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Boxicity of Halin graphs[☆]

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ABSTRACT

A k-dimensional box is the Cartesian product $R_1 \times R_2 \times \cdots \times R_k$ where each R_i is a closed interval on the real line. The boxicity of a graph G, denoted as $\text{box}(G)$ is the minimum integer k such that G is the intersection graph of a collection of k-dimensional boxes. Halin graphs are the graphs formed by taking a tree with no degree 2 vertex and then connecting its leaves to form a cycle in such a way that the graph has a planar embedding. We prove that if G is a Halin graph that is not isomorphic to K_4, then $\text{box}(G) = 2$. In fact, we prove the stronger result that if G is a planar graph formed by connecting the leaves of any tree in a simple cycle, then $\text{box}(G) = 2$ unless G is isomorphic to K_4 (in which case its boxicity is 1).

1. Introduction

Let $\mathcal{F} = \{S_i \mid i \in V\}$ be a collection of subsets of a universe U where V is an index set. The graph $\Lambda(\mathcal{F}) = (V, E)$ where $E = \{\{i, j\} \mid S_i \cap S_j \neq \emptyset\}$ is called the *intersection graph* of $\mathcal{F}$. It is a standard convention to denote an undirected edge $\{i, j\}$ in E as (i, j). When $\mathcal{F}$ is a collection of intervals on the real line, $\Lambda(\mathcal{F})$ is called an *interval graph*.

A *k-dimensional box* or *k-box* in short is the Cartesian product $R_1 \times R_2 \times \cdots \times R_k$ where each R_i is a closed interval on the real line. Two k-boxes, $(P_1, \ldots, P_k)$ and $(Q_1, \ldots, Q_k)$ are said to have a non-empty intersection if $P_i \cap Q_i \neq \emptyset$, for $1 \leq i \leq k$. The *boxicity* of a graph G, denoted as $\text{box}(G)$, is defined to be the minimum integer k such that G is the intersection graph of a collection of k-boxes. Since 1-boxes are nothing but closed intervals on the real line, interval graphs are the graphs with boxicity at most 1. We take the boxicity of a complete graph to be 1.

For a graph $G = (V, E)$, we write $G = T \cup C$ if $E = E(T) \cup E(C)$ where T is a tree on the vertex set V and C is a simple cycle on the leaves of T. Such a graph G is called a Halin graph if G has a planar embedding and T has no vertices of degree 2. The notion of Halin graphs were first used by Halin [9] in his study of minimally 3-connected graphs. Bondy and Lovasz [3] proved that these graphs are almost pancyclic – they contain a cycle of each length between 3 and n with the possible exception of one length, which must be even. Bondy [2] has also shown that Halin graphs are 1-Hamiltonian – i.e, they are Hamiltonian and if any one vertex or edge from the graph is removed, the resulting graph is also Hamiltonian. Lovasz and Plummer [11] show that every Halin graph with an even number of vertices is minimal bicritical (a graph is bicritical if the removal of any two vertices from the graph will result in a graph with a perfect matching). Halin graphs are also interesting because some problems that are NP-complete for general graphs have been shown to be polynomial-time solvable for Halin graphs. Examples are the travelling salesman problem [7] and the problem of finding a dominating cycle with at most l vertices [15].

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It has been shown in [17] that every Halin graph is a 2-interval graph – i.e., the intersection graph of sets, each of which is the union of at most 2 intervals. We show in this paper that the boxicity of a Halin graph (not isomorphic to K_4) is equal to 2 which means that every Halin graph is the intersection graph of axis-parallel rectangles on the plane as well. In fact, we show a stronger result – we show that our result holds for any graph $G = T \cup C$ that has a planar embedding, even if there are vertices of degree 2 in T. Since $\mathrm{box}(G) = 1$ when G is isomorphic to K_4, we show our result for graphs not isomorphic to K_4.

