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Mathematical analysis of the guided waves in photonic crystal fibers ☆

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Abstract

The propagation of guided waves in photonic crystal fibers (PCFs) is studied. The structure of a PCF can be regarded as a perfect two-dimensional photonic crystal with a line defect along the invariant direction. This problem can be treated as an eigenvalue problem for a family of noncompact self-adjoint operators. We prove that line defects do not change the essential spectrum of the associated "background" medium. This result plays a key role for studying the influence of line defects on the "background" spectrum. A modified Combes–Thomas estimate is also formulated.

Keywords: Maxwell's equations; Photonic crystal fibers; Band gap; Guided waves; Spectral properties; Combes–Thomas estimate

1. Introduction

Photonic crystals (PCs) are periodically structured dielectric media, which are designed to favor band gaps, i.e., monochromatic electromagnetic waves of certain frequencies cannot propagate through these structures. The fact that photonic crystals exhibit band gaps that bear a resemblance to semiconductors has great importance in physics. Since the first proposals of a photonic band gap effect by Yablonovitch [23] and John [12], lots of applications have been studied. Among these applications, photonic crystal fibers (PCFs) as fundamental transmission medium to guide electromagnetic waves have been intensively studied. See, e.g., [2,3,6,7,17,18]. Photonic crystal fibers consist of a periodic array of two different optical transparent materials running through the length of the fibers with a central defect which serve as cores for light guiding. Physically, guided waves (or guided modes) can be created in these structures, i.e., electromagnetic waves of certain frequencies propagate along the line defects of these structures may have finite transverse energy (or we can say that they are localized near the line defects) and radiating otherwise.

To the best of our acknowledge, although this phenomenon has been intensively studied in experiments and numerical simulations, theoretical studies are few. Recently, we noticed that in [22], both the transverse electric (TE) and transverse magnetic (TM) cases were studied. More precisely, in TM case, a guided wave has only longitudinal

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electric field and a purely transverse magnetic field. Similarly, in TE case, a guided wave has only longitudinal magnetic field and a purely transverse electric field. By dealing with the two 2D scalar differential equations, they proved the exponentially decay of the guided waves in the cladding. They also proved the possibility of opening gaps in the spectrum of the background spectrum making it possible to guide electromagnetic waves with suitable cores.

One can also deal with this problem under the assumption of weak guidance, i.e., small variations of electric permittivity and magnetic permeability of the medium. Then guided waves have only transverse electric and transverse magnetic fields, approximately. Under this assumption the problem can be reduced into a scalar problem in the transverse plane of a photonic crystal fiber. However, for photonic crystal fibers used in practice, this scalar approximation is generally not valid, due to great variations of electric permittivity and magnetic permeability of the medium. So it is very important to study the vectorial problem not only in theory but also in practice.

The goal of this paper is to give a mathematical framework for understanding this phenomenon. For this purpose, we use the theory developed in [8,9]. The distinguishing point of our work, is that the results here are also hold for the ordinary dielectric waveguides, where the ordinary waveguides is a cylindrical structure, with homogeneous electric permittivity and magnetic permeability in longitudinal direction and inhomogeneous electric permittivity and magnetic permeability in the transverse plane. This paper is a first step in rigorously explaining the spectral properties of guided modes in photonic crystal fibers. The existence of eigenvalues created by line defects, exponential decay property of the corresponding eigenfunctions and other interesting issues have been studied in [19].

The outline of the remainder of this paper is as follows: In Section 2, we show that this problem can be treated as an eigenvalue problem for a family of noncompact self-adjoint operators. We prove the self-adjointness of these operators in Section 3. In Section 4, we prove the stability of the essential spectrum, i.e., line defects do not change the essential spectrum of the associated "background" medium (in fact we only require background medium to be invariant in one direction, the periodic condition of the background medium in the transverse plane is unnecessary). This is a fundamental result for studying their point spectrum. Since the proof of the Combes–Thomas estimate used in Section 4 is complex, we will list it as Section 5 separately. It is worth noting that this estimate is also very useful for studying the exponential decay property of guided waves [19].

2. Mathematical formulation

First we will give a rigorous description of some special photonic crystals and photonic crystal fibers. We will adapt some notations for convenience in the following:

$$\vec{x} = (x, x_3) \in \mathbb{R}^3, \qquad x = (x_1, x_2) \in \mathbb{R}^2.$$

We consider lossless inhomogeneous dielectric medium occupying the whole space $\mathbb{R}^3$. The functions $\epsilon(\vec{x})$ and $\mu(\vec{x})$ which describe the medium are called electric permittivity and magnetic permeability, correspondingly. We assume that $\epsilon(\vec{x})$ and $\mu(\vec{x})$ are invariant under any translation in the x_3 direction

$$\epsilon(\vec{x}) = \epsilon(x), \qquad \mu(\vec{x}) = \mu(x). \tag{1}$$

It is reasonable physically that there exist constants c_1 and c_2 such that

$$0 < c_1 \leqslant \epsilon(\vec{x}), \mu(\vec{x}) \leqslant c_2 < \infty \quad \text{a.e.} \tag{2}$$

If they are periodic functions of the transverse variable x with period $Y = \mathbb{R}^2/\mathbb{Z}^2$, i.e.,

$$\epsilon(x + \vec{n}) = \epsilon(x), \quad \mu(x + \vec{n}) = \mu(x) \quad \text{for all } \vec{n} \in \mathbb{Z}^2,\ x \in \mathbb{R}^2, \tag{3}$$

these structures are often called (two-dimensional) photonic crystals, or photonic band gap materials [13]. Furthermore, a photonic crystal fiber is created if a line defect in parallel with x_3-direction is introduced (see Fig. 1). We describe the defect strip by

$$\tilde{\Omega} = \left\{\vec{x} = (x, x_3) \in \mathbb{R}^3 \;\middle|\; x_3 \in \mathbb{R},\ x \in \Omega\right\}, \tag{4}$$

where Ω is the support of the perturbation in the transverse plane. We assume that Ω is a measurable compact subset of $\mathbb{R}^2$. Without loss of generality, we also assume that $0 \in \Omega$. Inside the defect, the dielectric medium can be different from the background medium. We define the background medium and the perturbed medium rigorously in Section 4.

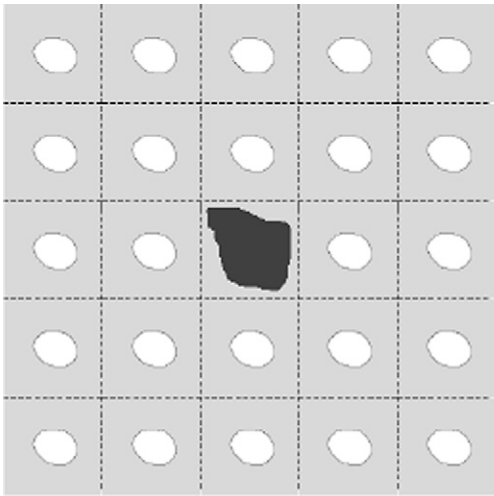

Fig. 1. The line defect is shown on the cross section of the photonic crystal fiber as a darker region.

It is worth noting that the results of this paper hold for all $\epsilon(x)$ and $\mu(x)$ which just satisfy (1) and (2). *That is to say, the condition* (3) *for* $\epsilon(x)$ *and* $\mu(x)$ *is unnecessary.*

The Maxwell's equations that govern light propagation in the medium in absence of free charges and currents look as follows:

$$\begin{cases} \nabla_{\vec{x}} \times E(\vec{x},t) + \dfrac{\partial B(\vec{x},t)}{\partial t} = 0, \quad \nabla_{\vec{x}} \cdot B(\vec{x},t) = 0, \\ \nabla_{\vec{x}} \times H(\vec{x},t) - \dfrac{\partial D(\vec{x},t)}{\partial t} = 0, \quad \nabla_{\vec{x}} \cdot D(\vec{x},t) = 0, \end{cases} \tag{5}$$

where $E(\vec{x},t)$, $H(\vec{x},t)$ are the electric and magnetic fields, and $D(\vec{x},t)$ and $B(\vec{x},t)$ are the displacement and magnetic induction fields, correspondingly. The so-called constitutive relations are

$$D(\vec{x},t) = \epsilon(\vec{x})E(\vec{x},t), \qquad B(\vec{x},t) = \mu(\vec{x})H(\vec{x},t).$$

We consider time-harmonic waves

$$E(\vec{x},t) = e^{i\omega t}\mathbb{E}(\vec{x}), \qquad H(\vec{x},t) = e^{i\omega t}\mathbb{H}(\vec{x}),$$

where $\omega > 0$ is the angular frequency. This leads from Eqs. (5) to

$$\begin{cases} \nabla \times \mathbb{E}(\vec{x}) + i\omega\mu\mathbb{H}(\vec{x}) = 0, \quad \nabla \cdot (\mu\mathbb{H}) = 0, \\ \nabla \times \mathbb{H}(\vec{x}) - i\omega\epsilon\mathbb{E}(\vec{x}) = 0, \quad \nabla \cdot (\epsilon\mathbb{E}) = 0. \end{cases} \tag{6}$$

Definition 2.1. A *guided mode* is the solution of (6) on the form

$$\begin{cases} \mathbb{E}(\vec{x}) = \big(E_1(x), E_2(x), E_3(x)\big)^\top e^{-i\beta x_3}, \\ \mathbb{H}(\vec{x}) = \big(H_1(x), H_2(x), H_3(x)\big)^\top e^{-i\beta x_3} \end{cases} \tag{7}$$

and

$$\int_{\mathbb{R}^2} \big(\epsilon|E|^2 + \mu|H|^2\big)\,dx < \infty,$$

where

$$E = \big(E_1(x), E_2(x), E_3(x)\big)^\top, \qquad H = \big(H_1(x), H_2(x), H_3(x)\big)^\top$$

and $\beta > 0$ is the wave number of the mode in the x_3-direction.

We will introduce some notations in the following:

$$\nabla_\beta = \begin{pmatrix} \partial_1 \\ \partial_2 \\ 0 \end{pmatrix} - i\beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \partial_1 \\ \partial_2 \\ -i\beta \end{pmatrix},$$

where $\partial_1 = \partial/\partial x_1$, $\partial_2 = \partial/\partial x_2$. Furthermore, we define

$$\nabla_\beta \phi = (\partial_1 \phi, \partial_2 \phi, -i\beta\phi)^\top,$$
$$\nabla_\beta \times \vec{u} = (\partial_2 u_3 + i\beta u_2, -\partial_1 u_3 - i\beta u_1, \partial_1 u_2 - \partial_2 u_1)^\top,$$
$$\nabla_\beta \cdot \vec{u} = \partial_1 u_1 + \partial_2 u_2 - i\beta u_3,$$

where $\vec{u} = (u_1, u_2, u_3)^\top$ and $\phi = \phi(x)$ is a scalar function.

Now plugging formula (7) into (6) and eliminating E or H, one obtains

$$\epsilon^{-1}\nabla_\beta \times \mu^{-1}\nabla_\beta \times E = \lambda E \tag{8}$$

and

$$\mu^{-1}\nabla_\beta \times \epsilon^{-1}\nabla_\beta \times H = \lambda H,$$

where $\lambda = \omega^2$.

