three elements, one of which is 1. Hence, possibly by renumbering the elements, if $\nu \neq 0$ then we can safely suppose that $2 \circ 2 = \{1, 3, 4\}$. As it will be shown later, we can limit ourselves to consider the following five cases:

(1) $\nu = 0$;
(2) $\nu = 1$, with $2 \circ 2 = \{1, 3, 4\}$;
(3) $\nu = 2$ with the following three subcases:
  (a) $2 \circ 2 = \{1, 3, 4\}$ and $3 \circ 3 = \{1, 2, 4\}$;
  (b) $2 \circ 2 = \{1, 3, 4\}$ and $3 \circ 3 = \{1, 4, 5\}$;
  (c) $2 \circ 2 = \{1, 3, 4\}$ and $5 \circ 5 = \{1, 3, 4\}$.
(4) $\nu = 3$ with the following four subcases:
  (a) $2 \circ 2 = \{1, 3, 4\}$, $3 \circ 3 = \{1, 2, 4\}$ and $4 \circ 4 = \{1, 2, 3\}$;
  (b) $2 \circ 2 = \{1, 3, 4\}$, $3 \circ 3 = \{1, 2, 4\}$ and $5 \circ 5 = \{1, 3, 4\}$;
  (c) $2 \circ 2 = \{1, 3, 4\}$, $3 \circ 3 = \{1, 2, 4\}$ and $5 \circ 5 = \{1, 2, 3\}$;
  (d) $2 \circ 2 = \{1, 3, 4\}$, $3 \circ 3 = \{1, 4, 5\}$ and $5 \circ 5 = \{1, 2, 4\}$.
(5) $\nu = 4$; in this case we can also set

$$2 \circ 2 = \{1, 3, 4\};\ 3 \circ 3 = \{1, 2, 4\};\ 4 \circ 4 = \{1, 2, 3\};\ 5 \circ 5 = \{1, 2, 4\}.$$

In what follows we will examine all the preceding cases and subcases. For each of them, we will produce some partially specified hyperproduct tables, derived from the analysis of the particular case and subcase. Whenever we encounter an incompletely specified hyperproduct, we will use a particular notation: a dot will be inserted as a placeholder, to indicate that such a hyperproduct may be possibly completed by other elements. For example, the notation $4 \circ 5 = \{1, 2, 3, .\}$ represents one of two possible hyperproducts, $4 \circ 5 = \{1, 2, 3\}$ and $4 \circ 5 = \{1, 2, 3, 5\}$ (remark that surely $4 \notin 4 \circ 5$).

4.1. *Case* 1: $\nu = 0$

If for every $x, y \in H - \{1\}$ we have $x \circ y = H - \{x\}$, we obtain immediately the following hypergroup, that we denote by B_5:

∘	1	2	3	4	5
1	{1}	{2}	{3}	{4}	{5}
2	{2}	{1, 3, 4, 5}	{1, 3, 4, 5}	{1, 3, 4, 5}	{1, 3, 4, 5}
3	{3}	{1, 2, 4, 5}	{1, 2, 4, 5}	{1, 2, 4, 5}	{1, 2, 4, 5}
4	{4}	{1, 2, 3, 5}	{1, 2, 3, 5}	{1, 2, 3, 5}	{1, 2, 3, 5}
5	{5}	{1, 2, 3, 4}	{1, 2, 3, 4}	{1, 2, 3, 4}	{1, 2, 3, 4}

Now suppose that there exists a hyperproduct $x \circ y$ $(x \neq y)$ such that $|x \circ y| = 3$. Then we have the following two possibilities:

a. For every $x, y \in H - \{1\}$ such that $x \neq y$ and $|x \circ y| = 3$, we have $y \notin x \circ y$.
b. There exist $x, y \in H - \{1\}$ such that $x \neq y$, $|x \circ y| = 3$ and $y \in x \circ y$.

In (a) we can suppose that $2 \circ 3 = \{1, 4, 5\}$. From the hypotheses, we obtain the following incomplete table:

∘	1	2	3	4	5
1	{1}	{2}	{3}	{4}	{5}
2	{2}	{1, 3, 4, 5}	{1, 4, 5}	{1, 3, 5, .}	{1, 3, 4, .}
3	{3}	{1, 4, 5, .}	{1, 2, 4, 5}	{1, 2, 5, .}	{1, 2, 4, .}
4	{4}	{1, 3, 5, .}	{1, 2, 5, .}	{1, 2, 3, 5}	{1, 2, 3, .}
5	{5}	{1, 3, 4, .}	{1, 2, 4, .}	{1, 2, 3, .}	{1, 2, 3, 4}

One can prove that the associative property is always verified, however we complete this table. Hence, we obtain $2^{11} = 2048$ hypergroups.

In (b) apart from isomorphisms, we can suppose that $2 \circ 3 = \{1, 3, 4\}$. Using Proposition 3.2, we can add some elements in certain hyperproducts and end up with the following incomplete table:

∘	1	2	3	4	5
1	{1}	{2}	{3}	{4}	{5}
2	{2}	{1, 3, 4, 5}	{1, 3, 4}	{1, 5, .}	{1, .}
3	{3}	{1, 4, .}	{1, 2, 4, 5}	{1, .}	{1, 4, .}
4	{4}	{1, 3, .}	{1, 5, .}	{1, 2, 3, 5}	{1, 3, .}
5	{5}	{1, .}	{1, .}	{1, .}	{1, 2, 3, 4}

Now we split case b into three subcases:

b_1. $2 \circ 4 = \{1, 4, 5\}$.

