The edge-isoperimetric problem for discrete tori

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Abstract

The edge-isoperimetric problem has long been solved for cartesian powers of the cycles $C_3$ and $C_4$, for which the lexicographic order is the optimal order, and powers of the cycles $C_n$ with $n > 5$, which do not have nested optimal subsets.

For powers of $C_5$, it is clear that the lexicographic order is not optimal. We present a solution to the edge-isoperimetric problem for powers of $C_5$ in the form of an optimal order for the vertices. We then prove that discrete tori of the forms $C_5^i \times C_4^j \times C_3^k$ and $C_n \times C_5^i \times C_4^j \times C_3^k$ have nested optimal subsets for $n > 5$, $i, j, k \geq 0$, and give an optimal order for members of that class. We conjecture that these are the only discrete tori which have nested optimal subsets.

Keywords: Edge-isoperimetric problem; Discrete tori

1. Introduction

The edge-isoperimetric problem (EIP) for a $G$ graph is to find, for a given $t$, a subset of $t$ vertices such that the number of edges in the induced subgraph is maximal among all induced subgraphs with the same number of vertices.

Let $G = (V_G, E_G)$ be a graph, and $A \subseteq V_G$. Using the notation from [3], let $I_G(A)$ be the edge set for the subgraph induced by $A$, and let $I_G(t)$ be the maximum number of edges in all induced subgraphs with $t$ vertices. In notation,

$$I_G(A) = \{(u, v) \in E_G: u, v \in A\},$$

$$I_G(t) = \max_{|A|=t} |I_G(A)|.$$
Let $\delta_G(t) = I_G(t) - I_G(t-1)$. If $A$ is a subset of vertices with $|A| = t$ and $|I_G(A)| = I_G(t)$, $A$ is called optimal. $G$ is said to have nested optimal subsets if there is a total order $\mathcal{O}$ on $V_G$ such that for $t = 1, \ldots, |V_G|$, the initial segment of $V_G$ of size $t$ is optimal, in which case we call $\mathcal{O}$ an optimal order.

The edge-isoperimetric problem has been solved for cartesian powers $G^n$ of several graphs $G$; for a survey of the work, see [1]. For tori, $C_k^n$ has nested optimal subsets for $k = 3, 4$, the optimal order being lexicographic [4,5]. For $k > 5$, $C_k^n$ does not have nested optimal subsets [4]. In [3], Bezrukov, Das, and Elsässer provide an optimal order for powers of the Petersen graph, of which $C_5$ is a subgraph, showing that powers of the Petersen graph have nested optimal subsets.

The edge-isoperimetric problem presented here is equivalent, for regular graphs, to the problem of minimizing, for a given $t$, the number of edges connecting a set $A$ with $t$ vertices to its complement, $V_G \setminus A$. This problem is closely related to the vertex-isoperimetric problem of minimizing the number of vertices with distance 1 from a set. In [6], Riordan presents an optimal order for discrete even tori, products of even cycles, for the vertex-isoperimetric problem.

In Section 2, we give some general lemmas regarding the edge-isoperimetric problem for the cartesian products of arbitrary graphs with nested optimal subsets. In Section 3, we use methods similar to those in [3] to prove that $C_5^n$ has nested optimal subsets. This yields an optimal order $\mathcal{O}^n$ that is a restriction to $C_5^n$ of the Petersen order $\mathcal{O}^n$ given in [3].

In Section 4, we prove, via several steps, that discrete tori of the forms $C_3^i \times C_4^j \times C_5^k$ and $C_n \times C_4^i \times C_4^j \times C_3^k$ have nested optimal subsets, and we give an optimal order for those tori. Finally, in Section 5, we give some conjectures and considerations for the edge-isoperimetric problem for discrete tori not of these forms and for cartesian powers of the dodecahedron.

2. General lemmas

The theory of edge-isoperimetric problems for cartesian products of arbitrary graphs is small. We present a few general lemmas which will be of use in the specific cases in later sections. The first lemma is found in [3], and a proof may be found there.

**Lemma 1** (Bezrukov et al. [3]). If a graph $G$ has nested optimal subsets, $\delta_G(t) \geq \delta_G(i+1) - 1$ for $i = 1, \ldots, |V_G| - 1$.

We now introduce some terminology to discuss the EIP for products of graphs in a general context. An ordered family $\{G_i\}_{i=1}^n$ is a collection of graphs such that, for all strictly increasing sequences $\{s_j\}_{j=1}^k$ in $\{1, \ldots, n\}$, there is a total order $\mathcal{O}(s_1, \ldots, s_k)$ on the vertices of $G_{s_1} \times G_{s_2} \times \cdots \times G_{s_k}$.