The concept of boxicity, introduced by Roberts [13], finds applications in fields such as ecology and operations research. Computing the boxicity of a graph was shown to be NP-hard by Cozzens [8]. This was improved by Yannakakis [18] and later by Kratochvil [10] who showed that deciding whether the boxicity of a graph is at most 2 itself is NP-complete. An upper bound on the boxicity of general graphs is given in [5] where it is shown that $\mathrm{box}(G) \le 2\Delta^2$ where G is any graph and Δ is the maximum degree of a vertex in G. Also, for any graph G on n vertices and maximum degree Δ, $\mathrm{box}(G) \le \lceil(\Delta+2)\ln n\rceil$ [4]. It was shown in [6] that $\mathrm{box}(G) \le \mathrm{tw}(G) + 2$ where $\mathrm{tw}(G)$ is the treewidth of G. Upper bounds on the boxicity of some special classes of graphs such as chordal graphs, circular-arc graphs, AT-free graphs, permutation graphs and co-comparability graphs are also given in [6]. The boxicity of planar graphs was shown to be at most 3 by Thomassen [16]. A better bound holds for outerplanar graphs, a subclass of planar graphs. Scheinerman [14] showed that the boxicity of outerplanar graphs is at most 2. But this bound does not hold for the class of series-parallel graphs, a slightly bigger subclass of planar graphs than the outerplanar graphs. Bohra et al. [1], showed that there exist series-parallel graphs with boxicity 3. In this paper, we consider another subclass of planar graphs, namely the class of Halin graphs. We show that the boxicity of Halin graphs is at most 2. Quest and Wegner [12] have characterized the graphs with boxicity at most 2 using the adjacency matrix and the "induced C–V matrices" of a graph. But as far as we can see, there is no straightforward way to use this characterization on Halin graphs to obtain our result.

2. Definitions and notations

As mentioned before, the notation $G = T \cup C$ is used to denote a graph that is formed by connecting the leaves of a tree T so that the subgraph induced by the leaves of T in G is the simple cycle C. Such a graph G is called a Halin graph if G is planar and T has no vertex of degree 2. For a graph G, $V(G)$ and $E(G)$ denote the vertex set and edge set of G respectively. For a vertex $u \in V(G)$, $N_G(u) = \{v \in V(G) \mid (u, v) \in E(G)\}$. This is often abbreviated to just $N(u)$ when the graph under consideration is clear. Given $H \subseteq V(G)$, we denote by G_H the subgraph induced by the vertices of H in G.

A graph G_1 is said to be a "supergraph" of a graph G_2 if $V(G_1) = V(G_2)$ and $E(G_1) \supseteq E(G_2)$. Also, given two graphs G_1 and G_2 on the same vertex set V we define $G_1 \cap G_2$ to be the graph with vertex set V and edge set $E(G_1) \cap E(G_2)$. As shown in [13], for any graph G, $\mathrm{box}(G) \le k$ if and only if there exists k interval graphs $I_1, \ldots, I_k$ such that $G = I_1 \cap \cdots \cap I_k$. Note that for this, each I_i has to be a supergraph of G.

3. Our result

We have our main result as the following theorem.

Theorem 1. *If $G = T \cup C$, where T is a tree and C is a simple cycle of the leaves of T such that G is planar, then* $\mathrm{box}(G) = 2$ *if G is not isomorphic to K_4.*

Corollary 1. *Every Halin graph has boxicity equal to* 2 *unless it is isomorphic to K_4, in which case it has boxicity equal to* 1.

4. The proof

Let $G = T \cup C$ where C is a simple cycle connecting the leaves of a tree T such that G is planar. Our strategy will be to construct two interval graphs G_1 and G_2 such that $G = G_1 \cap G_2$ thus proving that boxicity of G is at most 2. It can be easily seen that a cycle has boxicity 2 unless it is a triangle (in which case it has boxicity 1) and a wheel being just a universal vertex added to a cycle, has boxicity 2 unless it is a K_4 (in which case it has boxicity 1). Therefore, we will assume that G is not a wheel. For the sake of ease of presentation, a vertex will be called a "leaf" or "leaf vertex" if it is a leaf of the tree T.