By using Proposition 3.2 with $x = 2, y = 4, z = 5$ and $t = 3$, we obtain the following incomplete table:

$\circ$	1	2	3	4	5
1	{1}	{2}	{3}	{4}	{5}
2	{2}	{1, 3, 4, 5}	{1, 3, 4}	{1, 4, 5}	{1, 3, .}
3	{3}	{1, 4, .}	{1, 2, 4, 5}	{1, .}	{1, 4, .}
4	{4}	{1, 3, 5, .}	{1, 5, .}	{1, 2, 3, 5}	{1, 3, .}
5	{5}	{1, 4, .}	{1, 4, .}	{1, 3, .}	{1, 2, 3, 4}

Therefore, this subcase yields $3^8 \cdot 2^3 = 52\,488$ tables.

b_2. $2 \circ 4 = \{1, 3, 5\}$.

By using Proposition 3.3 with $x = 2, y = 4, z = 3$ and $t = 5$, we obtain $5 \in (2 \circ 5) \cap (3 \circ 4)$ and $3 \in 5 \circ 4$. Moreover, we have $3 \in 2 \circ 4 \subseteq 2 \circ (3 \circ 5) = (2 \circ 3) \circ 5 = \{1, 3, 4\} \circ 5 = \{5\} \cup (3 \circ 5) \cup (4 \circ 5)$ and so $3 \in 4 \circ 5$.

The following incomplete table arises:

$\circ$	1	2	3	4	5
1	{1}	{2}	{3}	{4}	{5}
2	{2}	{1, 3, 4, 5}	{1, 3, 4}	{1, 3, 5}	{1, 5, .}
3	{3}	{1, 4, .}	{1, 2, 4, 5}	{1, 5, .}	{1, 4, .}
4	{4}	{1, 3, .}	{1, 5, .}	{1, 2, 3, 5}	{1, 3, .}
5	{5}	{1, .}	{1, .}	{1, 3, .}	{1, 2, 3, 4}

Therefore this subcase leads to $4^2 \cdot 3^8 = 104\,976$ tables.

b_3. $2 \circ 4 = H - \{2\}$.

We obtain the following table:

$\circ$	1	2	3	4	5
1	{1}	{2}	{3}	{4}	{5}
2	{2}	{1, 3, 4, 5}	{1, 3, 4}	{1, 3, 4, 5}	{1, .}
3	{3}	{1, 4, .}	{1, 2, 4, 5}	{1, .}	{1, 4, .}
4	{4}	{1, 3, .}	{1, 5, .}	{1, 2, 3, 5}	{1, 3, .}
5	{5}	{1, .}	{1, .}	{1, .}	{1, 2, 3, 4}

So this subcase provides $4^5 \cdot 3^5 = 248\,832$ tables.

4.2. *Case* 2: $\nu = 1$

We can suppose without loss of generality that $2 \circ 2 = \{1, 3, 4\}$. By Proposition 3.1, we obtain the following table:

$\circ$	1	2	3	4	5
1	{1}	{2}	{3}	{4}	{5}
2	{2}	{1, 3, 4}	{1, .}	{1, .}	{1, .}
3	{3}	{1, 4, 5, .}	{1, 2, 4, 5}	{1, .}	{1, .}
4	{4}	{1, 3, 5, .}	{1, .}	{1, 2, 3, 5}	{1, .}
5	{5}	{1, .}	{1, .}	{1, .}	{1, 2, 3, 4}

We observe that

$$H = (2 \circ 2) \circ 2 = 2 \circ (2 \circ 2) = 2 \circ \{1, 3, 4\} = \{2\} \cup (2 \circ 3) \cup (2 \circ 4).$$

By axiom U_2, we obtain $(2 \circ 3) \cup (2 \circ 4) = H - \{2\}$. Then we have $5 \in (2 \circ 3) \cup (2 \circ 4)$.

Moreover, if $x \circ 2 = \{1, y, 5\}$, with $\{x, y\} = \{3, 4\}$, we have that $H = x \circ (2 \circ 2) = (x \circ 2) \circ 2 = \{1, y, 5\} \circ 2 = \{2\} \cup (y \circ 2) \cup (5 \circ 2)$ and so $y \in 5 \circ 2$. Thus, we are led to consider the following subcases:

(i) $3 \circ 2 = \{1, 4, 5\}$ and $4 \circ 2 = \{1, 3, 5\}$. Then $5 \circ 2 \supseteq \{1, 3, 4\}$ and, without loss of generality, we can suppose $5 \in 2 \circ 3$. We obtain:

$\circ$	1	2	3	4	5
1	{1}	{2}	{3}	{4}	{5}
2	{2}	{1, 3, 4}	{1, 5, .}	{1, .}	{1, .}
3	{3}	{1, 4, 5}	{1, 2, 4, 5}	{1, .}	{1, .}
4	{4}	{1, 3, 5}	{1, .}	{1, 2, 3, 5}	{1, .}
5	{5}	{1, 3, 4, .}	{1, .}	{1, .}	{1, 2, 3, 4}

This subcase provides $4^8 \cdot 2 \cdot 3 = 392\,216$ tables.