An ordered family $\{G_i\}_{i=1}^n$ is consistent if, for all strictly increasing sequences $\{s_j\}_{j=1}^k$ in $\{1, \ldots, n\}$, and all vertices $a = (a_1, \ldots, a_k)$, $b = (b_1, \ldots, b_k)$ of $G_{s_1} \times G_{s_2} \times \cdots \times G_{s_k}$ with $a_i = b_i$ for some $i$, $a \succ b$ if and only if $\tilde{a} \succ \tilde{b}$, where $\tilde{a} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k)$ and $\tilde{b} = (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_k)$.
An ordered family \( \{G_i\}_{i=1}^{n} \) has nested optimal subsets if for all strictly increasing sequences \( \{s_j\}_{j=1}^{k} \) in \( \{1, \ldots, n\} \), \( G_{s_1} \times G_{s_2} \times \cdots \times G_{s_k} \) has nested optimal subsets and \( C(s_1, \ldots, s_k) \) is an optimal order.

Notice that if \( \{G_i\}_{i=1}^{n} \) is an ordered family, then every subset \( \{G_i\}_{i=1}^{k} \) for \( k \leq n \) is an ordered family. If \( \{G_i\}_{i=1}^{n} \) is consistent (resp. has nested optimal subsets), then \( \{G_i\}_{i=1}^{k} \) is consistent (resp. has nested optimal subsets).

If \( \{G_i\}_{i=1}^{n} \) is a consistent ordered family, let \( G = G_1 \times G_2 \times \cdots \times G_n \), and for all \( A \subseteq V_G \), \( 1 \leq i \leq n \) and \( g \in V_{G_i} \), let \( A_i(g) = \{(x_1, \ldots, x_n) \in A: x_i = g\} \) and let \( V_i(g) = \{(x_1, \ldots, x_n) \in V_G: x_i = g\} \). \( A \) is called i-compressed if, for all \( g \in V_{G_i}, A_i(g) \) is an initial segment of \( V_i(g) \) under the order \( C(1, \ldots, n) \) restricted to \( V_i(g) \), and \( A \) is compressed if it is i-compressed for all \( i \). Let the i-compression of \( A \), denoted \( C(A, i) \), be the set obtained from \( A \) by replacing \( A_i(g) \) with the initial segment of \( V_i(g) \) of length \( |A_i(g)| \) for all \( g \in G_i \). Our definition of compressed subsets agrees with that in [1], but differs from the definition in [2] for \( n > 2 \).

The next two lemmas were proven for the version of compression presented in [2], and for some specific graphs in [1,3].

**Lemma 2.** If \( \{G_i\}_{i=1}^{n} \) is a consistent ordered family, and if for all \( i, G_i \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_n \) has nested optimal subsets with optimal order \( C(1, \ldots, i-1, i+1, \ldots, n) \), then for every optimal \( A \subseteq V_G \), where \( G = G_1 \times \cdots \times G_n \), there is a compressed optimal \( B \subseteq V_G \) with \( |B| = |A| \).

**Proof.** By consistency, applying \( C(\cdot, i) \) for all \( i \) a large, finite number of times to \( A \) leads to a compressed set \( B \) with \( |B| = |A| \). The claim that \( B \) is optimal follows from the claim that for all \( A' \subseteq V_G \) and all \( i \), we have \( |I_G(C(A', i))| \geq |I_G(A')| \). This latter claim is evident from the fact that, if we let \( G_i^+ = G_1 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_n \), then \( G_i^+ \) has nested optimal subsets, and

\[
|I_G(C(A', i))| = \sum_{g \in V_{G_i}} \left( I_{G_i^+}(|A'_i(g)|) + \sum_{g' > g} \max(|A'_i(g)|, |A'_i(g')|)\right).
\]

If \( G = G_1 \times \cdots \times G_n \), we can assume that \( V_{G_i} = \{0, \ldots, |V_{G_i}| - 1\} \) for all \( i \), and if each \( G_i \) has nested optimal subsets, then for each \( t = 1, \ldots, |V_{G_i}| \), the set \( \{0, \ldots, t - 1\} \) is optimal. Using this convention, we can state and prove the next lemma.

**Lemma 3.** Suppose \( \{G_i\}_{i=1}^{n} \) is a consistent ordered family such that \( G_i \) has nested optimal subsets for all \( i \), and the optimal order of \( G_i \) is the same as the total order on \( G_i \) as an element of the ordered family. Let \( G = G_1 \times \cdots \times G_n \). If \( A \subseteq V_G \) is compressed, then

\[
|I_G(A)| = \sum_{(x_1, \ldots, x_n) \in A} \sum_{i=1}^{n} \delta_G(x_i + 1).
\]

**Proof.** The proof is by induction on \( |A| \). If \( |A| = 1 \), then \( A = \{(0, \ldots, 0)\} \) and \( |I_G(A)| = 0 = \sum_{i=1}^{n} \delta_G(1) \).
If $|A| > 1$, let $a = (a_1, \ldots, a_n) = \max A$. Let $A' = A \setminus \{a\}$. Then $|A'| = |A| - 1$, and by consistency $A'$ is compressed, so by the inductive hypothesis we have $|I_G(A')| = \sum_{(x_1, \ldots, x_n) \in A'} \delta_G(x_i + 1)$. Adding $a$ to $A'$, since $A(a_i)$ is an initial segment of $V_G(a_i)$ for all $i$, adds $\sum_{i=1}^n \delta_G(x_i + 1)$, so

$$|I_G(A)| = |I_G(A')| + \sum_{i=1}^n \delta_G(x_i + 1)$$

$$= \sum_{(x_1, \ldots, x_n) \in A} \sum_{i=1}^n \delta_G(x_i + 1). \quad \square$$

If $A \subseteq V_G$ is compressed, let $a = \max A$ and $b = \min V_G \setminus A$. Then, it is clear that $A \setminus \{a\}$ and $(A \setminus \{a\}) \cup \{b\}$ are compressed.