4.1. Finding u'

Let $S = V(G) - V(C)$ denote the set of internal vertices of the tree T. We claim that there is a vertex $u' \in S$ such that $|N(u') \cap S| = 1$ and $|N(u') \cap V(C)| \ge 1$. If there is no such vertex, then G_S, the induced subgraph of G on S, has no vertices of degree 1 which is not possible since G_S is a tree (G_S has more than one vertex since G is not a wheel). Now, u' has at least one leaf of T as its neighbour since if it did not, then its degree in T is 1 implying that u' is a leaf of T – a contradiction since we have assumed that $u' \in S$.

4.2. Fixing the root of T

Designate the internal vertex of T adjacent to u', say r, to be the root of T. Given two vertices u and v, u is said to be an *ancestor* of v if u lies in the path rTv and u is said to be a *descendant* of v if v is an ancestor of u. Note that every vertex is an ancestor and a descendant of itself. Let $D(u)$ for any vertex $u \in V(G)$ be defined as the set of all leaves of T that are descendants of u. It can be easily seen that if u is a descendant of v, then $D(u) \subseteq D(v)$.

4.3. Ordering the vertices of C

Let $|V(C)| = k$ and let C be $p_0p_1 \dots p_{k-1}p_0$. Note that $D(u')$ cannot contain all the leaves since that would mean that $D(u') = D(r)$, implying that u' is the only neighbour of r in T. Then the degree of r in T would be 1, a contradiction since r is an internal vertex in T and not a leaf. Therefore, we can always find a leaf $p_i \in D(u')$ such that $p_{(i-1) \bmod k} \notin D(u')$ (recall that u' has at least one leaf of T as its neighbour and therefore, $D(u')$ is not empty). We define $l_j = p_{(i+j) \bmod k}$, for $0 \le j \le k-1$. This implies that $l_{k-1} \notin D(u')$ since $l_{k-1} = p_{(i-1) \bmod k}$. For $u \in V(C)$, we define $c(u) = i$ when $u = l_i$.

For the convenience of the reader, we summarize the construction as of now:

- We chose a vertex u' such that its neighbourhood contains exactly one internal vertex and at least one leaf of T.
- We chose the only internal vertex in the neighbourhood of u' to be the root r of T and defined the natural tree-order on T with r as the root. We also defined $D(u)$ to be the set of all leaves that are descendants (in our tree-order) of the vertex u.
- We defined a linear ordering $l_0, \dots, l_{k-1}$ of the vertices in $V(C)$ (the leaves of T) where $l_0 \in D(u')$ and $l_{k-1} \notin D(u')$.

Claim 1. *For any vertex $u \in V(G)$, the vertices in $D(u)$ will occur in consecutive places in the ordering $l_0, \dots, l_{k-1}$ of the vertices in C. In other words, if $u \in V(G)$ and $x, y, z \in V(C)$ such that $c(x) < c(z) < c(y)$ then it is not possible that $x, y \in D(u)$ and $z \notin D(u)$.*

Though the statement of the claim looks intuitive, its proof involves some technical details. Therefore, the reader might choose to skip the proof and continue with the rest of the construction so as not to get distracted from the main theme.

Proof (*Claim 1*). If u is a leaf of T, then the claim is true because $|D(u)| = 1$. Let us assume that this is not the case.

Consider any planar embedding of G. The cycle C divides the plane into a bounded region and an unbounded region. We claim that all the internal vertices of T will lie in one of these regions. Suppose there are two internal vertices of T such that they lie in different regions of C. Then, the path between them in T will have to pass the boundary of C. But the path cannot pass through a leaf of T and because the drawing is planar, no edge of the path can cross the boundary of C. We thus have a contradiction. Therefore, C forms the boundary of a face in any planar drawing of G.

Now, consider a planar embedding of G such that C forms the boundary of the unbounded face (i.e., all the internal vertices of T lie in the bounded region of C). Suppose $x, y \in D(u)$ and $z \notin D(u)$ such that $c(x) < c(z) < c(y)$ (recall that $c(l_i) = i$). Let $B = xCyTuTx$. It can be easily verified that B has exactly two regions – one bounded and the other unbounded. We say that a vertex is "inside" B if it lies in the bounded region bounded by B and say that it is "outside" B if it lies in the unbounded region whose boundary is B. We say that a vertex "lies on" B if it is in B.