(ii) Exactly one of the two hyperproducts $3 \circ 2$ and $4 \circ 2$ has size three. Without loss of generality, we can suppose $3 \circ 2 = \{1, 4, 5\}$ and $4 \circ 2 = \{1, 2, 3, 5\}$, whence $4 \in 5 \circ 2$. This results in the following incomplete table:

$\circ$	1	2	3	4	5
1	{1}	{2}	{3}	{4}	{5}
2	{2}	{1, 3, 4}	{1, .}	{1, .}	{1, .}
3	{3}	{1, 4, 5}	{1, 2, 4, 5}	{1, .}	{1, .}
4	{4}	{1, 2, 3, 5}	{1, .}	{1, 2, 3, 5}	{1, .}
5	{5}	{1, 4, .}	{1, .}	{1, .}	{1, 2, 3, 4}

Whence we have $4^9 \cdot 3 = 786\,432$ tables.

(iii) Both hyperproducts $3 \circ 2$ and $4 \circ 2$ have size four. We can suppose without loss of generality that $5 \in 2 \circ 3$.

$\circ$	1	2	3	4	5
1	{1}	{2}	{3}	{4}	{5}
2	{2}	{1, 3, 4}	{1, 5, .}	{1, .}	{1, .}
3	{3}	{1, 2, 4, 5}	{1, 2, 4, 5}	{1, .}	{1, .}
4	{4}	{1, 2, 3, 5}	{1, .}	{1, 2, 3, 5}	{1, .}
5	{5}	{1, .}	{1, .}	{1, .}	{1, 2, 3, 4}

Then also in this subcase there are $4^9 \cdot 3 = 786\,432$ tables.

4.3. *Case* 3: $\nu = 2$

We can suppose $2 \circ 2 = \{1, 3, 4\}$. By Proposition 3.1(2), we have $3 \in 4 \circ 4$ and $4 \in 3 \circ 3$. Then we arrive at the following possibilities:

(I) $3 \circ 3 = \{1, 2, 4\}$;
(II) $3 \circ 3 = \{1, 4, 5\}$;
(III) $4 \circ 4 = \{1, 2, 3\}$;
(IV) $4 \circ 4 = \{1, 3, 5\}$;
(V) $5 \circ 5 = \{1, 2, 3\}$;
(VI) $5 \circ 5 = \{1, 2, 4\}$;
(VII) $5 \circ 5 = \{1, 3, 4\}$.

One sees immediately that (III) reduces to (I) via the permutation (34). Moreover, (IV), (V) and (VI) are all equivalent to (II) via the permutations (34), (5234) and (235), respectively. Therefore, we can split the case $\nu = 2$ into three subcases:

(a) $2 \circ 2 = \{1, 3, 4\}$ and $3 \circ 3 = \{1, 2, 4\}$;
(b) $2 \circ 2 = \{1, 3, 4\}$ and $3 \circ 3 = \{1, 4, 5\}$;
(c) $2 \circ 2 = \{1, 3, 4\}$ and $5 \circ 5 = \{1, 3, 4\}$.

(a) From Proposition 3.1 we obtain the following incomplete table:

$\circ$	1	2	3	4	5
1	{1}	{2}	{3}	{4}	{5}
2	{2}	{1, 3, 4}	{1, 4, 5, .}	{1, .}	{1, .}
3	{3}	{1, 4, 5, .}	{1, 2, 4}	{1, .}	{1, .}
4	{4}	{1, 3, 5, .}	{1, 2, 5, .}	{1, 2, 3, 5}	{1, .}
5	{5}	{1, .}	{1, .}	{1, .}	{1, 2, 3, 4}

Moreover, we can prove that

$$2 \circ 4 \cap 3 \circ 4 \supseteq \{1, 4, 5\}.$$

In fact, $5 \in 4 \circ 3 \subseteq (2 \circ 3) \circ 3 = 2 \circ (3 \circ 3) = 2 \circ \{1, 2, 4\} = \{2\} \cup \{1, 3, 4\} \cup (2 \circ 4)$, whence $5 \in 2 \circ 4$. Now, if $4 \notin 2 \circ 4$, then $2 \circ 4 = \{1, 3, 5\}$ and so $(2 \circ 4) \circ 3 = \{1, 3, 5\} \circ 3 = \{3\} \cup \{1, 2, 4\} \cup (5 \circ 3)$. This implies $5 \notin (2 \circ 4) \circ 3 = 2 \circ (4 \circ 3)$, whence $3 \notin 4 \circ 3$ and $4 \circ 3 = \{1, 2, 5\}$. Thus $(4 \circ 3) \circ 2 = \{1, 2, 5\} \circ 2 = \{2\} \cup \{1, 3, 4\} \cup (5 \circ 2)$, whence $5 \notin (4 \circ 3) \circ 2 = 4 \circ (3 \circ 2) \supseteq 4 \circ \{1, 4, 5\} = H$. This is a contradiction and therefore $4 \in 2 \circ 4$. The inclusion $\{4, 5\} \subset 3 \circ 4$ is shown analogously, by exchanging the role of the elements 2 and 3.