**Corollary 1.** If $A$, $a$ and $b$ are as in the previous paragraph, and $B = (A \setminus \{a\}) \cup \{b\}$, then

$$|I_G(B)| - |I_G(A)| = \sum_{i=1}^n (\delta_G(b_i + 1) - \delta_G(a_i + 1)).$$

Finally, the following lemma and proof were pointed out to me by Sergei Bezrukov.

**Lemma 4.** If $n \geq 2$, $\{G_i\}_{i=1}^n$ is a consistent ordered family and the order $G = G(1,2,\ldots,n)$ is an optimal order on $G = G_1 \times \cdots \times G_n$, let $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$ be vertices of $G$ with $a \succ_G b$ and $a_i \neq b_i$ for all $i = 1, \ldots, n$. Then there is a vertex $c = (c_1, \ldots, c_n)$ with $a \succ_G c \succ_G b$ and $c_i = a_i$ for some $i$, or $\sum_{i=1}^n \delta_G(b_i + 1) \geq \sum_{i=1}^n \delta_G(a_i + 1)$.

**Proof.** Let $C = \{(c_1, \ldots, c_n) \in V_G: c_i = a_i - 1, \text{ and } c_j = a_j \text{ for all } j \neq i\}$. Clearly $b \notin C$, since $a_i \neq b_i$ for all $i$. By consistency, for all $c \in C$, $c \prec_G a$, and if for some $c \in C$ we have $c \succ_G b$, we’re done. Otherwise, let $D = \{c \in V_G: c \prec_G a\}$. Since $C \subseteq D$, $A = D \cup \{a\}$ and $B = D \cup \{b\}$ are compressed. Since $G$ is an optimal order, $|I_G(B)| \geq |I_G(A)|$, and by Lemma 1 $\sum_{i=1}^n (\delta_G(b_i + 1) - \delta_G(a_i + 1)) \geq 0$. \square

**3. The optimal order for $C^n_5$**

The lexicographic order is not optimal for $C^n_5$ for $n \geq 2$. To see why, consider subsets with 4 vertices. The optimal subsets with 4 vertices are clearly 4-cycles, which have 4 edges, but initial segments of the lexicographic order are paths with 4 vertices, which only have 3 edges. Instead, we inductively define a new order, though similar to the lexicographic order, on $C^n_5$. The order $\mathcal{C}^k_5$, shown in Fig. 1a, is clearly an optimal order for $C_5$.

Now we assume that $\mathcal{C}^k_5$ has been defined for $k < n$, and we define the order $\mathcal{C}^n_5$ on $V_{C^n_5}$ by defining the successor for any vector $(a_1, \ldots, a_n) \in V_{C^n_5}$. If $(a_2, \ldots, a_n) \neq (4, \ldots, 4)$,
let \((a'_2, \ldots, a'_n) = \text{succ}(a_2, \ldots, a_n)\) in the order \(\mathcal{C}_n^{n-1}\). Then, we define

\[
\text{succ}(a_1, \ldots, a_n) = \begin{cases} 
(a_1 + 1, a_2, \ldots, a_n) & \text{if } a_1 \in \{0, 3\}, \\
(a_1 - 1, a'_2, \ldots, a'_n) & \text{if } a_1 \in \{1, 4\} \text{ and } (a_2, \ldots, a_n) \neq (4, \ldots, 4), \\
(a_1, a'_2, \ldots, a'_n) & \text{if } a_1 = 2 \text{ and } (a_2, \ldots, a_n) \neq (4, \ldots, 4), \\
(a_1 + 1, 0, \ldots, 0) & \text{if } a_1 \in \{1, 2\} \text{ and } (a_2, \ldots, a_n) = (4, \ldots, 4).
\end{cases}
\]

The order \(\mathcal{C}_2^5\) is illustrated in Fig. 1b, where the edges that go off the top are the same as those on the bottom, and similarly for the sides. It is easy to show, using induction on \(n\), that any vector \((a_1, \ldots, a_n) \in \mathcal{V}_C^5\) has a unique predecessor, except for \((0, \ldots, 0)\) which has no predecessor, so every vector will be reached starting at \((0, \ldots, 0)\). The order \(\mathcal{C}_2^n\) is pictured schematically in Fig. 2, in which the ovals contain the vectors \((i; a_2, \ldots, a_n)\), where \(i\) is shown beneath the ovals, ordered upward by their last \(n - 1\) elements according to the order \(\mathcal{C}_2^{n-1}\). Thus, for all \(n\), \(\{\mathcal{C}_5^i\}_{i=1}^n\) is an ordered family.