Observation 1. Because of the planar embedding of G that we have chosen, it can be seen that any leaf vertex will have to either lie on xCy or outside B.

Observation 2. r does not lie on B.

We can assume that $r \neq u$ since that would contradict our assumption that $z \notin D(u)$. Also, r cannot lie on yTu or uTx since it contradicts our assumption that x and y are descendants of u and it cannot lie on xCy since it is not a leaf. Therefore, r does not lie on B.

Observation 3. u' is not inside B.

If u' is inside B, then l_0 cannot be outside B since u' is adjacent to l_0. From Observation 1, l_0 is in xCy which implies that $x = l_0$ (since $0 \le c(x) \le c(v)$, for any vertex $v \in xCy$, as $c(x) < c(y)$) and u', being the only internal vertex in $N(l_0)$, should lie on uTx. This contradicts our assumption that u' is inside B.

Observation 4. r is outside B.

Now suppose r is inside B. Then, u' cannot be outside B since r is adjacent to u' and it cannot be inside B due to Observation 3. Therefore, u' lies on B. If $u \neq u'$, then the fact that r is the only internal vertex adjacent to u' implies that r will have to lie on B, which contradicts Observation 2. Therefore, $u = u'$. Now, it can be seen that because of our choice of u' and r, $D(u') = N(u') - \{r\}$. This means that uTx and uTy are the edges $u'x$ and $u'y$ respectively and therefore, any path from r (inside B) to a vertex outside B will have to go through u'. Now, consider the leaf l_{k-1}. By our construction, $l_{k-1} \notin D(u')$. Therefore, $y \neq l_{k-1}$ and l_{k-1} does not lie on xCy and hence lies outside B (from Observation 1). The path from r to l_{k-1} will have to go through u' as we have noted before – but this implies that $l_{k-1} \in D(u')$ which is a contradiction. Therefore, r is outside B since we know from Observation 2 that r does not lie on B.

Because of Observation 4, the path zTr must contain a vertex v in B because of our planarity assumption. But if $v \neq u$, then x and y cannot both be descendants of u since either rTx or rTy will not contain u. If $v = u$, then rTz contains u and therefore, $z \in D(u)$, again a contradiction.

This proves our claim that for any vertex $u \in V(G)$, the vertices in $D(u)$ have to occur consecutively in the ordering $l_0, l_1, \ldots, l_{k-1}$. □

4.4. Construction of the interval graphs G_1 and G_2

We define f_1 and f_2 to be mappings of the vertex set $V(G)$ to closed intervals on the real line. Let G_1 and G_2 denote the interval graphs defined by f_1 and f_2 respectively.

For a vertex $u \in V(G)$, let $d(u)$ denote the number of ancestors of u other than itself (or "depth" of u in T). Let h denote the maximum depth of a vertex in T. Recall that $k = |V(C)|$ and S denotes the set of internal vertices of T.

Definition of f_1:

For $u \in V(G)$,

$f_1(l_0) = [0, k]$.

$f_1(u) = [c(u) - 1/2, c(u) + 1/2]$, if $u \in V(C)$ and $u \neq l_0$.

$f_1(u) = [\min_{v \in D(u)}\{c(v)\}, \max_{v \in D(u)}\{c(v)\}]$, if $u \in S$.

Definition of f_2:

For $u \in V(G)$,

$f_2(u') = [d(u'), h+2] = [1, h+2]$.

$f_2(u) = [d(u), d(u)+1]$, if $u \in S$ and $u \neq u'$.

$f_2(l_0) = [h+2, h+2]$.

$f_2(l_1) = [d(l_1), h+2]$.

$f_2(l_{k-1}) = [d(l_{k-1}), h+2]$.

$f_2(u) = [d(u), h+1]$, if $u \in V(C)$ and u is not l_0, l_1 or l_{k-1}.

Claim 2. *G_1 is a super graph of G.*

Proof. Consider an edge $(u, v) \in E(G)$. Clearly, $(u, v) \in E(T)$ or $(u, v) \in E(C)$.