The following incomplete table arises:

$\circ$	1	2	3	4	5
1	{1}	{2}	{3}	{4}	{5}
2	{2}	{1, 3, 4}	{1, 4, 5, .}	{1, 4, 5, .}	{1, .}
3	{3}	{1, 4, 5, .}	{1, 2, 4}	{1, 4, 5, .}	{1, .}
4	{4}	{1, 3, 5, .}	{1, 2, 5, .}	{1, 2, 3, 5}	{1, .}
5	{5}	{1, .}	{1, .}	{1, .}	{1, 2, 3, 4}

Hence, in this subcase we count $4^6 \cdot 2^6 = 262\,144$ tables.

(b) Here, we have the following incomplete table:

$\circ$	1	2	3	4	5
1	{1}	{2}	{3}	{4}	{5}
2	{2}	{1, 3, 4}	{1, .}	{1, .}	{1, .}
3	{3}	{1, 4, 5, .}	{1, 4, 5}	{1, 2, .}	{1, .}
4	{4}	{1, 3, 5, .}	{1, 2, 5, .}	{1, 2, 3, 5}	{1, .}
5	{5}	{1, .}	{1, 2, 4, .}	{1, .}	{1, 2, 3, 4}

where we have considered that $2 \in (3 \circ 2) \circ 2 = 3 \circ (2 \circ 2) = 3 \circ \{1, 3, 4\} = \{3\} \cup \{1, 4, 5\} \cup (3 \circ 4)$, whence $2 \in 3 \circ 4$. So there are $4^7 \cdot 2^4 \cdot 3 = 786\,432$ tables.

(c) In this case we have the following incomplete table:

$\circ$	1	2	3	4	5
1	{1}	{2}	{3}	{4}	{5}
2	{2}	{1, 3, 4}	{1, .}	{1, .}	{1, .}
3	{3}	{1, 4, 5, .}	{1, 2, 4, 5}	{1, 4, .}	{1, 2, 4, .}
4	{4}	{1, 3, 5, .}	{1, 3, .}	{1, 2, 3, 5}	{1, 2, 3, .}
5	{5}	{1, .}	{1, .}	{1, .}	{1, 3, 4}

where we have considered that $4 \in 3 \circ 4$ and $3 \in 4 \circ 3$. In fact if $4 \notin 3 \circ 4$ then $3 \circ 4 = \{1, 2, 5\}$ and so $(3 \circ 4) \circ 5 = \{1, 2, 5\} \circ 5 = \{5\} \cup (2 \circ 5) \cup \{1, 3, 4\}$, whence $2 \notin (3 \circ 4) \circ 5 = 3 \circ (4 \circ 5) \supseteq 3 \circ \{1, 2, 3\} = H$, a contradiction. By interchanging the role of the elements 3 and 4 we can prove analogously that $3 \in 4 \circ 3$. We obtain $4^6 \cdot 2^4 \cdot 3^2 = 589\,824$ tables.

4.4. *Case* 4: $\nu = 3$

Let us suppose again that $2 \circ 2 = \{1, 3, 4\}$. We have either $|3 \circ 3| = 3$ or $|3 \circ 3| = 4$. Obviously, by Proposition 3.1(2), we have $4 \in 3 \circ 3$ and $3 \in 4 \circ 4$. Hence, if $|3 \circ 3| = 3$ then we have the following possibilities:

- $2 \circ 2 = \{1, 3, 4\}$ and $3 \circ 3 = \{1, 2, 4\}$, whence $\{2, 3\} \subseteq 4 \circ 4$ by Proposition 3.1(2). The following subcases arise:
  (I) $3 \circ 3 = \{1, 2, 4\}$; $4 \circ 4 = \{1, 2, 3\}$.
  (II) $3 \circ 3 = \{1, 2, 4\}$; $5 \circ 5 = \{1, 2, 3\}$.
  (III) $3 \circ 3 = \{1, 2, 4\}$; $5 \circ 5 = \{1, 2, 4\}$.
  (IV) $3 \circ 3 = \{1, 2, 4\}$; $5 \circ 5 = \{1, 3, 4\}$.
- $2 \circ 2 = \{1, 3, 4\}$ and $3 \circ 3 = \{1, 4, 5\}$, whence $\{3, 5\} \subseteq 4 \circ 4$ and $4 \in 5 \circ 5$. We obtain:
  (V) $3 \circ 3 = \{1, 4, 5\}$; $4 \circ 4 = \{1, 3, 5\}$.
  (VI) $3 \circ 3 = \{1, 4, 5\}$; $5 \circ 5 = \{1, 2, 4\}$.
  (VII) $3 \circ 3 = \{1, 4, 5\}$; $5 \circ 5 = \{1, 3, 4\}$.

  On the other hand, if $|3 \circ 3| = 4$, since $3 \in 4 \circ 4$, then we obtain the following possibilities:
  (VIII) $4 \circ 4 = \{1, 2, 3\}$; $5 \circ 5 = \{1, 2, 3\}$.
  (IX) $4 \circ 4 = \{1, 2, 3\}$; $5 \circ 5 = \{1, 2, 4\}$.
  (X) $4 \circ 4 = \{1, 2, 3\}$; $5 \circ 5 = \{1, 3, 4\}$.
  (XI) $4 \circ 4 = \{1, 3, 5\}$; $5 \circ 5 = \{1, 2, 3\}$.
  (XII) $4 \circ 4 = \{1, 3, 5\}$; $5 \circ 5 = \{1, 3, 4\}$.