For vectors \(\mathbf{a}, \mathbf{b} \in \mathcal{V}_C^5\), we write \(\mathbf{a} \succ \mathbf{b}\) if \(\mathbf{b}\) precedes \(\mathbf{a}\) in the order \(\mathcal{C}_5^n\). The following lemma shows that the order \(\mathcal{C}_5^n\) behaves as we might intuitively expect on vectors that agree in some components.

**Lemma 5.** \(\{\mathcal{C}_5^i\}_{i=1}^n\) is a consistent ordered family.

**Proof.** We proceed by induction on \(n\). The lemma is vacuous for \(n = 1\) and trivial for \(n = 2\). For \(n > 2\), let \(\mathbf{a} = (a_1, \ldots, a_n)\), \(\mathbf{b} = (b_1, \ldots, b_n)\), and suppose that for some \(i\), \(a_i = b_i\). Let \(\tilde{\mathbf{a}}, \tilde{\mathbf{b}}\) be obtained from \(\mathbf{a}\) and \(\mathbf{b}\), respectively, by omitting their \(i\)th entries, that is, \(\tilde{\mathbf{a}} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)\) and \(\tilde{\mathbf{b}} = (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)\). The claim is equivalent to \(\mathbf{a} \succ \mathbf{b}\) if and only if \(\tilde{\mathbf{a}} \succ \tilde{\mathbf{b}}\), which we now prove.
If \( i = 1 \), then it is clear from the definition of \( C_5^n \) that \( a > b \) if and only if \( a = (a_2, \ldots, a_n) > (b_2, \ldots, b_n) = b \). If \( n > 2 \) and \( i > 1 \), there are several cases to consider:

- If \( a_1 > b_1 + 1 \), then \( a > b \) and \( \tilde{a} > \tilde{b} \).
- If \( a_1 = b_1 + 1 \) and \( b_1 \in \{1, 2\} \), then \( a > b \) and \( \tilde{a} > \tilde{b} \).
- If \( a_1 = b_1 + 1 \) and \( b_1 \in \{0, 3\} \), then by the definition of \( C_5^n \) and by the induction hypothesis, we have \( a > b \Leftrightarrow (a_2, \ldots, a_n) > (b_2, \ldots, b_n) \Leftrightarrow (\tilde{a}_2, \ldots, \tilde{a}_{n-1}) > (\tilde{b}_2, \ldots, \tilde{b}_{n-1}) \Leftrightarrow \tilde{a} > \tilde{b} \), where we let \( \tilde{a} = (\tilde{a}_1, \ldots, \tilde{a}_{n-1}) \) and \( \tilde{b} = (\tilde{b}_1, \ldots, \tilde{b}_{n-1}) \).
- If \( a_1 = b_1 \) or if \( a_1 = b_1 - 1 \) and \( b_1 \in \{1, 4\} \), then, using the notation of the previous case, by the definition of \( C_5^n \) and by the induction hypothesis, \( a > b \Leftrightarrow (a_2, \ldots, a_n) > (b_2, \ldots, b_n) \Leftrightarrow (\tilde{a}_2, \ldots, \tilde{a}_{n-1}) > (\tilde{b}_2, \ldots, \tilde{b}_{n-1}) \Leftrightarrow \tilde{a} > \tilde{b} \).
- In the remaining case, where \( a_1 < b_1 - 1 \) or \( a_1 = b_1 - 1 \) and \( b_1 \in \{2, 3\} \), \( a < b \) and \( \tilde{a} < \tilde{b} \). \( \square \)

When \( G = C_5 \), the functions \( I_G \) and \( \delta_G \) can be directly calculated.

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Now, we are ready to prove the first main result, namely that the order \( C_5^n \) is an optimal order for the graph \( C_5^n \).

**Theorem 1.** For any \( n \geq 1 \) and \( t = 1, \ldots, 5^n \), the set \( F^n(t) \) is optimal, where \( F^n(t) \) is the initial segment of \( V_{C_5^n} \) under order \( C_5^n \) with length \( t \). In other words, the ordered family \( \{C_5^n\}_{i=1}^n \) has nested optimal subsets.

**Proof.** The structure of this proof is roughly analogous to, and inspired by, the corresponding proof in [3] for the Petersen graph.
We prove the theorem by induction on \( n \). The case \( n = 1 \) is trivial, and the case \( n = 2 \) can be checked by hand since there are only \( \binom{10}{5} = 252 \) compressed subsets of \( V_{C_5}^n \). We then assume the theorem is true for the case \( n - 1 \), which satisfies the hypothesis for Lemma 2, and prove the theorem for \( n \geq 3 \).

Let \( A \subseteq V_{C_5}^n \) be an optimal compressed set, let \( a = (a_1, \ldots, a_n) = \max A \) and let \( b = (b_1, \ldots, b_n) = \min V_{C_5}^n \setminus A \) be the least vertex not in \( A \). If \( A = F^n(|A|) \), then we are done. Otherwise \( a > b \), in which case from the definition of \( \mathcal{E}^n_5 \) one of the five following (disjoint) cases occurs:

1. Assume \( a_1 - 1 > b_1 \).
2. \( a_1 - 1 = b_1 \) and \( b_1 \in \{1, 2\} \).
3. Assume \( a_1 = b_1 \) and \( (a_2, \ldots, a_n) \geq \mathcal{E}^{n-1} (b_2, \ldots, b_n) \).
4. \( a_1 + 1 = b_1 \), \( b_1 \in \{1, 4\} \), and \( (a_2, \ldots, a_n) \geq \mathcal{E}^{n-1} (b_2, \ldots, b_n) \).