We first consider the E-formulation (8). In the following, some functional spaces are useful. We shall denote for any 3D vector field $\vec{u} = (u_1(x), u_2(x), u_3(x))^\top$ the transverse field by $u = (u_1(x), u_2(x))^\top$, thus we have $\vec{u} = (u^\top, u_3(x))^\top$. We define

$$\operatorname{curl} u = \partial_1 u_2 - \partial_2 u_1$$

and

$$H(\operatorname{curl}; \mathbb{R}^2) = \{u \in L^2(\mathbb{R}^2) \mid \operatorname{curl} u \in L^2(\mathbb{R}^2)\}$$

with the norm

$$\|u\|^2_{H(\operatorname{curl};\mathbb{R}^2)} = \|u\|^2_{L^2(\mathbb{R}^2)^2} + \|\operatorname{curl} u\|^2_{L^2(\mathbb{R}^2)}.$$

A standard Sobolev space is also needed

$$H^1(\mathbb{R}^2) = \{\phi \in L^2(\mathbb{R}^2) \mid \nabla\phi \in L^2(\mathbb{R}^2)^2\}.$$

Furthermore, we also define

$$H_\epsilon = L^2(\mathbb{R}^2)^3 \tag{9}$$

equipped with the weighted inner product

$$(\vec{u}, \vec{v})_\epsilon = \int_{\mathbb{R}^2} \epsilon \vec{u} \cdot \bar{\vec{v}}\, dx$$

and the norm $\|\vec{u}\|_\epsilon = \sqrt{(\vec{u}, \vec{v})_\epsilon}$, where $\bar{\vec{v}}$ means the conjugate of $\vec{v}$.

We introduce

$$V_\epsilon = \{\vec{u} \in H_\epsilon \mid \nabla_\beta \times \vec{u} \in H_\epsilon\}.$$

The space V_ϵ is a Hilbert space equipped with the norm

$$\|\vec{u}\|^2_{V_\epsilon} = \int_{\mathbb{R}^2} \epsilon(|\vec{u}|^2 + |\nabla_\beta \times \vec{u}|^2)\, dx.$$

Lemma 2.1. *V_ϵ is isomorphic to $H(\mathrm{curl};\mathbb{R}^2)\times H^1(\mathbb{R}^2)$ and the norm $\|\cdot\|_{V_\epsilon}$ is equivalent to the norm $\|\cdot\|_{H(\mathrm{curl};\mathbb{R}^2)\times H^1(\mathbb{R}^2)}$, i.e.,*

$$V_\epsilon=\{\vec{u}\mid \vec{u}=(u^\top,u_3)^\top\in H(\mathrm{curl};\mathbb{R}^2)\times H^1(\mathbb{R}^2)\}.$$

Proof. We notice that

$$\begin{aligned}\|\vec{u}\|_{V_\epsilon}^2&=\|\vec{u}\|_{H_\epsilon}^2+\|\nabla_\beta\times\vec{u}\|_{H_\epsilon}^2\\&=\int_{\mathbb{R}^2}\epsilon\big(|\vec{u}|^2+|\partial_2u_3+i\beta u_2|^2+|\partial_1u_3+i\beta u_1|^2+|\partial_1u_2-\partial_2u_1|^2\big)\,dx\\&=\int_{\mathbb{R}^2}\epsilon\big(|\vec{u}|^2+|\nabla u_3+i\beta u|^2+|\mathrm{curl}\,u|^2\big)\,dx.\end{aligned}\tag{10}$$

This implies

$$\|\vec{u}\|_{V_\epsilon}\leqslant C\|\vec{u}\|_{H(\mathrm{curl};\mathbb{R}^2)\times H^1(\mathbb{R}^2)}$$

for some constant $C<\infty$.

On the other hand, for $\vec{u}\in V_\epsilon$, using (10) and for some integer $n\geqslant 1+2\beta^2$, we have

$$\begin{aligned}\|\vec{u}\|_{V_\epsilon}^2&=\|\vec{u}\|_{H_\epsilon}^2+\|\nabla_\beta\times\vec{u}\|_{H_\epsilon}^2\\&=\int_{\mathbb{R}^2}\epsilon\big(|\vec{u}|^2+|\nabla u_3+i\beta u|^2+|\mathrm{curl}\,u|^2\big)\,dx\\&=\int_{\mathbb{R}^2}\epsilon\big(|\vec{u}|^2+|\nabla u_3|^2+\beta^2|u|^2-2\beta\,\mathrm{Im}(u\cdot\overline{\nabla u_3})+|\mathrm{curl}\,u|^2\big)\,dx\\&\geqslant\int_{\mathbb{R}^2}\epsilon\big(|\vec{u}|^2+|\nabla u_3|^2+\beta^2|u|^2-2\beta|u||\nabla u_3|+|\mathrm{curl}\,u|^2\big)\,dx\\&=\int_{\mathbb{R}^2}\epsilon\left(\frac{1}{n}|u_3|^2+\frac{1}{n}|\nabla u_3|^2\right)dx+\int_{\mathbb{R}^2}\epsilon\left(\frac{1}{2}|u|^2+\frac{1}{2}|\mathrm{curl}\,u|^2\right)dx\\&\quad+\int_{\mathbb{R}^2}\epsilon\left(\frac{n-1}{n}|u_3|^2+\frac{1}{2}|\mathrm{curl}\,u|^2\right)dx\\&\quad+\int_{\mathbb{R}^2}\epsilon\left(\frac{n-1}{n}|\nabla u_3|^2+\left(\frac{1}{2}+\beta^2\right)|u|^2-2\beta|u||\nabla u_3|\right)dx\\&\geqslant\frac{1}{n}c_1\|u_3\|_{H^1(\mathbb{R}^2)}^2+\frac{1}{2}c_1\|u\|_{H(\mathrm{curl};\mathbb{R}^2)}^2\\&\quad+\int_{\mathbb{R}^2}\epsilon\left(\frac{n-1}{n}|\nabla u_3|^2+\left(\frac{1}{2}+\beta^2\right)|u|^2-2\beta|u||\nabla u_3|\right)dx\\&\geqslant\frac{1}{n}c_1\|u_3\|_{H^1(\mathbb{R}^2)}^2+\frac{1}{2}c_1\|u\|_{H(\mathrm{curl};\mathbb{R}^2)}^2\\&\quad+\int_{\mathbb{R}^2}\epsilon\left(2\sqrt{\frac{n-1}{n}}\sqrt{\beta^2+\frac{1}{2}}-2\beta\right)|u||\nabla u_3|\,dx\\&\geqslant\frac{1}{n}c_1\|u_3\|_{H^1(\mathbb{R}^2)}^2+\frac{1}{2}c_1\|u\|_{H(\mathrm{curl};\mathbb{R}^2)}^2\end{aligned}$$

where c_1 is defined in (2). This implies

$$\|\vec{u}\|_{V_\epsilon} \geqslant \tilde{C}\|\vec{u}\|_{H(\mathrm{curl};\mathbb{R}^2)\times H^1(\mathbb{R}^2)}$$

for some constant $\tilde{C} < \infty$. This completes the proof. □

Some simple properties about the operators $\nabla_\beta\times$, $\nabla_\beta\cdot$ and ∇_β should be noticed:

Lemma 2.2.

$$\begin{aligned} &\text{(i)} && \nabla_\beta \cdot (\nabla_\beta \times) = 0, \\ &\text{(ii)} && \nabla_\beta \times (\nabla_\beta) = 0, \\ &\text{(iii)} && \nabla_\beta \cdot (\epsilon E) = 0 \quad \textit{for } E = (E_1, E_2, E_3)^\top \textit{ satisfies } (8) \textit{ and } \lambda \neq 0. \end{aligned} \tag{11}$$

Proof. One can easily check (i) and (ii). From Eq. (8),

$$\nabla_\beta \times \mu^{-1}\nabla_\beta \times E = \lambda\epsilon E, \tag{12}$$

applying (i) to (12) for $\lambda \neq 0$, one obtains (iii). □

Remark 2.1. Identity (iii) of Lemma 2.2 means that all physical solutions must satisfy the divergence free condition for $\lambda > 0$.

3. Self-adjointness

In the following, we will first give a space decomposition which is analogous to the classical Hodge decomposition (also called Helmholtz decomposition or Weyl decomposition in some literature).

Lemma 3.1. *The space H_ϵ can be decomposed to the direct sum of the spaces $H_\epsilon(\beta)$ and $G(\beta)$,*

$$H_\epsilon = H_\epsilon(\beta) \oplus G(\beta), \tag{13}$$

where

$$H_\epsilon(\beta) = \{\vec{u} \in H_\epsilon \mid \nabla_\beta \cdot (\epsilon\vec{u}) = 0\} \tag{14}$$

and

$$G(\beta) = \{\nabla_\beta\phi \mid \phi \in H^1(\mathbb{R}^2)\}.$$

The sum (13) *is orthogonal with respect to the scalar product with the weight $\epsilon(x)\,dx$.*

Proof. For arbitrary $\vec{u} \in H_\epsilon$, introduce the unique weak solution $\phi \in H^1(\mathbb{R}^2)$ of $\nabla_\beta \cdot (\epsilon\nabla_\beta\phi) = \nabla_\beta \cdot (\epsilon\vec{u})$, i.e., ϕ solves the weak formulation

$$\int_{\mathbb{R}^2} (\epsilon\nabla_\beta\phi) \cdot \nabla_\beta\overline{\psi}\,dx = \int_{\mathbb{R}^2} (\epsilon\vec{u}) \cdot \nabla_\beta\overline{\psi}\,dx$$

for any $\psi \in H^1(\mathbb{R}^2)$. We set $\vec{v} = \vec{u} - \nabla_\beta\phi$, then we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^2} (\epsilon\vec{u} - \epsilon\nabla_\beta\phi) \cdot \overline{\nabla_\beta\psi}\,dx \\ &= \int_{\mathbb{R}^2} \big(\nabla_\beta \cdot (\epsilon\vec{u} - \epsilon\nabla_\beta\phi)\big)\overline{\psi}\,dx \\ &= \int_{\mathbb{R}^2} \big(\nabla_\beta \cdot (\epsilon\vec{v})\big)\overline{\psi}\,dx \quad \text{for all } \psi \text{ in } H^1(\mathbb{R}^2). \end{aligned}$$

This implies $\vec{v} \in H_\epsilon(\beta)$ and orthogonality of the spaces between $H_\epsilon(\beta)$ and $G(\beta)$. Thus the lemma is proved. □

Definition 3.1. The unbounded operator $\mathscr{A}_\epsilon(\beta)$ is defined by

$$\mathscr{A}_\epsilon(\beta)\vec{u} = \epsilon^{-1}\nabla_\beta \times \mu^{-1}\nabla_\beta \times \vec{u}$$

with

$$D\big(\mathscr{A}_\epsilon(\beta)\big) = \big\{\vec{u} \in V_\epsilon \,\big|\, \nabla_\beta \times \mu^{-1}\nabla_\beta \times \vec{u} \in H_\epsilon\big\}.$$

We can describe the structure of $\mathscr{A}_\epsilon(\beta)$ by

Lemma 3.2.

(i) $\mathcal{K}\mathrm{er}\,\mathscr{A}_\epsilon(\beta) = G(\beta)$,

(ii) $\mathcal{I}\mathrm{m}\,\mathscr{A}_\epsilon(\beta) \subset H_\epsilon(\beta)$.