It is not difficult to see that (IV) and (IX) reduce to (II) via the permutations (425) and (34), respectively. Moreover, (III), (*VIII*), (*X*) and (*XII*) are all equivalent to (IV) via the permutations (23), (234), (34) and (34)(25), respectively. Finally, (*XI*) reduces to (VI) via the transposition (34). Therefore, when $\nu = 3$, the remaining possibilities are the following:

(a) $2 \circ 2 = \{1, 3, 4\}$, $3 \circ 3 = \{1, 2, 4\}$ and $4 \circ 4 = \{1, 2, 3\}$;
(b) $2 \circ 2 = \{1, 3, 4\}$, $3 \circ 3 = \{1, 2, 4\}$ and $5 \circ 5 = \{1, 3, 4\}$;
(c) $2 \circ 2 = \{1, 3, 4\}$, $3 \circ 3 = \{1, 2, 4\}$ and $5 \circ 5 = \{1, 2, 3\}$;
(d) $2 \circ 2 = \{1, 3, 4\}$, $3 \circ 3 = \{1, 4, 5\}$ and $5 \circ 5 = \{1, 2, 4\}$.

Whence the following four respective tables arise:

	$\circ$	1	2	3	4	5
(a)	1	{1}	{2}	{3}	{4}	{5}
	2	{2}	{1, 3, 4}	{1, 4, 5, .}	{1, 3, 5, .}	{1, .}
	3	{3}	{1, 4, 5, .}	{1, 2, 4}	{1, 2, 5, .}	{1, .}
	4	{4}	{1, 3, 5, .}	{1, 2, 5, .}	{1, 2, 3}	{1, .}
	5	{5}	{1, .}	{1, .}	{1, .}	{1, 2, 3, 4}
(b)	1	{1}	{2}	{3}	{4}	{5}
	2	{2}	{1, 3, 4}	{1, 4, 5, .}	{1, .}	{1, .}
	3	{3}	{1, 4, 5, .}	{1, 2, 4}	{1, .}	{1, 2, 4, .}
	4	{4}	{1, 3, 5, .}	{1, 2, 5, .}	{1, 2, 3, 5}	{1, 2, 3, .}
	5	{5}	{1, .}	{1, .}	{1, .}	{1, 3, 4}
(c)	1	{1}	{2}	{3}	{4}	{5}
	2	{2}	{1, 3, 4}	{1, 4, 5, .}	{1, .}	{1, 3, 4, .}
	3	{3}	{1, 4, 5, .}	{1, 2, 4}	{1, .}	{1, 2, 4, .}
	4	{4}	{1, 3, 5, .}	{1, 2, 5, .}	{1, 2, 3, 5}	{1, .}
	5	{5}	{1, .}	{1, .}	{1, .}	{1, 2, 3}
(d)	1	{1}	{2}	{3}	{4}	{5}
	2	{2}	{1, 3, 4}	{1, .}	{1, .}	{1, 3, 4, .}
	3	{3}	{1, 4, 5, .}	{1, 4, 5}	{1, .}	{1, .}
	4	{4}	{1, 3, 5, .}	{1, 2, 5, .}	{1, 2, 3, 5}	{1, 2, 3, .}
	5	{5}	{1, .}	{1, 2, 4, .}	{1, .}	{1, 2, 4}

We obtain $4^6 \cdot 2^6 = 262\,144$ tables altogether.

4.5. *Case* 5: $\nu = 4$

We proceed analogously to the preceding case. Let $2 \circ 2 = \{1, 3, 4\}$. We have either $3 \circ 3 = \{1, 2, 4\}$ or $3 \circ 3 = \{1, 4, 5\}$ (note that $4 \in 3 \circ 3$, as in the case $\nu = 3$). Hence, again by Proposition 3.1(2), the only possibilities are:

(I) $2 \circ 2 = \{1, 3, 4\}$; $3 \circ 3 = \{1, 2, 4\}$; $4 \circ 4 = \{1, 2, 3\}$; $5 \circ 5 = \{1, 2, 4\}$;
(II) $2 \circ 2 = \{1, 3, 4\}$; $3 \circ 3 = \{1, 2, 4\}$; $4 \circ 4 = \{1, 2, 3\}$; $5 \circ 5 = \{1, 2, 3\}$;
(III) $2 \circ 2 = \{1, 3, 4\}$; $3 \circ 3 = \{1, 2, 4\}$; $4 \circ 4 = \{1, 2, 3\}$; $5 \circ 5 = \{1, 3, 4\}$;
(IV) $2 \circ 2 = \{1, 3, 4\}$; $3 \circ 3 = \{1, 4, 5\}$; $4 \circ 4 = \{1, 3, 5\}$; $5 \circ 5 = \{1, 3, 4\}$.

We see immediately that we can deal just with (I) since (II), (III) and (IV) are transformed into (I) by applying the permutations (34), (23) and (2534), respectively. Then, taking into account Proposition 3.1(1), we arrive at the following incomplete table:

$\circ$	1	2	3	4	5
1	{1}	{2}	{3}	{4}	{5}
2	{2}	{1, 3, 4}	{1, 4, 5, .}	{1, 3, 5, .}	{1, 3, 4, .}
3	{3}	{1, 4, 5, .}	{1, 2, 4}	{1, 2, 5, .}	{1, .}
4	{4}	{1, 3, 5, .}	{1, 2, 5, .}	{1, 2, 3}	{1, 2, 3, .}
5	{5}	{1, .}	{1, .}	{1, .}	{1, 2, 4}

Now we prove a lemma that allows us to add more elements in some hyperproducts of the preceding table.