In many of the above cases, we can show that \( b \in A \) by compression, which is a contradiction, so \( A = F^n(|A|) \). To show this, it suffices to find a vertex \( c \) such that \( a > c > b \) with the vector pairs \( a, c \) and \( c, b \) each having an equal entry. If no such vertex \( c \) exists, then using Lemma 3 we will show that replacing \( a \) with \( b \) gives a set \( B \) with \( |\mathcal{L}_G(B)| \geq |\mathcal{L}_G(A)| \). Then, after a finite number of such transformations, we obtain \( F^n(|A|) \), and \( F^n(|A|) \) is shown to be optimal. We proceed by cases:

**Case a:** \( a_1 - 1 > b_1 \).

a1. Assume \( a_1 > 2 \) and \( b_1 < 2 \). Then

\[
a = (a_1, \ldots, a_n) > (2, a_2, b_3, \ldots, b_n) > (b_1, \ldots, b_n) = b,
\]

so by compression \( b \in A \), a contradiction.

For the rest of this case we assume that \( a_1 - b_1 = 2 \) and \( b_1 \in \{0, 2\} \).

a2. Assume \( a_i > 1 \) for some \( i, 2 \leq i \leq n - 1 \). Then \( (a_i, a_{i+1}, \ldots, a_n) \geq \mathcal{E}^{n-1} (1, b_{i+1}, \ldots, b_n) \), and \( a_i > b_i + 1 \) implies \( (a_1, \ldots, a_{i-1}, 1, b_{i+1}, \ldots, b_n) \geq \mathcal{E}^{n-1} (b_1, \ldots, b_i) \). Thus

\[
a = (a_1, \ldots, a_n) > (a_1, \ldots, a_{i-1}, 1, b_{i+1}, b_n) > (b_1, \ldots, b_n) = b,
\]

so by compression \( b \in A \), a contradiction.

a3. Assume \( b_i < 3 \) for some \( i, 2 \leq i \leq n - 1 \). Then \( (4, a_{i+1}, \ldots, a_n) \geq \mathcal{E}^{n-1} (b_1, \ldots, b_n) \). Similarly, to case a2, \( (a_1, \ldots, a_i) \geq \mathcal{E}^{n-1} (b_1, \ldots, b_{i-1}, 4) \). Thus,

\[
a = (a_1, \ldots, a_n) > (b_1, \ldots, b_{i-1}, 4, a_{i+1}, \ldots, a_n) > (b_1, \ldots, b_n) = b,
\]

so by compression \( b \in A \), a contradiction.

a4. Assume \( a_i = 0 \) or \( b_i = 4 \) for some \( i, 2 \leq i \leq n \), and \( a_k \leq 1, b_k \geq 3 \) for all \( k, 2 \leq k \leq n - 1 \). If \( a_n \geq b_n \),

\[
a = (a_1, \ldots, a_n) \geq (a_1, \ldots, a_{n-1}, b_n) > (b_1, \ldots, b_n) = b,
\]

so by compression \( b \in A \), a contradiction.
If \( a_n < b_n \), let \( B = (A \setminus \{a\}) \cup \{b\} \). Then by Lemma 3,

\[
|I_{C^*}(B)| - |I_{C^*}(A)| = \sum_{k=1}^{n} (\delta_C(b_k + 1) - \delta_C(a_k + 1)) \\
\geq -1 + \delta_C(b_1 + 1) + \delta_C(a_1 + 1) \\
\geq -1 + 1 = 0
\]

because \( \delta_C(b_1 + 1) - \delta_C(a_1 + 1) \geq -1 \) for \( a_1 - b_1 = 2 \), \( \delta_C(b_k + 1) - \delta_C(a_k + 1) \geq \delta_C(3) - \delta_C(1) = 0 \) for \( 2 \leq k \leq n - 1 \), \( \delta_C(b_n + 1) - \delta_C(a_n + 1) \geq 0 \) since \( b_n > a_n \), and \( \delta_C(b_1 + 1) - \delta_C(a_1 + 1) \geq 1 \) for \( a_1 = 0, b_1 \geq 3 \) or \( a_i \leq 1, b_i = 4 \).

a5. The remaining case is where \( a_i = 1, b_i = 3 \) for all \( i, 2 \leq i \leq n - 1 \), \( a_n \neq 0, b_n \neq 4 \). If \( a_n > b_n \), then

\[
a = (a_1, \ldots, a_{n-1}, a_n) \geq (a_1, \ldots, a_{n-1}, b_n) > (b_1, \ldots, b_n) = b,
\]

so by compression \( b \in A \), a contradiction.