Proof. (i) By (ii) of Lemma 2.2 we have $G(\beta) \subset \mathcal{K}\mathrm{er}\,\mathscr{A}_\epsilon(\beta)$. Conversely, if $\vec{u} \in \mathcal{K}\mathrm{er}\,\mathscr{A}_\epsilon(\beta)$, by Green's formula, we have

$$0 = \int_{\mathbb{R}^2} \nabla_\beta \times \big(\mu^{-1}\nabla_\beta \times \vec{u}\big) \cdot \bar{\vec{u}}\,dx = \int_{\mathbb{R}^2} \mu^{-1}|\nabla_\beta \times \vec{u}|^2\,dx,$$

this implies

$$\nabla_\beta \times \vec{u} = 0,$$

i.e.,

$$\begin{cases} \partial_2 u_3 + i\beta u_2 = 0, \\ -\partial_1 u_3 - i\beta u_1 = 0, \\ \partial_1 u_2 - \partial_2 u_1 = 0. \end{cases}$$

Hence we have

$$\begin{aligned} \vec{u} &= \left(-\frac{1}{i\beta}\partial_1 u_3, -\frac{1}{i\beta}\partial_2 u_3, u_3\right)^\top \\ &= \frac{i}{\beta}(\partial_1 u_3, \partial_2 u_3, -i\beta u_3)^\top \\ &= \frac{i}{\beta}\nabla_\beta u_3. \end{aligned}$$

Since $\nabla_\beta u_3 \in G(\beta)$, this implies $\mathcal{K}\mathrm{er}\,\mathscr{A}_\epsilon(\beta) \subset G(\beta)$. Thus, $\mathcal{K}\mathrm{er}\,\mathscr{A}_\epsilon(\beta) = G(\beta)$.

(ii) is the immediate consequence of (i) of Lemma 2.2. □

Since $\mathscr{A}_\epsilon(\beta)|_{G(\beta)} = 0$, we have $\sigma(\mathscr{A}_\epsilon(\beta)) = \{0\} \cup \sigma(\mathscr{A}_\epsilon(\beta)|_{H_\epsilon(\beta)\cap V_\epsilon})$. It is natural to work on the restriction of $\mathscr{A}_\epsilon(\beta)$ to the space $H_\epsilon(\beta) \cap V_\epsilon$, i.e.,

$$A_\epsilon(\beta) \equiv \mathscr{A}_\epsilon(\beta)|_{V_\epsilon(\beta)},$$

where

$$V_\epsilon(\beta) = H_\epsilon(\beta) \cap V_\epsilon = \big\{\vec{u} \in V_\epsilon \,\big|\, \nabla_\beta \cdot (\epsilon\vec{u}) = 0\big\}.$$

The nonnegative closed quadratic form $a_\epsilon(\beta;\cdot,\cdot)$ corresponding to $A_\epsilon(\beta)$ is

$$a_\epsilon(\beta;\vec{u},\vec{v}) = \int_{\mathbb{R}^2} \big(\mu^{-1}\nabla_\beta \times \vec{u}\big) \cdot \overline{\nabla_\beta \times \vec{v}}\,dx \quad \text{for all } (\vec{u},\vec{v}) \in V_\epsilon(\beta) \times V_\epsilon(\beta). \tag{15}$$

Next, a two-dimensional scalar valued operator div is defined by

$$\operatorname{div} u = \partial_1 u_1 + \partial_2 u_2 \quad \text{for } u = (u_1, u_2)^\top.$$

Theorem 3.1. *For any $\beta > 0$, the operator $A_\epsilon(\beta)$ is self-adjoint, uniformly positive and*

$$\sigma\big(A_\epsilon(\beta)\big) \subset \big[\rho_- \beta^2, \infty\big),$$

where

$$\rho_- = \inf_{x\in\mathbb{R}^2}\big(\epsilon^{-1}(x)\mu^{-1}(x)\big) > 0.$$

Proof.

$$\begin{aligned} a_\epsilon(\beta;\vec{u},\vec{u}) &= \int_{\mathbb{R}^2} \mu^{-1}|\nabla_\beta \times \vec{u}|^2\,dx \\ &\geqslant \rho_- \int_{\mathbb{R}^2} \epsilon|\nabla_\beta \times \vec{u}|^2\,dx \\ &= \rho_- \int_{\mathbb{R}^2} \epsilon\big(|\partial_2 u_3 + i\beta u_2|^2 + |\partial_1 u_3 + i\beta u_1|^2 + |\partial_1 u_2 - \partial_2 u_1|^2\big)\,dx \\ &= \rho_- \int_{\mathbb{R}^2} \epsilon\big(|\nabla u_3 + i\beta u|^2 + |\operatorname{curl} u|^2\big)\,dx \\ &= \rho_- \int_{\mathbb{R}^2} \epsilon\big(|\nabla u_3|^2 + \beta^2|u|^2 - 2\beta\,\mathrm{Im}(u\cdot\overline{\nabla u_3}) + |\operatorname{curl} u|^2\big)\,dx. \end{aligned} \tag{16}$$

Notice that $\vec{u} \in H_\epsilon(\beta)$,

$$\nabla_\beta \cdot (\epsilon\vec{u}) = 0,$$

it implies that

$$\operatorname{div}(\epsilon u) = i\beta(\epsilon u_3).$$

By Green's formula, we have

$$-\int_{\mathbb{R}^2} \epsilon u \cdot \overline{\nabla u_3}\,dx = \int_{\mathbb{R}^2} \operatorname{div}(\epsilon u)\overline{u_3}\,dx = i\beta \int_{\mathbb{R}^2} \epsilon|u_3|^2\,dx.$$

Since ϵ is a real number, we have

$$-\mathrm{Im}\int_{\mathbb{R}^2} \epsilon u \cdot (\overline{\nabla u_3})\,dx = -\int_{\mathbb{R}^2} \epsilon\,\mathrm{Im}\big(u\cdot(\overline{\nabla u_3})\big)\,dx = \beta\int_{\mathbb{R}^2} \epsilon|u_3|^2\,dx. \tag{17}$$

Plugging identity (17) into (16) leads to

$$\begin{aligned} a_\epsilon(\beta;\vec{u},\vec{u}) &\geqslant \rho_- \int_{\mathbb{R}^2} \epsilon\big(|\nabla u_3|^2 + |\operatorname{curl} u|^2\big)\,dx + \rho_-\beta^2 \int_{\mathbb{R}^2} \epsilon\big(|u|^2 + 2|u_3|^2\big)\,dx \\ &\geqslant \rho_-\beta^2\|\vec{u}\|_\epsilon^2. \qquad \square \end{aligned}$$

Remark 3.1.

(i) Theorem 3.1 is just the first step for studying the spectral properties of $A_\epsilon(\beta)$. It is well known that the spectrum of $A_\epsilon(\beta)$ consists of an essential spectrum corresponding to a continuum of radiating modes (i.e., plane wave-like modes) and a point spectrum corresponding to guided modes. Of course the radiating modes have no finite energy in the transverse plane.

(ii) The results of Lemmas 3.1, 3.2 and Theorem 3.1 are similar to the versions of Lemmas 1.1, 1.2 and 2.1 in [14]. However, we should notice that the operator $\mathscr{A}_\epsilon(\beta)$ is different to the counterpart defined in [14].

4. Stability of essential spectrum

In the following we set $\mathscr{L}(H)$ as the space of all bounded linear operators, where H is a Hilbert space, and $\mathrm{Com}(H)$ as the subspace of $\mathscr{L}(H)$ of all compact operators.

We will also describe the background medium by ϵ_0 and μ_0, and the perturbed medium by ϵ and μ. However, it should be noticed that we do not require ϵ_0 and μ_0 satisfying (3) until we give a statement.

We adapt $A_\epsilon(\beta)$ as the perturbed operator according to $A_{\epsilon_0}(\beta)$. We also introduce

$$\eta(x)=\mu^{-1}(x)-\mu_0^{-1}(x), \qquad \xi(x)=\epsilon^{-1}(x)-\epsilon_0^{-1}(x)$$

and

$$\eta_\pm=\max\{\pm\eta(x),0\}, \qquad \xi_\pm=\max\{\pm\xi(x),0\},$$

then we have

$$\begin{aligned}
A_\epsilon(\beta)-A_{\epsilon_0}(\beta) &= \left(\epsilon^{-1}\nabla_\beta\times\mu^{-1}\nabla_\beta\times\right)-\left(\epsilon_0^{-1}\nabla_\beta\times\mu_0^{-1}\nabla_\beta\times\right)\\
&= \left(\left(\epsilon^{-1}\nabla_\beta\times\mu^{-1}\nabla_\beta\times\right)-\left(\epsilon_0^{-1}\nabla_\beta\times\mu^{-1}\nabla_\beta\times\right)\right)\\
&\quad+\left(\left(\epsilon_0^{-1}\nabla_\beta\times\mu^{-1}\nabla_\beta\times\right)-\left(\epsilon_0^{-1}\nabla_\beta\times\mu_0^{-1}\nabla_\beta\times\right)\right)\\
&= \left(\xi\nabla_\beta\times\mu^{-1}\nabla_\beta\times\right)+\left(\epsilon_0^{-1}\nabla_\beta\times\eta\nabla_\beta\times\right).
\end{aligned}$$

By our hypotheses (4), both ξ and η are bounded measurable functions and they are supported inside Ω. $\xi_\pm\nabla_\beta\times\mu^{-1}\nabla_\beta\times$, $\epsilon_0^{-1}\nabla_\beta\times\eta_\pm\nabla_\beta\times$ and $A_\epsilon(\beta)-A_{\epsilon_0}(\beta)$ is denoted by $A_{\xi\pm}$, $A_{\eta\pm}$ and S, respectively. It is easy to see $A_{\epsilon\pm}$, $A_{\eta\pm}$ are nonnegative self-adjoint operators. Since we have

$$S=A_{\xi+}-A_{\xi-}+A_{\eta+}-A_{\eta-}$$

and

$$A_\epsilon(\beta)=\left(A_{\epsilon_0}(\beta)+A_{\xi+}+A_{\eta+}\right)-(A_{\xi-}+A_{\eta-}),$$

it suffices to prove Theorem 4.1 below in the case that both $\xi(x)$ and $\eta(x)$ do not change their signs for all x. Without loss of generality, we only consider the case $\xi\geqslant 0$ and $\eta\geqslant 0$.

Theorem 4.1 *(Stability of essential spectrum).*

$$\sigma_{\mathrm{ess}}\left(A_\epsilon(\beta)\right)=\sigma_{\mathrm{ess}}\left(A_{\epsilon_0}(\beta)\right).$$

Proof. Since $(A_{\epsilon_0}(\beta)+I)^{-n}S(A_{\epsilon_0}(\beta)+I)^{-n}$ is a Hilbert–Schmidt operator for arbitrary $n\geqslant 1$ which will be proved in Theorem 4.2 below, this implies that $(A_{\epsilon_0}(\beta)+I)^{-n}S(A_{\epsilon_0}(\beta)+I)^{-n}\in\mathrm{Com}(H)$. This theorem follows from Corollary XIII.4 in [20]. □

Theorem 4.2. *For any $\epsilon(x)$ and $\mu(x)$ satisfying condition* (2)*, any bounded measurable functions $\xi(x)$, $\eta(x)$ with the same compact support and any $n\geqslant 1$, $(A_{\epsilon_0}(\beta)+I)^{-n}S(A_{\epsilon_0}(\beta)+I)^{-n}$ is a Hilbert–Schmidt operator.*

Proof. We first prove this theorem in the case $S=\epsilon_0^{-1}\nabla_\beta\times\eta\nabla_\beta\times$, where $\eta\geqslant 0$ has a compact support.