Lemma 4.1. *In the preceding incomplete table, we have:*

1. $4 \circ 2 = H - \{4\}$
2. $2 \circ 4 = H - \{2\}$

3. $3 \circ 4 = 3 \circ 2 = H - \{3\}$
4. $3 \circ 5 \supseteq \{1, 2, 4\}$.

Proof. 1. Suppose by absurd that $4 \circ 2 = \{1, 3, 5\}$. Then $(4 \circ 2) \circ 5 = \{1, 3, 5\} \circ 5 = \{5\} \cup (3 \circ 5) \cup \{1, 2, 4\}$, whence $3 \notin (4 \circ 2) \circ 5$, while $4 \circ (2 \circ 5) \supseteq 4 \circ (2 \circ 5) \supseteq 4 \circ \{1, 3, 4\} = H$, a contradiction.
2. Similarly, starting from $(2 \circ 4) \circ 5 = 2 \circ (4 \circ 5)$, it must be $4 \in 2 \circ 4$ and so $2 \circ 4 = H - \{2\}$.
3. By 1, we have $3 \circ (4 \circ 2) = H \Rightarrow (3 \circ 4) \circ 2 = H$ and, with some computations, $3 \circ 4 = H - \{3\}$. Similarly from 2. we have $3 \circ (2 \circ 4) = H \Rightarrow (3 \circ 2) \circ 4 = H$ and so $3 \circ 2 = H - \{3\}$.
4. $2 \circ (2 \circ 5) = H \Rightarrow (2 \circ 2) \circ 5 = \{1, 3, 4\} \circ 5 = \{5\} \cup (3 \circ 5) \cup (4 \circ 5) = H$ and so $4 \in 3 \circ 5$. Moreover $H = 4 \circ (4 \circ 5) = (4 \circ 4) \circ 5 = \{1, 2, 3\} \circ 5 = \{5\} \cup (2 \circ 5) \cup (3 \circ 5) = H \Rightarrow 2 \in 3 \circ 5$. □

Therefore, by Lemma 4.1 the following table arises:

$\circ$	1	2	3	4	5
1	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{5\}$
2	$\{2\}$	$\{1, 3, 4\}$	$\{1, 4, 5, .\}$	$\{1, 3, 4, 5\}$	$\{1, 3, 4, .\}$
3	$\{3\}$	$\{1, 2, 4, 5\}$	$\{1, 2, 4\}$	$\{1, 2, 4, 5\}$	$\{1, 2, 4, .\}$
4	$\{4\}$	$\{1, 2, 3, 5\}$	$\{1, 2, 5, .\}$	$\{1, 2, 3\}$	$\{1, 2, 3, .\}$
5	$\{5\}$	$\{1, .\}$	$\{1, .\}$	$\{1, .\}$	$\{1, 2, 4\}$

Finally, when $\nu = 4$ we obtain $2^5 \cdot 4^3 = 2048$ tables altogether.

5. The number of isomorphism classes

In this section we resume the results obtained with our two algorithms, that we called `findHgroups` ans `sieveHgroups` (see Appendix), in order to determine the number of isomorphism classes we are interested in.

We applied the algorithm `findHgroups` to the incomplete hyperproduct tables deduced in the previous section. For any given incomplete table, it finds all hypergroups of type U on the right that complete it.

The output of each run of `findHgroups` is stored into a file and given as input to the algorithm `sieveHgroups`. This algorithm applies a suitable set of isomorphisms (to be described hereafter) to all hypergroups in the given file, compares these transformed hypergroups to all other hypergroups in the file, discards the ones that are isomorphic, and outputs the number of the remaining non-isomorphic hypergroups. Obviously, since hypergroups belonging to distinct subfamilies cannot be isomorphic, we obtain the final number of isomorphism classes by summing the partial results obtained by `sieveHgroups`.

In order to improve the performance of the selection of non-isomorphic hypergroups with the algorithm `sieveHgroups`, it is necessary to identify the smallest possible set of candidate isomorphisms for each subfamily. Indeed, the computational complexity of the algorithm increases with the square of the number of isomorphisms that must be tried. For clarity, suppose that we want to extract all pairwise non-isomorphic hypergroups from a given subfamily of hypergroups with n elements; if the number of candidate isomorphisms is m, then we must produce (at most) mn hypergroups and compare for equality each of them against all others. Hence, we have to perform m^2n^2 comparisons, in the worst case.