If \( 0 < a_n < b_n < 4 \), let

\[
X = \{(a_1, x_2, \ldots, x_n): \text{for } i = 2, \ldots, n, \ a_i - 1 \leq x_i \leq a_i \},
\]

\[
Y = \{(b_1, y_2, \ldots, y_n): \text{for } i = 2, \ldots, n, \ b_i \leq y_i \leq b_i + 1 \}.
\]

Then \( |X| = |Y| = 2^{n-1} \), and since \( A \) is compressed, \( X \subseteq A \) (since \( a \in A \)) and \( Y \cap A = \emptyset \) (since \( b \notin A \)). Let \( B = (A \setminus X) \cup Y \). It is easy to see that \( B \) is compressed. Also,

\[
|I_{C^*}(B)| - |I_{C^*}(A)| = \sum_{y \in Y} \sum_{i=2}^{n} \delta_C(y_i + 1) - \sum_{x \in X} \sum_{i=2}^{n} \delta_C(x_i + 1) \\
+ 2^{n-1}(\delta_C(b_1 + 1) - \delta_C(a_1 + 1)) \\
= (n - 2)2^{n-2}(\delta_C(4) + \delta_C(5) - \delta_C(1) - \delta_C(2)) \\
+ 2^{n-2}(\delta_C(b_n + 1) + \delta_C(b_n + 2) - \delta_C(a_n) \\
- \delta_C(a_n + 1)) + 2^{n-1}(\delta_C(b_1 + 1) - \delta_C(a_1 + 1)) \\
\geq (n - 2)2^{n-1} + 2^{n-2} - 2^{n-1} \\
= (n - 3)2^{n-1} + 2^{n-2} > 0,
\]

which contradicts the optimality of \( A \).

Case b: \( a_1 - 1 = b_1 \) and \( b_1 \in \{1, 2\} \). The analysis of this case is identical to that of cases a2–a4, with the difference that now we can guarantee \( \delta_C(b_1 + 1) - \delta_C(a_1 + 1) \geq 0 \), so cases a4 and a5 can both be analyzed by the method of a4.

Case c: \( a_1 - 1 = b_1, b_1 \in \{0, 3\} \), and \( (a_2, \ldots, a_n) \geq a_{n-1}(b_2, \ldots, b_n) \).
Using Lemma 5,

\[ a = (a_1, a_2, \ldots, a_n) \geq (a_1, b_2, \ldots, b_n) > (b_1, \ldots, b_n) = b, \]

so by compression \( b \in A \), a contradiction.

Case d: \( a_1 = b_1 \) and \( (a_2, \ldots, a_n) \gg_{\varphi_{n-1}} (b_2, \ldots, b_n) \). Since \( A \) is 1-compressed, \( b \in A \), a contradiction.

Case e: \( a_1 + 1 = b_1 \) and \( (a_2, \ldots, a_n) \gg_{\varphi_{n-1}} (b_2, \ldots, b_n) \).

e1. Assume \( b_2 \in \{0, 3\} \). Then \( \text{succ}(b_2, \ldots, b_n) = (b_2 + 1, b_3, \ldots, b_n) \). Since \( (a_1, b_2 + 1) \gg_{\varphi_{n-1}} (a_1 + 1, b_2, \ldots, b_n) \),

\[ a = (a_1, \ldots, a_n) \geq (a_1, b_2 + 1, \ldots, b_n) > (a_1 + 1, b_2, \ldots, b_n) = b, \]

so by compression \( b \in A \), a contradiction.

e2. Assume \( b_2 = 2 \). If \( (b_3, \ldots, b_n) \neq (4, \ldots, 4) \), let \( (c_3, \ldots, c_n) \) be the successor of \( (b_3, \ldots, b_n) \). Then \( \text{succ}(b_2, \ldots, b_n) = (b_2, c_3, \ldots, c_n) \), and \( (a_1, c_3, \ldots, c_n) \gg_{\varphi_{n-1}} (a_1 + 1, b_3, \ldots, b_n) \), so by Lemma 5, we have

\[ a = (a_1, \ldots, a_n) \geq (a_1, b_2, c_3, \ldots, c_n) > (a_1 + 1, b_2, \ldots, b_n) = b, \]

so by compression \( b \in A \), a contradiction.

There are two cases:
- If \( (a_2, \ldots, a_n) \geq \text{succ}(\text{succ}(b_2, \ldots, b_n)) = (b_2 + 2, 0, \ldots, 0) \), then \( (a_1, b_2, 0, \ldots, 0) \in A \), since \( A \) is 1-compressed. Thus,

\[ a \geq (a_1, b_2 + 2, 0, \ldots, 0) \]

\[ > (a_1 + 1, b_2 + 1, 0, \ldots, 0) \]

\[ > (a_1 + 1, b_2, 4, \ldots, 4) = b, \]

so by compression \( b \in A \), a contradiction.
- If \( (a_2, \ldots, a_n) = \text{succ}(b_2, \ldots, b_n) = (b_2 + 1, 0, \ldots, 0) \) and \( b = (b_1, b_2, 4, \ldots, 4) \). Let \( B = (A \setminus \{a\}) \cup \{b\} \). Then