As we know that, if $0\leqslant A\leqslant B$, where A and B are two self-adjoint operators with B Hilbert–Schmidt, A is also Hilbert–Schmidt. Hence, we can always find infinitely differentiable function $\tilde\eta$ with compact support such that $\eta(x)\mu_0(x)\leqslant\tilde\eta(x)$ for $x\in\mathbb{R}^2$. Because

$$0\leqslant\epsilon_0^{-1}\nabla_\beta\times\eta(x)\nabla_\beta\times\leqslant\epsilon_0^{-1}\nabla_\beta\times\left(\frac{\tilde\eta}{\mu_0}\right)\nabla_\beta\times,$$

it is sufficient to consider the case $\eta=\tilde\eta/\mu_0$ with $\tilde\eta\in C_0^\infty(\Omega_\eta)$, where Ω_η represents the support of η. Also, let χ_η denote the characteristic function of Ω_η, then we have

$$\begin{aligned}
&\big(A_{\epsilon_0}(\beta)+I\big)^{-n}\big(\epsilon_0^{-1}\nabla_\beta\times\eta\nabla_\beta\times\big)\big(A_{\epsilon_0}(\beta)+I\big)^{-n}\\
&\quad=\big(A_{\epsilon_0}(\beta)+I\big)^{-n}\big(\epsilon_0^{-1}(\nabla_\beta\tilde{\eta})\times\mu_0^{-1}\nabla_\beta\times\big)\big(A_{\epsilon_0}(\beta)+I\big)^{-n}\\
&\qquad+\big(A_{\epsilon_0}(\beta)+I\big)^{-n}\big(\epsilon_0^{-1}\tilde{\eta}\nabla_\beta\times\mu_0^{-1}\nabla_\beta\times\big)\big(A_{\epsilon_0}(\beta)+I\big)^{-n}\\
&\quad=\big(A_{\epsilon_0}(\beta)+I\big)^{-n}\chi_\eta\epsilon_0^{-1}(\nabla_\beta\tilde{\eta})\times\mu_0^{-1}\nabla_\beta\times\big(A_{\epsilon_0}(\beta)+I\big)^{-1}\big(A_{\epsilon_0}(\beta)+I\big)^{-(n-1)}\\
&\qquad+\big(A_{\epsilon_0}(\beta)+I\big)^{-n}\chi_\eta\epsilon_0^{-1}\tilde{\eta}\nabla_\beta\times\mu_0^{-1}\nabla_\beta\times\big(A_{\epsilon_0}(\beta)+I\big)^{-1}\big(A_{\epsilon_0}(\beta)+I\big)^{-(n-1)}.
\end{aligned}$$

Some notations are needed in the following

$$\begin{aligned}
&\epsilon_0^+=\sup_{x\in\mathbb{R}^2}\epsilon_0(x),\qquad \mu_0^+=\sup_{x\in\mathbb{R}^2}\mu_0(x),\\
&\epsilon_0^-=\inf_{x\in\mathbb{R}^2}\epsilon_0(x),\qquad \mu_0^-=\inf_{x\in\mathbb{R}^2}\mu_0(x).
\end{aligned}\tag{18}$$

Because $A_{\epsilon_0}(\beta)$ is a nonnegative operator on H_{ϵ_0}, it is easy to see

$$\big\|\nabla_\beta\times\mu_0^{-1}\nabla_\beta\times\big(A_{\epsilon_0}(\beta)+I\big)^{-1}\big\|=\big\|A_{\epsilon_0}(\beta)\big(A_{\epsilon_0}(\beta)+I\big)^{-1}\big\|_{\epsilon_0}\leqslant\epsilon_0^+.\tag{19}$$

Next, we will estimate the term $\|\nabla_\beta\times(A_{\epsilon_0}(\beta)+I)^{-1}\|$. For all $\vec{v}\in H_{\epsilon_0}$, we have

$$\begin{aligned}
\big\|\nabla_\beta\times\big(A_{\epsilon_0}(\beta)+I\big)^{-1}\vec{v}\big\|^2&=\int_{\mathbb{R}^2}\big|\nabla_\beta\times\big(A_{\epsilon_0}(\beta)+I\big)^{-1}\vec{v}\big|^2\,dx\\
&\leqslant\mu_0^+\int_{\mathbb{R}^2}\big(\nabla_\beta\times\big(A_{\epsilon_0}(\beta)+I\big)^{-1}\vec{v}\big)\cdot\overline{\big(\mu_0^{-1}\nabla_\beta\times\big(A_{\epsilon_0}(\beta)+I\big)^{-1}\vec{v}\big)}\,dx\\
&=\mu_0^+\int_{\mathbb{R}^2}\big(\big(A_{\epsilon_0}(\beta)+I\big)^{-1}\vec{v}\big)\cdot\overline{\big(\nabla_\beta\times\mu_0^{-1}\nabla_\beta\times\big(A_{\epsilon_0}(\beta)+I\big)^{-1}\vec{v}\big)}\,dx\\
&=\mu_0^+\int_{\mathbb{R}^2}\big(\big(A_{\epsilon_0}(\beta)+I\big)^{-1}\vec{v}\big)\cdot\overline{\big(\epsilon_0A_{\epsilon_0}(\beta)\big(A_{\epsilon_0}(\beta)+I\big)^{-1}\vec{v}\big)}\,dx\\
&\leqslant\mu_0^+\epsilon_0^+\big\|\big(A_{\epsilon_0}(\beta)+I\big)^{-1}\vec{v}\big\|\big\|A_{\epsilon_0}(\beta)\big(A_{\epsilon_0}(\beta)+I\big)^{-1}\vec{v}\big\|\\
&\leqslant\mu_0^+\epsilon_0^+\|\vec{v}\|^2.
\end{aligned}$$

Hence we have

$$\big\|\nabla_\beta\times\big(A_{\epsilon_0}(\beta)+I\big)^{-1}\big\|\leqslant\sqrt{\mu_0^+\epsilon_0^+},\tag{20}$$

where operator norm $\|\cdot\|$ is defined by

$$\|A\|=\sup_{\vec{0}\neq\vec{v}\in H_\epsilon}\frac{\|A\vec{v}\|}{\|\vec{v}\|}.$$

Besides, it is easy to see

$$\big\|\big(A_{\epsilon_0}(\beta)+I\big)^{-1}\big\|\leqslant 1.\tag{21}$$

Since $\operatorname{tr}|\chi_\eta(A_{\epsilon_0}(\beta)+I)^{-1}\chi_\eta|^2<\infty$ by Theorem 4.3 below, we know that $(A_{\epsilon_0}(\beta)+I)^{-1}\chi_\eta$ is a Hilbert–Schmidt operator. Then for this case the theorem is proved by employing (19), (20) and (21).

We then deal with the case

$$S=\xi\nabla_\beta\times\mu^{-1}\nabla_\beta\times.$$

Since $\mu^{-1}=\eta+\mu_0^{-1}$, we can rewrite S as

$$S=(\xi\nabla_\beta\times\eta\nabla_\beta\times)+\big(\xi\nabla_\beta\times\mu_0^{-1}\nabla_\beta\times\big).$$

For the first part, we can proof $(A_{\epsilon_0}(\beta)+I)^{-n}\xi\nabla_\beta\times\eta\nabla_\beta\times(A_{\epsilon_0}(\beta)+I)^{-n}$ is Hilbert–Schmidt by mimicking the proof above. In the following, we only need to deal with the second part. Let χ_ξ be the characteristic function of ξ, then we have

$$\begin{aligned}&\big(A_{\epsilon_0}(\beta)+I\big)^{-n}\xi\nabla_\beta\times\mu_0^{-1}\nabla_\beta\times\big(A_{\epsilon_0}(\beta)+I\big)^{-n}\\&\quad=\big(A_{\epsilon_0}(\beta)+I\big)^{-n}\chi_\xi\frac{\xi}{\epsilon_0}A_{\epsilon_0}(\beta)\big(A_{\epsilon_0}(\beta)+I\big)^{-1}\big(A_{\epsilon_0}(\beta)+I\big)^{-(n-1)}.\end{aligned}$$

Also, it follows from Theorem 4.3 below that $(A_{\epsilon_0}(\beta)+I)^{-n}\chi_\xi$ is Hilbert–Schmidt. Furthermore, we have

$$\big\|A_{\epsilon_0}(\beta)\big(A_{\epsilon_0}(\beta)+I\big)^{-1}\big\|\leqslant 1.$$

Hence we know that the theorem also holds in this case by employing the inequality above and (21). Thus we complete the proof of this theorem. □

For simplicity of notations, the operators $A_{\epsilon_0}(\beta)$ and $\mathscr{A}_{\epsilon_0}(\beta)$ are abbreviated to $A_0(\beta)$ and $\mathscr{A}_0(\beta)$ in the following, correspondingly.

We define

$$\overline{\mu_0}=\sup_{x\in\mathbb{R}^2}\mu_0^{-1}(x),\qquad \underline{\mu_0}=\inf_{x\in\mathbb{R}^2}\mu_0^{-1}(x) \tag{22}$$

and

$$\overline{\epsilon_0}=\sup_{x\in\mathbb{R}^2}\epsilon_0^{-1}(x),\qquad \underline{\epsilon_0}=\inf_{x\in\mathbb{R}^2}\epsilon_0^{-1}(x). \tag{23}$$

We shall also formally define some auxiliary operators

$$\mathscr{Y}_0(\beta)\vec{u}=-\underline{\epsilon_0}\epsilon_0^{-1}\nabla_\beta\big(\mu_0^{-1}\nabla_\beta\cdot(\epsilon_0\vec{u})\big)$$

and

$$W_0(\beta)\vec{u}=\mathscr{A}_0(\beta)\vec{u}+\mathscr{Y}_0(\beta)\vec{u}$$

for any $\vec{u}\in H_{\epsilon_0}$. $W_0(\beta)$ is rigorously defined by the nonnegative self-adjoint operator on weighted Hilbert space $L^2(\mathbb{R}^2;\epsilon_0(x)\,dx)$ given by the nonnegative quadratic form

$$w_0(\beta;\vec{u},\vec{u})=\int_{\mathbb{R}^2}\mu_0^{-1}|\nabla_\beta\times\vec{u}|^2\,dx+\underline{\epsilon_0}\int_{\mathbb{R}^2}\mu_0^{-1}\big|\nabla_\beta\cdot(\epsilon_0\vec{u})\big|^2\,dx$$

for $\vec{u}\in\{\vec{u}\mid\nabla_\beta\times\vec{u}\in L^2(\mathbb{R}^2)^3,\ \nabla_\beta\cdot(\epsilon_0\vec{u})\in L^2(\mathbb{R}^2)\}$.