For that reason, we determined the minimal set of candidate isomorphisms for each subfamily found in the preceding section, by direct inspection. For any incomplete table, we selected from the permutation group S_5 those permutations that leave that table invariant. Let $S_5^{(1)} = \{\pi \in S_5 | \pi(1) = 1\}$ and let id denote the identity in S_5. The result is illustrated as follows:

Case	Candidate isomorphisms
$\nu = 0$ (a)	$S_5^{(1)}$
$\nu = 0$ (b)	$S_5^{(1)} - \{(24), (34), (45), (23)(45), (24)(35), (2354), (2453)\}$
$\nu = 1$	$\{id, (34)\}$
$\nu = 2$ (a)	$\{id, (23)\}$
$\nu = 2$ (b)	$\{id\}$
$\nu = 2$ (c)	$\{id, (34), (25), (25)(34)\}$
$\nu = 3$ (a)	$\{id, (34), (23), (243), (234), (24)\}$
$\nu = 3$ (b)	$\{id\}$
$\nu = 3$ (c)	$\{id, (23)\}$
$\nu = 3$ (d)	$\{id, (235), (253)\}$
$\nu = 4$	$\{id, (24)\}$

The next table reports our computational results. The first column lists all subfamilies found in the preceding section. For each of them, we report in the second column the number of (possibly non-associative) hyperproduct tables belonging to it; in the third column the number of hypergroups (associative hyperproduct tables) found by `findHgroups`; and in the fourth column the output of `sieveHgroups`.

Case	Tables	`findHgroups`	`sieveHgroups`
$\nu = 0$ (a)	2 048	2048	217
$\nu = 0$ (b)	406 296	4913	1841
$\nu = 1$	1965 080	8735	6325
$\nu = 2$ (a)	262 144	1504	782
$\nu = 2$ (b)	786 432	2505	2505
$\nu = 2$ (c)	589 824	5876	1523
$\nu = 3$ (a)	262 144	296	68
$\nu = 3$ (b)	262 144	1016	1016
$\nu = 3$ (c)	262 144	272	148
$\nu = 3$ (d)	262 144	512	176
$\nu = 4$	2 048	272	148

By adding the group $\mathbb{Z}_5$ and the hypergroup B_5, found at the beginning of Section 4, to the results shown in the preceding table, we are led to the following result:

Theorem 5.1. *There exist* 14 751 *isomorphism classes of hypergroups of type U on the right having size five, with bilateral scalar identity.*

Finally we recall that, as shown in [16,17], there exist 114 isomorphism classes of hypergroups of type U on the right of size 5 in which the right scalar identity is not also a left scalar identity. Hence, we have:

Theorem 5.2. *There exist* 14 865 *isomorphism classes of hypergroups of type U on the right having size five.*

Appendix

In this section we describe the software routines employed in the present work. Our software is entirely written in Matlab. With respect to the codes shown here below, our actual routines differ only in minor details, that we introduced only for efficiency purposes. For space reasons, we omit the description of those program parts that are devoted to data input and output, file management, memory allocation, and data initialization. All our computations were performed on a laptop PC endowed by Matlab 4.1.

In our implementations, hyperproduct tables are described by 5×5 matrices with integer entries. Each subset of $\{1, 2, 3, 4, 5\}$ is associated to an integer from 0 to 31 by a binary encoding of its characteristic function: If $\{h_1, \ldots, h_k\} \subseteq \{1, 2, 3, 4, 5\}$ then the associated representation is the integer

$$\sum_{i=1}^{k} 2^{h_i - 1}.$$

For example,

Subset	Representation
$\emptyset$	0
$\{1\}$	1
$\{2\}$	2
$\{1, 2\}$	3
$\{3\}$	4
$\{1, 3\}$	5
...	...
$\{1, 2, 3, 4\}$	30
$\{1, 2, 3, 4, 5\}$	31

Note that the singletons $\{1\}, \{2\}, \ldots, \{5\}$ are represented as powers of two, and the union of disjoint sets is associated to the sum of the two subset representations. In this way, subset manipulation is reduced to bitwise binary arithmetic. The following Matlab function inverts the representation function of a subset; given an integer p in the range $0, \ldots, 31$, it produces the array $[h_1, \ldots, h_k]$, $h_i \in \{1, 2, 3, 4, 5\}$, whose associated representation is p:

```
function v = listofsingletons(p)
% returns the list of elements of a subset from its representation
count = 0;
v = [];
while p > 0
    count = count + 1;
    if rem(p,2) == 1
        v = [v count];
    end
    p = fix(p/2);
end
```

The (i, j)-entry of a matrix H representing an hyperproduct table is the representation of $i \circ j$, for $i, j = 1, \ldots, 5$. For example, the representation of the hypergroup B_5 in Section 4.1 is the following:

$$\begin{bmatrix} 1 & 2 & 4 & 8 & 16 \\ 2 & 29 & 29 & 29 & 29 \\ 4 & 27 & 27 & 27 & 27 \\ 8 & 23 & 23 & 23 & 23 \\ 16 & 15 & 15 & 15 & 15 \end{bmatrix}.$$

The reproducibility of a hyperproduct table Hcan be checked by the following Matlab function:

```
function bool = isreproducible(H)
% H = a hyperproduct table (input)
% returns true iff H is reproducible
allH = 31 * ones(1,5);
temp = zeros(5,1);          % check rows
for j = 1:5
    temp = bitor(temp,H(:,j));
end
bool = all(temp == allH');
store = zeros(1,5);         % check columns
for i = 1:5
    temp = bitor(temp,H(i,:));
end
bool = bool & all(temp == allH);
```