\[ |I_{C_5}^1(B)| - |I_{C_5}^1(A)| = \delta_{C_5}(b_1 + 1) - \delta_{C_5}(a_1 + 1) + \delta_{C_5}(b_2 + 1) - \delta_{C_5}(a_2 + 1) \]

\[ + (n - 2)(\delta_{C_5}(4 + 1) - \delta_{C_5}(0 + 1)) \]

\[ = \delta_{C_5}(b_1 + 1) - \delta_{C_5}(b_1) + \delta_{C_5}(3) - \delta_{C_5}(4) \]

\[ + (n - 2)(\delta_{C_5}(5) - \delta_{C_5}(1)) \]

\[ \geq 1 + 0 + 2(n - 2) > 0, \]

contradicting the optimality of \( A \).
e3. Assume \( b_2 \in \{1, 4\} \) and \((b_3, \ldots, b_4) = (4, \ldots, 4)\). Then \( b_2 \neq 4 \) since \((a_2, \ldots, a_n) >_{\text{SOC}} (b_2, \ldots, b_6)\), so \( b_2 = 1 \). Also, \( \text{succ}(b_2, \ldots, b_6) = (b_2 + 1, 0, \ldots, 0) \). If \((a_2, \ldots, a_n) \geq \text{succ}(b_2, \ldots, b_6)) = (b_2 + 1, 1, 0, \ldots, 0)\), we have
\[
a \geq (a_1, b_2 + 1, 1, 0, \ldots, 0)
\]
\[
> (a_1 + 1, b_2 + 1, 0, \ldots, 0)
\]
\[
> (a_1 + 1, b_2, 4, \ldots, 4) = b,
\]
so \( b \in A \) by compression.

If \((a_2, \ldots, a_n) = \text{succ}(b_2, \ldots, b_6)\), we have \( a = (b_1 - 1, b_2 + 1, 0, \ldots, 0) \) and \( b = (b_1, b_2, 4, \ldots, 4) \). Let \( B = (A \setminus \{a\}) \cup \{b\} \), so
\[
|I_{C^5}(B)| - |I_{C^5}(A)| = \delta_{C^5}(b_1 + 1) - \delta_{C^5}(a_1 + 1) + \delta_{C^5}(b_2 + 1) - \delta_{C^5}(a_2 + 1)
\]
\[
+ (n - 2) (\delta_{C^5}(4 + 1) - \delta_{C^5}(0 + 1))
\]
\[
= \delta_{C^5}(b_1 + 1) - \delta_{C^5}(b_1) + \delta_{C^5}(2) - \delta_{C^5}(3)
\]
\[
+ (n - 2) (\delta_{C^5}(5) - \delta_{C^5}(1))
\]
\[
\geq 1 + 0 + 2(n - 2) > 0,
\]
contradicting the optimality of \( A \).

e4. All that remains is the case \( b_2 \in \{1, 4\} \) and \((b_3, \ldots, b_4) \neq (4, \ldots, 4)\). Let \((d_3, \ldots, d_n) = \text{succ}(b_3, \ldots, b_6)\), so \( \text{succ}(b_2, \ldots, b_6) = (b_2 - 1, d_3, \ldots, d_n) \). If \((a_2, \ldots, a_n) \geq (b_2, d_3, \ldots, d_n) = \text{succ}(\text{succ}(b_2, \ldots, b_6))\), then since \( A \) is 1-compressed, we have \((a_1, b_2, d_3, \ldots, d_n) \in A\), and
\[
a \geq (a_1, b_2, d_3, \ldots, d_n)
\]
\[
> (a_1 + 1, b_2 - 1, d_3, \ldots, d_n)
\]
\[
> (a_1 + 1, b_2, \ldots, b_n) = b,
\]
so by compression \( b \in A \), a contradiction.

If \((a_2, \ldots, a_n) = \text{succ}(b_2, \ldots, b_6)\), then \( a = (b_1 - 1, b_2 - 1, a_3, \ldots, a_n) \), and we can apply to the vectors \((a_2, \ldots, a_n)\) and \((b_2, \ldots, b_6)\) the same analysis as in cases e1–e3 and the other part of e4. This will take care of every case except for \( b_1 \in \{1, 4\} \), \((a_3, \ldots, a_n) = \text{succ}(b_3, \ldots, b_6)\), and \((b_4, \ldots, b_n) \neq (4, \ldots, 4)\). Continuing in this manner, the only remaining case is when \( a = \text{succ}(b) \), \( a = (b_1 - 1, b_2 - 1, \ldots, b_{n-1} - 1, b_n + 1) \), where \( b_1 \in \{1, 4\} \) for \( 1 \leq i \leq n - 1 \) and \( b_n \neq 4 \). In this case let \( B = (A \setminus \{a\}) \cup \{b\} \). Then,
\[
|I_{C^5}(B)| - |I_{C^5}(A)| = \sum_{i=1}^{n-1} (\delta_{C^5}(b_1 + 1) - \delta_{C^5}(b_1)) + \delta_{C^5}(b_n + 1) - \delta_{C^5}(b_n + 2)
\]
\[
\geq (n - 1) - 1 = n - 2 > 0
\]
which contradicts the optimality of \( A \). \( \square \)
4. The optimal orders for discrete tori

We now prove, through several stages, that discrete tori of the forms $C^i_j \times C^j_d \times C^k_f$ and $C^j_d \times C^i_j \times C^k_f$ have nested optimal subsets for $i, j, k \geq 0, n > 5$.