By Green' formula, one has

$$\big(W_0(\beta)\vec{u},\vec{u}\big)_{\epsilon_0}=w_0(\beta;\vec{u},\vec{u}).$$

It follows from Lemma 3.1 that $\mathscr{Y}_0(\beta)|_{H_{\epsilon_0}(\beta)}=0$. Furthermore, if we set

$$Y_0(\beta)=\mathscr{Y}_0(\beta)|_{G(\beta)},$$

we have

$$W_0(\beta)=A_0(\beta)\oplus Y_0(\beta) \tag{24}$$

by the decomposition (13). Particularly, if $\epsilon_0(x)\equiv 1$ and $\mu_0(x)\equiv 1$, we have

$$\begin{aligned}\Theta&\equiv(\nabla_\beta\times\nabla_\beta\times)-\nabla_\beta(\nabla_\beta\cdot)\\&=\nabla_\beta(\nabla_\beta\cdot)-(\nabla_\beta\cdot\nabla_\beta)\otimes I_3-\nabla_\beta(\nabla_\beta\cdot)\\&=-\Delta_\beta\otimes I_3,\end{aligned}$$

where $\Delta_\beta=\partial_1^2+\partial_2^2-\beta^2$ is the operator in H_{ϵ_0} and I_3 is the identity operator on $\mathbb{C}^3$.

Now we will give an auxiliary theorem needed in the proof of Theorem 4.2.

Theorem 4.3. *$(A_0(\beta)+I)^{-1}\chi_D$ is a Hilbert–Schmidt operator, where D can be any bounded measurable subset of $\mathbb{R}^2$.*

Since

$$\begin{aligned}0 &\leqslant \big(A_0(\beta)+I|_{V_{\epsilon_o}(\beta)}\big)^{-1}\oplus 0|_{G(\beta)}\\ &\leqslant \big(A_0(\beta)+I|_{V_{\epsilon_o}(\beta)}\big)^{-1}\oplus \big(Y_0(\beta)+I|_{G(\beta)}\big)^{-1}\\ &= \big(W_0(\beta)+I\big)^{-1},\end{aligned}$$

Theorem 4.3 is the immediate consequence of the following theorem:

Theorem 4.4. *The operator $(W_0(\beta)+I)^{-1}\chi_D$ is a Hilbert–Schmidt operator, where $W_0(\beta)$ is defined in* (24) *and D can be any bounded measurable subset of $\mathbb{R}^2$.*

In order to prove this theorem, we need some preparing work in the following.
We shall introduce some needed notations.

$$R := \big(W_0(\beta)+I\big)^{-1}, \qquad T(t) := (t\Theta+I)^{-1} \quad \text{for } t>0.$$

Lemma 4.1.

$$T(\underline{\epsilon_0}\underline{\mu_0}) \leqslant R \leqslant T(\overline{\epsilon_0}\overline{\mu_0})$$

Proof. As we know that if A and B are self-adjoint operators with $0<A<B$, we have $A^{-1}>B^{-1}$. Hence the lemma follows from (22) and (23). □

Theorem 4.5. *For arbitrary $r>1$ and $t>0$, there exists a constant $M_1=M_1(r,\beta,t)<\infty$, such that*

$$\operatorname{tr}\big(\chi_\Omega T^r(t)\chi_\Omega\big) \leqslant M_1 \tag{25}$$

for any bounded measurable subset $\Omega\subset\mathbb{R}^2$.

Proof. Since

$$T^r(t) = (t\Theta+I)^{-r} = t^{-r}\big(-\Delta+\beta^2+t^{-1}\big)^{-r},$$

it is sufficient to prove

$$\operatorname{tr}\big(\chi_\Omega\big(-\Delta+\beta^2+t^{-1}\big)^{-r}\chi_\Omega\big) \leqslant t^r M_1(r,t),$$

where $\Delta=\partial_1^2+\partial_2^2$. Let $G(x,y;\beta^2+t^{-1})$ be the kernel according to the operator $(-\Delta+\beta^2+t^{-1})^{-r}$. Since for any number $k>0$, $(-\Delta+k)^{-r}$ is a positive operator, we have $G(x,y;k)\geqslant 0$. The following formula is due to Simon [21], formally

$$(H+E)^{-\alpha} = c_\alpha\int_0^\infty e^{-tH}e^{-tE}t^{\alpha-1}\,dt$$

for $\alpha>0$, where c_α is a constant expressible as a Γ function. Particularly, let $H=-\Delta$, $E=k$ and $\alpha=r$. Then we have

$$(-\Delta+k)^{-r} = c_r\int_0^\infty e^{t\Delta}e^{-tk}t^{r-1}\,dt.$$

Using the Fourier transform, we obtain

$$G(x,y;k)=c\int_0^\infty e^{-s}e^{-\frac{k|x-y|^2}{4s}}s^{r-2}\,ds \tag{26}$$

for some constant $c>0$, as desired. Here $G(x,y;k)\geqslant 0$ is the kernel of $(-\Delta+k)^{-r}$.

First of all, we shall show that for $0<r<1$,

$$0\leqslant G(x,y;k)\leqslant \tilde{c}\big(\sqrt{k}|x-y|\big)^{-(2-2r)}e^{-\varrho|x-y|}, \tag{27}$$

where $\tilde{c}$ and ϱ are two positive constants. Using a scaling argument, one can use the inequality (26) to obtain

$$\begin{aligned}
G(x,y;k)&=c\int_0^\infty e^{-s}e^{-\frac{k|x-y|^2}{4s}}s^{r-2}\,ds\\
&=c\int_0^\infty e^{-k|x-y|^2t}e^{-\frac{1}{4t}}\big(k|x-y|^2t\big)^{r-2}k|x-y|^2\,dt\\
&=c\big(\sqrt{k}|x-y|\big)^{2r-2}\int_0^\infty e^{-(k|x-y|^2t+\frac{1}{8t})}e^{-\frac{1}{8t}}t^{r-2}\,dt\\
&\leqslant c\big(\sqrt{k}|x-y|\big)^{2r-2}e^{-\sqrt{\frac{k}{2}}|x-y|}\int_0^\infty e^{-\frac{1}{8t}}t^{r-2}\,dt\\
&\leqslant \tilde{c}\big(\sqrt{k}|x-y|\big)^{2r-2}e^{-\varrho|x-y|},
\end{aligned}$$

for some constant $\tilde{c}>0$, where $\varrho=\sqrt{\frac{k}{2}}$. Note that we used the fact $0<r<1$ to obtain the last inequality. In order to prove $\operatorname{tr}(\chi_\Omega(-\Delta+k)^{-r}\chi_\Omega)\leqslant\infty$ for $r>1$, it suffices to prove that $\operatorname{tr}(\chi_\Omega(-\Delta+k)^{-\frac{r}{2}}\chi_\Omega)$ is a Hilbert–Schmidt operator for $\frac{r}{2}>\frac{1}{2}$, or equivalently, to prove

$$\int_\Omega\int_\Omega K^2(x,y;k)\,dx\,dy<\infty,$$

where $K(x,y;k)$ is the kernel of $(-\Delta+k)^{-\frac{r}{2}}$.

We first consider the case $\frac{1}{2}<\frac{r}{2}<1$. We can use the estimate (27) to obtain

$$0\leqslant K(x,y;k)\leqslant c'\big(\sqrt{k}|x-y|\big)^{-(2-2\frac{r}{2})}e^{-\varrho'|x-y|}$$

for suitable c' and $\varrho'>0$. A simple calculation shows that

$$\int_\Omega\int_\Omega\Big(\big(\sqrt{k}|x-y|\big)^{-(2-r)}e^{-\varrho'|x-y|}\Big)^2\,dx\,dy\leqslant\int_\Omega\int_\Omega\big(\sqrt{k}|x-y|\big)^{-2(2-r)}\,dx\,dy<\infty.$$

(Note that $\frac{r}{2}>\frac{1}{2}$, i.e., $r>1$.) Thus $\operatorname{tr}(\chi_\Omega(-\Delta+k)^{-\frac{r}{2}}\chi_\Omega)$ is proved to be a Hilbert–Schmidt operator for $\frac{1}{2}<\frac{r}{2}<1$. As a consequence, $\operatorname{tr}(\chi_\Omega(-\Delta+k)^{-r}\chi_\Omega)\leqslant\infty$ for $1<r<2$.

Moreover, recalling that if A and B are self-adjoint operators and $0\leqslant A\leqslant B$ with $\operatorname{tr}B<\infty$, then $\operatorname{tr}A<\infty$. Note that $(-\Delta+k)^{-p}\leqslant k^{q-p}(-\Delta+k)^{-q}$ for $0<q<p$. Thus we have $\operatorname{tr}(\chi_\Omega(-\Delta+k)^{-r}\chi_\Omega)<\infty$ for $r\geqslant 2$. This completes the proof of Theorem 4.5. □

Remark 4.1. Similar results appear in [1] and [9] for 3D case.

We will also introduce some notations needed in the following. We set

$$\vec{m} = (m_1, m_2) \in \mathbb{Z}^2$$

and $\chi_{\vec{m}}$ as the characteristic function of the set

$$\Omega_{\vec{m}} = \left\{ x \in \mathbb{R}^2 \;\Big|\; -\frac{1}{2} \leqslant x_1 - m_1 < \frac{1}{2},\ -\frac{1}{2} \leqslant x_2 - m_2 < \frac{1}{2} \right\} \quad \text{for } \vec{m} \in \mathbb{Z}^2,$$

and

$$R_{\vec{m}\vec{n}} = \chi_{\vec{m}} R \chi_{\vec{n}}, \quad \chi_{\vec{m}\vec{n}} = \max\{\chi_{\vec{m}}, \chi_{\vec{n}}\}. \tag{28}$$

It is easy to see that $\sum_{m \in \mathbb{Z}^2} \chi_{\vec{m}} \equiv 1$.

Theorem 4.6. *There exists a positive number $M_2 = M_2(\overline{\epsilon_0}, \overline{\mu_0}) < \infty$, such that*

$$\operatorname{tr}|R_{\vec{m}\vec{n}}|^2 = \operatorname{tr} R^*_{\vec{m}\vec{n}} R_{\vec{m}\vec{n}} \leqslant M_2 \quad \textit{for all } \vec{m}, \vec{n} \in \mathbb{Z}^2.$$

Proof. Using Theorem 4.5 and Lemma 4.1, we have

$$\begin{aligned}
\operatorname{tr}|R_{\vec{m}\vec{n}}|^2 &= \operatorname{tr} R^*_{\vec{m}\vec{n}} R_{\vec{m}\vec{n}} \\
&= \operatorname{tr} \chi_{\vec{n}} R \chi_{\vec{m}} \chi_{\vec{m}} R \chi_{\vec{n}} \\
&= \operatorname{tr} \chi_{\vec{n}} R \chi_{\vec{m}} R \chi_{\vec{n}} \\
&\leqslant \operatorname{tr} \chi_{\vec{n}} R \chi_{\vec{m}\vec{n}} R \chi_{\vec{n}} \\
&= \operatorname{tr} \chi_{\vec{n}} \chi_{\vec{m}\vec{n}} R \chi_{\vec{m}\vec{n}} R \chi_{\vec{m}\vec{n}} \chi_{\vec{n}} \\
&\leqslant \operatorname{tr} \chi_{\vec{m}\vec{n}} R \chi_{\vec{m}\vec{n}} R \chi_{\vec{m}\vec{n}} \\
&= \operatorname{tr}(\chi_{\vec{m}\vec{n}} R \chi_{\vec{m}\vec{n}})^2 \\
&\leqslant \operatorname{tr}\big(\chi_{\vec{m}\vec{n}} T(\overline{\epsilon_0}\overline{\mu_0}) \chi_{\vec{m}\vec{n}}\big)^2 \\
&= \operatorname{tr} \chi_{\vec{m}\vec{n}} T(\overline{\epsilon_0}\overline{\mu_0}) \chi_{\vec{m}\vec{n}} T(\overline{\epsilon_0}\overline{\mu_0}) \chi_{\vec{m}\vec{n}} \\
&\leqslant \operatorname{tr} \chi_{\vec{m}\vec{n}} T^2(\overline{\epsilon_0}\overline{\mu_0}) \chi_{\vec{m}\vec{n}} \\
&\leqslant M_1(2, \overline{\epsilon_0}\overline{\mu_0}) \\
&= M_2(\overline{\epsilon_0}, \overline{\mu_0}) < \infty. \qquad \square
\end{aligned}$$