1. $(u, v) \in E(T)$.
   In this case, either u is an ancestor of v or vice versa as T is a tree. Let us assume without loss of generality that u is the ancestor of v. Therefore, $D(v) \subseteq D(u)$. There are two possibilities now:
   (a) u and v are both internal vertices of T.
   Since $D(v) \subseteq D(u)$, we have $\min_{x \in D(u)}\{c(x)\} \leq \min_{x \in D(v)}\{c(x)\} \leq \max_{x \in D(v)}\{c(x)\} \leq \max_{x \in D(u)}\{c(x)\}$. Therefore, $f_1(u) \cap f_1(v) \neq \emptyset$, which implies that $(u, v) \in E(G_1)$.
   (b) u is an internal vertex of T and v is a leaf vertex of T.
   Since $v \in D(u)$, $\min_{x \in D(u)}\{c(x)\} \leq c(v) \leq \max_{x \in D(u)}\{c(x)\}$. Thus, both $f_1(u)$ and $f_1(v)$ contain the point $c(v)$ and therefore, $(u, v) \in E(G_1)$ (Note that $c(l_0) = 0$ and thus $c(l_0) \in f_1(l_0)$).
2. $(u, v) \in E(C)$.
   Without loss of generality, we can assume that $u = l_i$, for some i, and $v = l_{(i+1) \bmod k}$. For $1 \leq i \leq k-2$, $f_1(u)$ and $f_1(v)$ contain the point $i + 1/2$. If $u = l_0$ or $v = l_0$, then it is clear that $(u, v) \in E(G_1)$, since $f_1(l_0)$ contains $f_1(u), \forall u \in V(G)$.

Therefore, G_1 is a supergraph of G. □

Claim 3. *G_2 is a supergraph of G.*

Proof. Consider an edge $(u, v) \in E(G)$. We have the following three cases now.

1. u or v is l_0.
   By our choice of l_0, it is adjacent only to l_1, l_{k-1} and u' in G. Since $f_2(l_0), f_2(l_1), f_2(l_{k-1})$ and $f_2(u')$ contain the point $h+2$, all the edges incident on l_0 in G are also present in G_2.
2. $(u, v) \in E(T)$, $u \neq l_0$ and $v \neq l_0$.
   Let us assume without loss of generality that u is the parent of v. It is easily seen that $d(v) = d(u) + 1$. Since $u \neq l_0$ and $v \neq l_0$, the point $d(u) + 1$ is contained in both $f_2(u)$ and $f_2(v)$ (Recall that $d(u) \leq h, \forall u \in V(G)$).
3. $(u, v) \in E(C)$, $u \neq l_0$ and $v \neq l_0$.
   Since u and v are leaf vertices, $f_2(u)$ and $f_2(v)$ both contain the point $h + 1$ and therefore $(u, v) \in E(G_2)$.

This shows that G_2 is a supergraph of G. □

Claim 4. $G = G_1 \cap G_2$.

Proof. Since Claims 2 and 3 have established that G_1 and G_2 are supergraphs of G, it is sufficient to show that, for any pair of vertices $u, v \in V(G)$, $(u, v) \notin E(G)$ implies $(u, v) \notin E(G_1)$ or $(u, v) \notin E(G_2)$. Consider such a pair of vertices. There are three cases to be considered.