We can check the associativity of a hyperproduct table by verifying the equality $a \circ (b \circ c) = (a \circ b) \circ c$ for all triples $a, b, c \in \{1, 2, 3, 4, 5\}$. If a triple is found such that $a \circ (b \circ c) \neq (a \circ b) \circ c$, then the hyperproduct is not associative. This can be implemented in Matlab as follows:

```
function bool = isassociative(H)
% H = a hyperproduct table (input)
% returns true iff H is associative
bool = 1;
vpow2 = 2.^[0:4];
for i = vpow2
    for j = vpow2
        for h = vpow2
            l1 = listofsingletons(H(j,k));
            l2 = listofsingletons(H(i,j));
            bool = bool & (bitor(H(i,l1)) == bitor(H(l2,h)'));
            if ~bool, break, end
        end
        if ~bool, break, end
    end
    if ~bool, break, end
end
```

A.1. The routine *findHgroups*

The first and most computation-demanding task is to enumerate all hypergroups belonging to a given subfamily. This is accomplished by an exhaustive search for all hyperproduct tables H completing a given incomplete table $\mathcal{H}$, and the examination of which of these tables actually describe an hypergroup of type U on the right. Hence, the high-level description of the routine `findHgroups` is the following:

• **for all** *hyperproduct tables* $H \in \mathcal{H}$ **do**
• **if** [`isreproducible` (H) *and* `isassociative`(H)] **then** *output* H
• **end**

The actual code depends on the particular incomplete table $\mathcal{H}$. In fact, there is no simple way to devise a data structure describing an incomplete hyperproduct table. Hence, our program has no input, and the enumeration of all hyperproduct tables $H \in \mathcal{H}$ is coded using a sequence of nested `for` loops, carefully designed to explore all possibilities in the given incomplete table. These loops must be reprogrammed for each given incomplete table. For example, the code for the analysis of the case $\nu = 0$ (a) in Section 4.1, is the following:

```
for I24 = [21, 29]
  for I25 = [13, 29]
    for I32 = [25, 27]
      for I34 = [19, 27]
        for I35 = [11, 27]
          for I42 = [21, 23]
            for I43 = [19, 23]
              for I45 = [7, 23]
                for I52 = [13, 15]
                  for I53 = [11, 15]
                    for I54 = [7, 15]

H = ([1  2   4   8   16
      2  29  25  I24 I25
      4  I32 27  I34 I35
      8  I42 I43 23  I45
      16 I52 I53 I54 15 ]);

    if isreproducible(H)
       if isassociative(H)
          disp(H)            % output H on screen
        end
    end

end; end; end; end; end; end; end; end; end; end; end;
```

The output is displayed on the computer screen and saved into a file using the Matlab `diary` function.

A.2. The routine `sieveHgroups`

The purpose of the routine `sieveHgroups` is to scan a list of hypergroups built by means of `findHgroups` and select among them the largest subset of pairwise non-isomorphic hypergroups, with respect to a given set of candidate isomorphisms.

Firstly, we need a function that, given a hypergroup H and an isomorphism π, computes the transformed hypergroup $K = \pi(H)$. Our implementation is the following:

```
function K = isomorph(perm,H)
% applies an isomorhism (permutation) to a given hyperproduct table
% and computes the transformed table
% perm = permutation (input)
% H = a hyperproduct table (input)
% K = the transformed hyperproduct table (output)
K = zeros(5,5);
vpow2 = 2.^[0:4]';
[I,invperm] = sort(perm);  % computes the inverse permutation
for i = 1:5
   for j = 1:5
      s = listofsingletons(H(invperm(i),invperm(j)));
      K(i,j) = sum(vpow2(perm(s)));
   end
end
```

Based on the preceding function, we designed the routine `sieveHgroups` according to the following high-level description. Let $\{\pi_1, \ldots, \pi_m\}$ be a set of candidate isomorphisms, and let $\{H_1, \ldots, H_n\}$ be a set of hypergroups. To each hypergroup, associate a binary-valued variable *flag* whose value is set initially to zero. Then, perform the following operations:

- **for** $k = 1, \ldots, m$
- **for** $i = 1, \ldots, n$
- **if** [$flag(i) = 0$ *and* $\exists j > i : H_j = \pi_k(H_i)$] **then** $flag(j) := 1$
- **end**
- **end**

Upon termination, those hypergroups whose corresponding flag is zero are pairwise non-isomorphic. Hence, the number of flag variables whose value is zero gives the number of distinct isomorphism classes within the given set of hypergroups.

The Matlab code of the computational core of the routine `sieveHgroups` is listed here below. The three-way array `Hlist(5,5,n)` is the list of hypergroups that must be examined (the *i*th hypergroup is `Hlist(:,:,i)`); it is initialized by reading the file produced by one run of the routine `findHgroups`; the array `permlist(m,5)` is the list of permutations that must be applied to the hypergroups (the *k*th permutation is `permlist(k,:)`); it is initialized with the list of candidate isomorphisms (i.e., permutations) corresponding to each subfamily as specified in Section 5. We also use a vector `flag(n)` with one entry for each hypergroup; initially, all its entries are initially zeroed.

```
for k = 1:m                     % outer loop: choose a permutation
   perm = permlist(k,:);
   for i = 1:n                  % inner loop: choose a hypergroup
      if flag(i) == 0
         K = isomorph(perm,Hlist(:,:,i));
         for j = i+1:n
            if flag(j) == 0
               if Hlist(:,:,j) == K
                  flag(j) = 1; break
               end
            end
         end
      end
   end
end
```

When this code fragment terminates, the number of isomorphism classes is given by the number of tables whose associated flag is zero, that is, `sum(flag == 0)`.

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