Lemma 4.2. *Suppose $A \in \mathscr{L}(H)$ is a positive operator, where H is a Hilbert space. Then for any number $s \in (0, 1)$,*

$$\operatorname{tr}(A) \leqslant \|A\|^s \operatorname{tr}\big(A^{1-s}\big).$$

Proof. One can prove this inequality easily by using the definition of the trace. More precisely, for any given orthonormal basis $\{u_n\}_{n=1}^{\infty}$,

$$\begin{aligned}
\operatorname{tr}(A) &= \sum_{n=1}^{\infty} (u_n, Au_n) = \sum_{n=1}^{\infty} \|\sqrt{A}u_n\|^2 \leqslant \sum_{n=1}^{\infty} \big\|(\sqrt{A})^s (\sqrt{A})^{1-s} u_n\big\|^2 \\
&\leqslant \|\sqrt{A}\|^{2s} \sum_{n=1}^{\infty} \big\|(\sqrt{A})^{1-s} u_n\big\|^2 = \|A\|^s \operatorname{tr}\big(A^{1-s}\big). \qquad \square
\end{aligned}$$

Lemma 4.3. *Let $A \geqslant 0$ be a bounded operator and P an orthogonal projection on a Hilbert space H. For any $\gamma > 1$, we have*

$$\operatorname{tr}(PAP)^\gamma \leqslant \operatorname{tr} PA^\gamma P. \tag{29}$$

Proof. For the proof, we refer to Lemma 21 in [9]. □

Theorem 4.7. *There exists a positive number $M_3 < \infty$ such that*

$$\operatorname{tr} \chi_{\vec m} R^2 \chi_{\vec m} \leqslant M_3, \quad \forall \vec m \in \mathbb{Z}^2.$$

Proof. Since $1 = \sum_{\vec n \in \mathbb{Z}^2} \chi_{\vec n} = \sum_{\vec n \in \mathbb{Z}^2} \chi_{\vec n}^2$, we have

$$\operatorname{tr} \chi_{\vec m} R^2 \chi_{\vec m} = \sum_{\vec n \in \mathbb{Z}^2} \operatorname{tr} \chi_{\vec m} R \chi_{\vec n} \chi_{\vec n} R \chi_{\vec m} = \sum_{\vec n \in \mathbb{Z}^2} \operatorname{tr} |R_{\vec m \vec n}|^2.$$

For $\alpha \in (0,1)$, it follows from Lemma 4.2 that

$$\operatorname{tr}|R_{\vec m \vec n}|^2 = \operatorname{tr}|R_{\vec m \vec n}|^{\alpha} |R_{\vec m \vec n}|^{2-\alpha} \leqslant \|R_{\vec m \vec n}\|^{\alpha} \operatorname{tr}|R_{\vec m \vec n}|^{2-\alpha}.$$

For $\alpha \in (0,1)$, we can use Corollary 5.3 of Section 5 to obtain

$$\sum_{\vec n \in \mathbb{Z}^2} \|R_{\vec m \vec n}\|^{\alpha} \leqslant \frac{1}{(\epsilon_0^-)^{\alpha}} \sum_{\vec n \in \mathbb{Z}^2} \|\chi_{\vec m} R_0 \chi_n\|_{\epsilon_0}^{\alpha} \leqslant M_3 \quad \text{for all } \vec m \in \mathbb{R}^2,$$

where ϵ_0^- is defined in (18). Hence we have

$$\begin{aligned}
\operatorname{tr} \chi_{\vec m} R^2 \chi_{\vec m} &\leqslant \sum_{\vec n \in \mathbb{Z}^2} \|R_{\vec m \vec n}\|^{\alpha} \operatorname{tr}|R_{\vec m \vec n}|^{2-\alpha} \\
&\leqslant \sup_{\vec m \in \mathbb{Z}^2} \bigl(\operatorname{tr}|R_{\vec m \vec n}|^{2-\alpha}\bigr) \sum_{\vec n \in \mathbb{Z}^2} \|R_{\vec m \vec n}\|^{\alpha} \\
&\leqslant M_3 \sup_{\vec m \in \mathbb{Z}^2} \bigl(\operatorname{tr}|R_{\vec m \vec n}|^{2-\alpha}\bigr).
\end{aligned}$$

Next, we need to prove $\sup_{\vec m \in \mathbb{Z}^2}(\operatorname{tr}|R_{\vec m \vec n}|^{2-\alpha}) < \infty$. Note $\lambda_j(A)$, $j = 1, 2, \ldots$ (counting multiplicity) the singular values of $A \in \operatorname{Com}(H)$, then we can easily verify that $\lambda_j(A) = \lambda_j(|A|) = \lambda_j(A^*)$, $\lambda_j(BA) \leqslant \|B\| \lambda_j(A)$ for all $B \in \mathscr{L}(H)$ (for the proof, see, e.g., [9,11]). Using these properties we have

$$\begin{aligned}
\lambda_j^2(R_{\vec m \vec n}) &= \lambda_j\bigl(|R_{\vec m \vec n}|^2\bigr) \\
&= \lambda_j\bigl(\chi_{\vec n} R \chi_{\vec m} R \chi_{\vec n}\bigr) \\
&\leqslant \lambda_j(\chi_{\vec n} R \chi_{\vec m \vec n} R \chi_{\vec n}) \\
&\leqslant \lambda_j(\chi_{\vec m \vec n} R \chi_{\vec m \vec n} R \chi_{\vec m \vec n}) \\
&= \lambda_j\bigl((\chi_{\vec m \vec n} R \chi_{\vec m \vec n})^2\bigr) \\
&= \bigl(\lambda_j(\chi_{\vec m \vec n} R \chi_{\vec m \vec n})\bigr)^2 \\
&\leqslant \bigl(\lambda_j\bigl(\chi_{\vec m \vec n} T(\overline{\epsilon_0} \overline{\mu_0}) \chi_{\vec m \vec n}\bigr)\bigr)^2.
\end{aligned}$$

Hence we have

$$\lambda_j(R_{\vec m \vec n}) \leqslant \lambda_j\bigl(\chi_{\vec m \vec n} T(\overline{\epsilon_0} \overline{\mu_0}) \chi_{\vec m \vec n}\bigr).$$

Note that $0 < \alpha < 1$, so $2 - \alpha > 1$. Applying (25) and (29), we have

$$\begin{aligned}
\operatorname{tr}|R_{\vec m \vec n}|^{2-\alpha} &= \sum_{j=1}^{\infty} \bigl(\lambda_j(R_{\vec m \vec n})\bigr)^{2-\alpha} \\
&\leqslant \sum_{j=1}^{\infty} \bigl(\lambda_j\bigl(\chi_{\vec m \vec n} T(\overline{\epsilon_0} \overline{\mu_0}) \chi_{\vec m \vec n}\bigr)\bigr)^{2-\alpha} \\
&= \operatorname{tr}\bigl(\chi_{\vec m \vec n} T(\overline{\epsilon_0} \overline{\mu_0}) \chi_{\vec m \vec n}\bigr)^{2-\alpha}
\end{aligned}$$

$$\leqslant \operatorname{tr}\bigl(\chi_{\vec{m}\vec{n}} T^{2-\alpha}(\overline{\epsilon_0}\,\overline{\mu_0})\chi_{\vec{m}\vec{n}}\bigr)$$
$$\leqslant M_1(2-\alpha, \beta, t).$$

Thus the theorem is proved. □

It should be noticed that the proof of this theorem is just a similar version of Lemma 23 in [9].

Now we can complete the proof of Theorem 4.4.

Proof of Theorem 4.4. Since η has a compact support, we can conclude that there exists an index set J with $|J| < \infty$ (where $|J|$ is the cardinality of the set J), such that

$$\overline{\operatorname{supp} D} \subset \Omega_J,$$

where $\Omega_J = \bigcup_{\vec{m}\in J} \Omega_{\vec{m}}$. Hence we have $R\chi_D \leqslant R\chi_{\Omega_J}$.

On the other hand, by applying Theorem 4.7, we have

$$\begin{aligned}\operatorname{tr}|R\chi_{\Omega_J}|^2 &\leqslant \sum_{\vec{m}\in J} \operatorname{tr}|R\chi_{\vec{m}}|^2 \\ &= \sum_{\vec{m}\in J} \operatorname{tr} \chi_{\vec{m}} R^2 \chi_{\vec{m}} \\ &\leqslant |J| \sup_{\vec{m}\in J} \operatorname{tr} \chi_{\vec{m}} R^2 \chi_{\vec{m}} \\ &< \infty.\end{aligned}$$

Thus we know $R\chi_D$ is a Hilbert–Schmidt operator. □

5. A Combes–Thomas estimate

We first introduce some notations.

Let $\chi_{x,h}$ be the characteristic function of a square of side $2h$ centered at x, i.e.,

$$\chi_{x,h} = \chi_{\Omega_{x,h}}$$

with

$$\Omega_{x,h} = \bigl\{ y \in \mathbb{R}^2 \bigm| |y_1 - x_1| \leqslant h,\ |y_2 - x_2| \leqslant h \bigr\},$$

and

$$R(z) = \bigl(A_0(\beta) - zI\bigr)^{-1}.$$

We also denote $\langle\cdot,\cdot\rangle$ as the inner product of Hilbert space H with the norm $\|\cdot\|$.

Classical wave operators, e.g., acoustic operators and Maxwell operators, can be regarded as generalized Schrödinger operators. Usually they satisfy a resolvent decay estimate which is called Combes–Thomas estimate in mathematical physics. See, e.g., [5,8–10,16,21].

Theorem 5.1. *Let $z \in \rho(A_0(\beta))$, $n \in \mathbb{N}$, $h > 0$ and $0 < \nu < 1$, then we have*

$$\bigl\| \chi_{x,h} R^n(z) \chi_{y,h} \bigr\|_{\epsilon_0} \leqslant \left(\left(\frac{1+\nu}{1-\nu} \right)^2 \frac{1}{d} \right)^n e^{2\sqrt{2}h\nu\theta_0} e^{-\nu\theta_0|x-y|} \quad \textit{for all } x, y \in \mathbb{R}^2,$$

with

$$\theta_0 = \frac{d}{4}\sqrt{\frac{\mu_0^-}{d+|z|}},$$

where

$$d \equiv \operatorname{dist}\bigl(z, \sigma\bigl(A_0(\beta)\bigr)\bigr) = \inf_{\vec{u} \in D(A_0(\beta)),\, \|\vec{u}\|_{\epsilon_0} = 1} \bigl\| \bigl(A_0(\beta) - zI\bigr)\vec{u} \bigr\|_{\epsilon_0}$$

and μ_0^- *is defined in* (18). *The norm in the left-hand side is the operator norm in* H_{ϵ_0}, *where* H_{ϵ_0} *is analogous to* H_ϵ *defined in* (9).