1. One of u or v is l_0.
   Let us assume without loss of generality that $u = l_0$. $(u, v) \notin E(G)$ now implies that $v \in V(G) - \{l_1, l_{k-1}, u'\}$ since l_0 is only adjacent to l_1, l_{k-1} and u' in G. It can be easily verified that only $f_2(u')$, $f_2(l_1)$ and $f_2(l_{k-1})$ have a non-empty intersection with $f_2(l_0)$. Therefore, $(u, v) \notin E(G_2)$.
2. $u \neq l_0$, $v \neq l_0$ and one of u and v is the ancestor of the other.
   Let us assume without loss of generality that u is the ancestor of v. This implies that $d(v) \geq d(u)+2$ since $(u, v) \notin E(G)$. We know that $u \neq u'$ since all the descendants of u' are its neighbours by our choice of u' and the root r. Now, since $u \neq u'$, the right end-point of $f_2(u)$ is $d(u) + 1$ and for all possible choices of v (excluding l_0), the left end-point of $f_2(v)$ is $d(v) \geq d(u) + 2$. Therefore, $f_2(u) \cap f_2(v) = \emptyset$ by the definition of f_2. Thus, in this case, $(u, v) \notin E(G_2)$.
3. $u \neq l_0$, $v \neq l_0$ and neither one of u and v is an ancestor of the other.
   One of the following three subcases hold.
   (a) u and v are both leaves of T.
   Let $u = l_i$ and $v = l_j$. Assume without loss of generality that $i < j$. Since neither of u or v is l_0, we have $1 \leq i < j \leq k - 1$. Also, $j > i + 1$ as $(u, v) \notin E(G)$. Therefore, $f_1(l_i) \cap f_1(l_j) = \emptyset$, from the definition of f_1. Thus, we have $(u, v) \notin E(G_1)$.
   (b) u and v are both internal vertices of T.
   Since $u \notin rTv$ and $v \notin rTu$, we have $D(u) \cap D(v) = \emptyset$ (To see this, suppose there is a vertex $z \in D(u) \cap D(v)$. Then both u and v would lie on rTz, implying that either $u \in rTv$ or $v \in rTu$). Now, from Claim 1, we have $\max_{x \in D(u)}\{c(x)\} < \min_{x \in D(v)}\{c(x)\}$ or $\max_{x \in D(v)}\{c(x)\} < \min_{x \in D(u)}\{c(x)\}$. By the definition of f_1, it can be seen that $f_1(u) \cap f_1(v) = \emptyset$, implying that $(u, v) \notin E(G_1)$.
   (c) One of u and v is a leaf of T and the other is an internal vertex of T.
   Let us assume that u is an internal vertex and v is a leaf of T. Since we are considering the case when neither of u and v is an ancestor of the other and neither is l_0, we have $v \notin D(u)$ and $v \neq l_0$. From Claim 1, we know that either $c(v) < \min_{x \in D(u)}\{c(x)\}$ or $c(v) > \max_{x \in D(u)}\{c(x)\}$. Therefore, by definition of f_1 and because $v \neq l_0, f_1(u) \cap f_1(v) = \emptyset$ and thus we have $(u, v) \notin E(G_1)$.

Since we have considered all possible cases when $(u, v) \notin E(G)$ and have shown that in each case, (u, v) is not present either in $E(G_1)$ or in $E(G_2)$, it follows that $G = G_1 \cap G_2$. □

Now, to complete the proof, we show that if G is not isomorphic to K_4, then $\text{box}(G) \geq 2$. Suppose G is not isomorphic to K_4. We will show that G is not an interval graph. By definition of G, $|V(C)| \geq 3$. If $|V(C)| > 3$, then C is an induced cycle with more than 3 vertices which means that G cannot be an interval graph and therefore $\text{box}(G) \geq 2$. If $|V(C)| = 3$, then C is a triangle. Now, all the leaves in $V(C)$ cannot be adjacent to the same internal vertex of T. To see this, look at G_S, the subgraph induced by S in G (recall that $S = V(G) - V(C)$, or the set of internal vertices of T). Since G is not isomorphic to K_4, G_S is a tree with more than one vertex. Therefore, there are at least two vertices of degree 1 in G_S. But if all the vertices in $V(C)$ are adjacent only to one vertex of S in G, there should be at least one vertex in S with degree 1 in G – which is a contradiction since all vertices of S, being internal vertices of T, have degree more than 1 in G. Therefore, we can find two leaves, say x and y, of T such that they are adjacent to different internal vertices in T. Let u and v denote the internal vertices of T adjacent to x and y respectively. Now, $xuTvyx$ forms an induced cycle of length greater than or equal to 4 (note that x and y are adjacent since $|V(C)| = 3$). Therefore, G cannot be an interval graph. Thus, we have $\text{box}(G) \geq 2$. ■

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