Proof. We formally define the operators parameterized by α,

$$A_\alpha(\beta) = e^{-\alpha\cdot\tilde{x}} A_0(\beta) e^{\alpha\cdot\tilde{x}}, \quad \alpha = (\alpha', 0),\ \alpha' \in \mathbb{R}^2 \text{ and } \tilde{x} = (x, 0),\ x \in \mathbb{R}^2,$$

as the closed densely operators on $\{\vec{u} \in C_0^1(\mathbb{R}^2; \mathbb{C}^3) \mid \nabla_\beta \cdot \vec{u} = 0\}$ uniquely defined by the corresponding quadratic form

$$\begin{aligned} a_\alpha(\vec{u}, \vec{u}) &= \int_{\mathbb{R}^2} \mu_0^{-1} (\nabla_\beta + \alpha) \times \vec{u} \cdot \overline{(\nabla_\beta - \alpha) \times \vec{u}}\, dx \\ &= \bigl\langle \mu_0^{-1} (\nabla_\beta + \alpha) \times \vec{u}, (\nabla_\beta - \alpha) \times \vec{u} \bigr\rangle. \end{aligned}$$

We denote $a_0[\vec{u}]$ and $a_\alpha[\vec{u}]$ as the abbreviation of $a_{\epsilon_0}(\vec{u}, \vec{u})$ and $a_\alpha(\vec{u}, \vec{u})$, respectively (where $a_{\epsilon_0}(\vec{u}, \vec{u})$ is defined in the same way as $a_\epsilon(\vec{u}, \vec{u})$ defined in (15)). Notice that

$$\begin{aligned} a_\alpha[\vec{u}] - a_0[\vec{u}] &= \bigl\langle \mu_0^{-1} (\nabla_\beta + \alpha) \times \vec{u}, (\nabla_\beta - \alpha) \times \vec{u} \bigr\rangle - \bigl\langle \mu_0^{-1} \nabla_\beta \times \vec{u}, \nabla_\beta \times \vec{u} \bigr\rangle \\ &= -\bigl\langle \mu_0^{-1} \nabla_\beta \times \vec{u}, \alpha \times \vec{u} \bigr\rangle + \overline{\bigl\langle \mu_0^{-1} \nabla_\beta \times \vec{u}, \alpha \times \vec{u} \bigr\rangle} - \bigl\langle \mu_0^{-1} \alpha \times \vec{u}, \alpha \times \vec{u} \bigr\rangle \\ &= -2i \operatorname{Im}\bigl(\bigl\langle \mu_0^{-1} \nabla_\beta \times \vec{u}, \alpha \times \vec{u} \bigr\rangle\bigr) - \bigl\langle \mu_0^{-1} \alpha \times \vec{u}, \alpha \times \vec{u} \bigr\rangle, \end{aligned}$$

then we have

$$\bigl|a_\alpha[\vec{u}] - a_0[\vec{u}]\bigr| = \bigl(4\bigl(\operatorname{Im}\bigl(\bigl\langle \mu_0^{-1} \nabla_\beta \times \vec{u}, \alpha \times \vec{u} \bigr\rangle\bigr)\bigr)^2 + \bigl(\bigl\langle \mu_0^{-1} \alpha \times \vec{u}, \alpha \times \vec{u} \bigr\rangle\bigr)^2\bigr)^{\frac{1}{2}}.$$

Using the inequality

$$ab \leqslant \frac{1}{2\gamma} a^2 + \frac{\gamma}{2} b^2 \quad \text{for all } \gamma > 0,$$

we have

$$\begin{aligned} \bigl|a_\alpha[\vec{u}] - a_0[\vec{u}]\bigr| &\leqslant \bigl(4|\alpha|^2 \|\vec{u}\|^2 \bigl(\mu_0^-\bigr)^{-1} a_0[\vec{u}] + \bigl(\mu_0^-\bigr)^{-2} |\alpha|^4 \|\vec{u}\|^4\bigr)^{\frac{1}{2}} \\ &= |\alpha| \|\vec{u}\| \bigl(4\bigl(\mu_0^-\bigr)^{-1} a_0[\vec{u}] + \bigl(\mu_0^-\bigr)^{-2} |\alpha|^2 \|\vec{u}\|^2\bigr)^{\frac{1}{2}} \\ &\leqslant \frac{1}{2} |\alpha| \left(\left(\frac{1}{\gamma} \|\vec{u}\|^2 \right) + \gamma \bigl(4\bigl(\mu_0^-\bigr)^{-1} a_0[\vec{u}] + \bigl(\mu_0^-\bigr)^{-2} |\alpha|^2 \|\vec{u}\|^2\bigr) \right) \\ &= 2|\alpha| \gamma \bigl(\mu_0^-\bigr)^{-1} a_0[\vec{u}] + \frac{1}{2} |\alpha| \left(\frac{1}{\gamma} + \gamma \bigl(\mu_0^-\bigr)^{-2} |\alpha|^2 \right) \|\vec{u}\|^2. \end{aligned}$$

Since we can choose γ sufficiently small such that $2|\alpha|\gamma(\mu_0^-)^{-1} < 1$ for any fixed α, it follows Theorem VI 3.9 in [15] that $a_\alpha[\cdot]$ is sectorial and closed for $|\alpha| > 0$. Then by the first representation theorem (Theorem VI 2.1 in [15]) we can define $A_\alpha(\beta)$ as the unique m-sectorial operator corresponding to $a_\alpha[\cdot]$.

Assume $z \in \rho(A_0(\beta))$, if there exists $0 < \nu < 1$ such that

$$2\bigl\| \bigl(e + f A_0(\beta)\bigr) R(z) \bigr\|_{\epsilon_0} \leqslant \nu, \tag{30}$$

where

$$e = \frac{1}{2} |\alpha| \left(\frac{1}{\gamma} + \bigl(\mu_0^-\bigr)^{-2} |\alpha|^2 \gamma \right), \qquad f = 2|\alpha| \bigl(\mu_0^-\bigr)^{-1} \gamma.$$

We can apply Theorem VI 3.9 in [15], further take into account the fact $d = \operatorname{dist}(z, \sigma(A_0(\beta))) \leqslant \frac{1}{\|R(z)\|_{\epsilon_0}}$ to conclude that $z \in \rho(A_\alpha(\beta))$ and

$$\begin{aligned}\left\|R_\alpha(z)-R(z)\right\|_{\epsilon_0} &\leqslant \frac{8\|(e+fA_0(\beta))R(z)\|_{\epsilon_0}}{(1-2\|(e+fA_0(\beta))R(z)\|_{\epsilon_0})^2}\left\|R(z)\right\|_{\epsilon_0}\\ &\leqslant \frac{4\nu}{(1-\nu)^2}\frac{1}{d},\end{aligned}$$

where $R_\alpha(z) := (A_\alpha(\beta)-zI)^{-1}$. Hence we have

$$\left\|R_\alpha(z)\right\|_{\epsilon_0} \leqslant \left(1+\frac{4\nu}{(1-\nu)^2}\right)\frac{1}{d} = \left(\frac{1+\nu}{1-\nu}\right)^2\frac{1}{d}. \tag{31}$$

On the other hand,

$$\begin{aligned}2\left\|\left(e+fA_0(\beta)\right)R(z)\right\|_{\epsilon_0} &\leqslant \left(2e+2f\left(d+|z|\right)\right)\frac{1}{d}\\ &= |\alpha|\left(\frac{1}{\gamma}+\left(|\alpha|\left(\mu_0^-\right)^{-2}+4\left(d+|z|\right)\left(\mu_0^-\right)^{-1}\right)\gamma\right)\frac{1}{d}.\end{aligned}$$

Define

$$\Phi(\gamma) = \frac{1}{\gamma}+\left(|\alpha|^2\left(\mu_0^-\right)^{-2}+4\left(d+|z|\right)\left(\mu_0^-\right)^{-1}\right)\gamma.$$

One can easily find that

$$\gamma_0 = \left(|\alpha|^2\left(\mu_0^-\right)^{-2}+4\left(d+|z|\right)\left(\mu_0^-\right)^{-1}\right)^{-\frac{1}{2}} \tag{32}$$

minimizes the function $\Phi(\gamma)$ for $\gamma > 0$, so we have

$$\Phi(\gamma_0) = 2\gamma_0^{-1} = \min_{\gamma>0}\Phi(\gamma).$$

Hence

$$2\gamma_0^{-1}\frac{|\alpha|}{d} \leqslant \nu \tag{33}$$

ensures the inequality (30). Furthermore, plugging (32) into (33) and solving the inequality, we can conclude it suffices to require

$$\begin{aligned}|\alpha|^2 &\leqslant \frac{1}{2}\mu_0^-\sqrt{16\left(d+|z|\right)^2+\nu^2d^2}-2\left(d+|z|\right)\mu_0^-\\ &= 2\left(d+|z|\right)\mu_0^-\left(\sqrt{1+\frac{\nu^2d^2}{16(d+|z|^2)}}-1\right).\end{aligned}$$

We can also give a simple condition on $|\alpha|$ by applying Taylor expansion. Since $\sqrt{1+x} \leqslant 1+\frac{x}{2}$ for $x > 0$, we can conclude that if

$$\begin{aligned}|\alpha|^2 &\leqslant 2\left(d+|z|\right)\mu_0^-\left(-1+1+\frac{1}{2}\frac{\nu^2d^2}{16(d+|z|)^2}\right)\\ &= \frac{\mu_0^-\nu^2d^2}{16(d+|z|)},\end{aligned}$$

i.e., $|\alpha| \leqslant \frac{\nu d}{4}\sqrt{\frac{\mu_0^-}{d+|z|}}$, (30) holds.

We set $\theta_0 = \frac{d}{4}\sqrt{\frac{\mu_0^-}{d+|z|}}$. In the following we assume that $|\alpha| \leqslant \nu\theta_0$, then (31) holds. For any $x_0, y_0 \in \mathbb{R}^2$, $n \in \mathbb{N}$ and $h > 0$, let $\alpha = (\frac{\nu\theta_0}{|x_0-y_0|}(x_0-y_0), 0)^\top$, we have

$$\begin{aligned}\|\chi_{x_0,h}R^n(z)\chi_{y_0,h}\|_{\epsilon_0} &= \|\chi_{x_0,h}e^{-\alpha\cdot\tilde{x}}R_\alpha^n(z)e^{\alpha\cdot\tilde{x}}\chi_{y_0,h}\|_{\epsilon_0}\\ &= \|e^{-\alpha'\cdot(x_0-y_0)}\chi_{x_0,h}e^{-\alpha'\cdot(x-x_0)}R_\alpha^n(z)e^{\alpha'\cdot(x-y_0)}\chi_{y_0,h}\|_{\epsilon_0}\\ &\leqslant e^{-\nu\theta_0|x_0-y_0|}\|\chi_{x_0,h}e^{-\alpha'\cdot(x-x_0)}\|_\infty\|R_\alpha(z)\|_{\epsilon_0}^n\|\chi_{y_0,h}e^{\alpha'\cdot(x-y_0)}\|_\infty.\end{aligned}$$

Since

$$\|\chi_{x_0,h}e^{\pm\alpha'\cdot(x-x_0)}\|_\infty \leqslant e^{\sqrt{2}|\alpha|h}$$

(notice that $|\alpha'| = |\alpha|$), we have

$$\begin{aligned}\|\chi_{x_0,h}R^n(z)\chi_{y_0,h}\|_{\epsilon_0} &\leqslant e^{-\nu\theta_0|x_0-y_0|}e^{\sqrt{2}|\alpha|h}\left(\left(\frac{1+\nu}{1-\nu}\right)^2\frac{1}{d}\right)^n e^{\sqrt{2}|\alpha|h}\\ &= \left(\left(\frac{1+\nu}{1-\nu}\right)^2\frac{1}{d}\right)^n e^{2\sqrt{2}h|\alpha|}e^{-\nu\theta_0|x_0-y_0|}\\ &\leqslant \left(\left(\frac{1+\nu}{1-\nu}\right)^2\frac{1}{d}\right)^n e^{2\sqrt{2}h\nu\theta_0}e^{-\nu\theta_0|x_0-y_0|}.\end{aligned}$$

Thus the theorem is proved. □

Corollary 5.1. *For any number $s > 0$ and any $z \in \rho(A_0(\beta))$, there holds*

$$\sum_{\vec{n}\in\mathbb{Z}^2}\|\chi_{\vec{m}}R(z)\chi_{\vec{n}}\|_{\epsilon_0}^s \leqslant C_0 < \infty \quad \textit{for all } \vec{m}\in\mathbb{R}^2,$$

where $\chi_{\vec{m}}$ and $\chi_{\vec{n}}$ are defined in (28) *and $R(z) = (A_0(\beta) - zI)^{-1}$.*

Proof. For $\vec{m} \in \mathbb{R}^2$ fixed, since $\nu\theta_0$ is a positive constant, we have

$$\begin{aligned}\sum_{\vec{n}\in\mathbb{Z}^2}\|\chi_{\vec{m}}R(z)\chi_{\vec{n}}\|_{\epsilon_0}^s &\leqslant \sum_{\vec{n}\in\mathbb{Z}^2}\left(\left(\frac{1+\nu}{1-\nu}\right)^2\frac{1}{d}\right)^s e^{2s\sqrt{2}h\nu\theta_0}e^{-s\nu\theta_0|\vec{m}-\vec{n}|}\\ &\leqslant K_0\sum_{\vec{n}\in\mathbb{Z}^2}e^{-s\nu\theta_0\sqrt{(n_1-m_1)^2+(n_2-m_2)^2}}\\ &\leqslant K_0\sum_{\vec{n}\in\mathbb{Z}^2}e^{-s\nu\theta_0\frac{|n_1-m_1|+|n_2-m_2|}{\sqrt{2}}}\\ &\leqslant C_0 < \infty,\end{aligned}$$

where $K_0 = ((\frac{1+\nu}{1-\nu})^2\frac{1}{d})^s e^{2s\sqrt{2}h\nu\theta_0} < \infty$. □

We can also give Combes–Thomas estimates on the resolvent of operators $Y_0(\beta)$ and $W_0(\beta)$ in the following. We can prove them by mimicking the proof of Theorem 5.1 and Corollary 5.1.

Let $\tilde{R}(z) = (Y_0(\beta) - zI)^{-1}$ for $z \in \rho(Y_0(\beta))$. Then we have

Theorem 5.2. *For any $z \in \rho(Y_0(\beta))$, $n \in \mathbb{N}$, $h > 0$ and $0 < \nu < 1$, there holds*

$$\|\chi_{x,h}\tilde{R}^n(z)\chi_{y,h}\|_{\epsilon_0} \leqslant \left(\left(\frac{1+\nu}{1-\nu}\right)^2\frac{1}{d'}\right)^n e^{2\sqrt{2}h\nu\theta_1}e^{-\nu\theta_1|x-y|} \quad \textit{for all } x, y\in\mathbb{R}^2,$$

with

$$\theta_1 = \frac{d'}{4}\sqrt{\frac{\mu_0^-}{d'+|z|}},$$

where $d' = \operatorname{dist}(z, \sigma(Y_0(\beta)))$.

Corollary 5.2. *For any number $s > 0$ and any $z \in \rho(Y_0(\beta))$, there holds*

$$\sum_{\vec{n}\in\mathbb{Z}^2} \left\| \chi_{\vec{m}} \tilde{R}(z) \chi_{\vec{n}} \right\|_{\epsilon_0}^{s} \leqslant C_0' < \infty \quad \textit{for all } \vec{m} \in \mathbb{R}^2.$$

Because $W_0(\beta) = A_0(\beta) \oplus Y_0(\beta)$, we have

$$R_0 = \left(A_0(\beta) - zI|_{V_{\epsilon_0}(\beta)}\right)^{-1} \oplus \left(Y_0(\beta) - zI|_{G(\beta)}\right)^{-1} = R(z) \oplus \tilde{R}(z),$$

where $R_0 = (W_0(\beta) - zI)^{-1}$. Then Theorem 5.3 below follows from Theorems 5.1 and 5.2.

Theorem 5.3. *For any $z \in \rho(W_0(\beta))$, $n \in \mathbb{N}$, $h > 0$, and $0 < \nu < 1$, there holds*

$$\left\| \chi_{x,h} R_0^n \chi_{y,h} \right\|_{\epsilon_0} \leqslant \left(\left(\frac{1+\nu}{1-\nu} \right)^2 \frac{1}{d''} \right)^n e^{2\sqrt{2}h\nu\theta_2} e^{-\nu\theta_2|x-y|} \quad \textit{for all } x, y \in \mathbb{R}^2,$$

where $\theta_2 = \frac{d''}{4}\sqrt{\frac{\mu_0^-}{d''+|z|}}$ with $d'' = \operatorname{dist}(z, \sigma(W_0(\beta)))$.

Remark 5.1. It is worth noting that the resolvent decay exponentially fast, depending on d (the distance from z to the edge of $\sigma(W_0(\beta))$).

Furthermore, we also have

Corollary 5.3. *For any number $s > 0$ and any $z \in \rho(W_0(\beta))$, there holds*

$$\sum_{\vec{n}\in\mathbb{Z}^2} \| \chi_{\vec{m}} R_0 \chi_n \|_{\epsilon_0}^{s} \leqslant C_0'' < \infty \quad \textit{for all } \vec{m} \in \mathbb{R}^2.$$

Remark 5.2.

(i) It is worth noting that $\epsilon(x)$ and $\mu(x)$ may be any bounded measurable functions, thus the periodicity conditions of $\epsilon(x)$ and $\mu(x)$ are unnecessary in the proof of Theorem 4.1. Our result is general, and the theorem of "stability of the essential spectrum" presented in [4,14,22] can be regarded as a special case of ours. In [4,14], they considered an infinite dielectric cylinder with an air cladding. In that case both $\epsilon(x) - 1$ and $\mu(x) - 1$ have compact supports, noted as Ω. So there is a sufficiently large disk B_R (where R means it's radius), such that $\overline{\Omega} \subset B_R$. Since outside the disk, the medium is homogeneous, they proposed a constructive method to prove that the essential spectrum is stable. However, since the medium can be inhomogeneous in the whole space that we considered here, their method is failed here.

(ii) Generally speaking, Theorem 4.1 can be proved by verifying $A_\epsilon(\beta) - A_{\epsilon_0}(\beta)$ is relatively compact with respect to $A_{\epsilon_0}(\beta)$, but unfortunately, this is not right.

(iii) The existence of eigenvalues in the band gap of photonic crystal fibers created by defects, exponentially decaying property of the corresponding eigenfunctions and other interesting issues (e.g., embedding of eigenvalues in the essential spectrum) have been studied in [19].

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References

[1] S. Alama, P. Deift, R. Hempel, Eigenvalue branches of Schrödinger operator $H - \lambda W$ in a gap of $\sigma(H)$, Comm. Math. Phys. 121 (1989) 291–321.

[2] H. Benisty, Modal analysis of optical guides with two-dimensional photonic band-gap boundaries, J. Appl. Phys. 79 (1996) 7483–7492.

[3] T.A. Birks, P.J. Roberts, P.St.J. Russel, D.M. Atkin, T.J. Shepherd, Full 2-D photonic crystal bandgaps in silica/air structure, Electron. Lett. 31 (1995) 1941–1943.
[4] A.S. Bonnet-Ben Dhia, P. Joly, Mathematical analysis and numerical approximation of optical waveguides, in: G. Bao, L. Cowsar, W. Masters (Eds.), Mathematical Modeling in Optical Science, SIAM, Philadelphia, 2001 (Chapter 8).
[5] J.M. Combes, L. Thomas, Asymptotic behavior of eigenfunctions for multi-particle Schrödinger operators, Comm. Math. Phys. 34 (1973) 251–270.
[6] R.F. Cregan, B.J. Mangan, J.C. Knight, T.A. Birks, P.S.J. Russell, P.J. Roberts, D.C. Allan, Single-mode photonic band gap guidance of light in air, Science 285 (1999) 1537–1539.
[7] A. Ferrando, E. Silvestre, J.J. Miret, P. Andrs, M.V. Andrs, Full-vector analysis of realistic photonic crystal fiber, Opt. Lett. 24 (1999) 276–278.
[8] A. Figotin, A. Klein, Localization of classical waves I: Acoustic waves, Comm. Math. Phys. 180 (1996) 439–482.
[9] A. Figotin, A. Klein, Localization of classical waves II: Electromagnetic waves, Comm. Math. Phys. 184 (1997) 411–441.
[10] F. Germinet, A. Klein, Operator kernel estimates for functions of generalized Schrödinger operators, Proc. Amer. Math. Soc. 131 (2003) 911–920.
[11] I.C. Gohberg, M.G. Krein, Introduction to the Theory of Nonselfadjoint Operators, Providence, 1969.
[12] S. John, Strong localization of photons in certain disordered dielectric superlattices, Phys. Rev. Lett. 58 (1987) 2486–2489.
[13] S.G. John, J.G. Joannopoulos, Photonic Crystals: The Road from Theory to Practice, Kluwer Academic Publishers, 2002.
[14] P. Joly, C. Poirier, Electromagnetic open waveguides: Mathematical analysis, Research report, 1994.
[15] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, 1984.
[16] A. Klein, A. Koines, A general framework for localization of classical waves: I. Inhomogeneous media and defect eigenmode, Math. Phys. Anal. Geom. 4 (2001) 97–130.
[17] J.C. Knight, Photonic crystal fibres, Nature 424 (2003) 847–851.
[18] J.C. Knight, T.A. Birks, P.S.J. Russell, D.M. Atkin, All-silica single mode optical fiber with photonic crystal cladding, Opt. Lett. 21 (1996) 1547–1549.
[19] D. Miao, F. Ma, On guided waves created by line defects, J. Statist. Phys., submitted for publication.
[20] M. Reed, B. Simon, Methods of Modern Mathematical Physics, vol. IV, Analysis of Operators, Academic Press, New York, 1978.
[21] B. Simon, Schrödinger semi-groups, Bull. Amer. Math. Soc. 7 (1982) 447–526.
[22] S. Soussi, Modeling photonic crystal fibers, Adv. Appl. Math. 36 (2006) 288–317.
[23] E. Yablonovitch, Inhibited spontaneous emission in solid-state physics and electronics, Phys. Rev. Lett. 58 (1987) 2059–2062.