First, we look at the seemingly tangential problem of $G \times Q_k$, where $Q_k$ is the $k$-dimensional hypercube, and $G = G_1 \times \cdots \times G_n$ for some consistent ordered family $\{G_i\}_{i=1}^n$ with nested optimal subsets. Let $\mathcal{G}$ be the total order on $G$. We regard $Q_k$ as the $k$th power of $P_2$, the path on two vertices, so if $a, b \in V_{G \times Q_k}$, we can let $a = (a_1, \ldots, a_n, \overline{a_1}, \ldots, \overline{a_k})$ and $b = (b_1, \ldots, b_n, \beta_1, \ldots, \beta_k)$. We define an order on $G \times Q_k$ by letting $a > b$ if and only if:

(i) $(a_1, \ldots, a_n) > (b_1, \ldots, b_n)$, or
(ii) $(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$ and $(\overline{a_1}, \ldots, \overline{a_k}) > (\beta_1, \ldots, \beta_k)$,

where $\leq$ denotes the lexicographic order on $Q_k$. Notice that since $\{P_2\}_{i=1}^k$ and $\{G_i\}_{i=1}^n$ are consistent ordered families, then $\{G_i\}_{i=1}^{k+n}$ is consistent, where $G_i = P_2$ for $i > n$.

**Theorem 2.** If $\{G_i\}_{i=1}^n$ is a consistent ordered family with nested optimal subsets, such that $G_i \times P_2$ has nested optimal subsets for all $i$, given by the lexicographic order, then the ordered family $\{G_i\}_{i=1}^{k+n}$, where $G_i = P_2$ for $i > n$, has nested optimal subsets.

**Proof.** The proof is by induction on $n + k$. When $n + k = 2$, if $n = 0$ or $k = 0$, it follows from the fact that $Q_2$ has nested optimal subsets and $\{G_i\}_{i=1}^n$ has nested optimal subsets. If $n = k = 1$, by assumption $G_1 \times P_2$ has nested optimal subsets, with the optimal order being the lexicographic order.

Assume $n + k \geq 3$. If $n = 0$ or $k = 0$, the result follows from the fact that $Q_f$ has nested optimal subsets [4] and $\{G_i\}_{i=1}^n$ has nested optimal subsets, respectively, so assume $n, k \geq 1$. Let $G = G_1 \times \cdots \times G_n$. Let $A$ be an optimal compressed set, and let $a = \max A$ and $b = \min V_{G \times Q_k} \setminus A$. If $A$ is an initial segment of $V_{G \times Q_k}$, we are done; otherwise $a > b$. Since $A$ is compressed, $a_n \neq b_n$ and $a_i \neq a_m$ for $1 \leq n \leq i \leq j$, because otherwise by compression that $b \in A$. Thus, $(a_1, \ldots, a_j)$ is the binary complement of $(b_1, \ldots, b_j)$.

If $(x_1, \ldots, x_j) \neq (0, \ldots, 0)$, let $(x_1', \ldots, x_j')$ be its predecessor in the lexicographic order. Since $(x_1', \ldots, x_j')$ is not the binary complement of $(b_1, \ldots, b_j)$, they agree in some position, so

$$a = (a_1, \ldots, a_i, x_1, \ldots, x_j)$$
$$> (a_1, \ldots, a_i, x_1', \ldots, x_j')$$
$$> (b_1, \ldots, b_i, b_1, \ldots, b_j) = b,$$

so by compression $b \in A$, a contradiction.

Similarly, if $(b_1, \ldots, b_j) \neq (1, \ldots, 1)$, let $(b_1', \ldots, b_j')$ be its successor in the lexicographic order, which agrees with $(x_1, \ldots, x_j)$ in some position because they are not...
binary complements, so

\[ a = (a_1, \ldots, a_i, a_{i+1}, \ldots, a_j) \]

> \((b_1, \ldots, b_i, b'_1, \ldots, b'_j)\)

> \((b_1, \ldots, b_i, b_j, \ldots, b_j) = b,\]

so by compression \(b \in A\), a contradiction.

Hence, we may assume that \((a_1, \ldots, a_i) = (0, \ldots, 0)\) and \((b_1, \ldots, b_j) = (1, \ldots, 1)\). If \(n = 1\), and \(a_1 > b_1 + 1\), we have

\[ a = (a_1, 0, \ldots, 0) \]

> \((b_1, 1, \ldots, 1)\)

so by compression \(b \in A\), a contradiction.

If \(n \geq 2\), by Lemma 4, there is a \(c = (c_1, \ldots, c_n)\) with \(a > c > b\) and \(a_i = c_i\) for some \(i\), or \(\sum_{i=1}^n \delta_G(b_i + 1) - \sum_{i=1}^n \delta_G(a_i + 1)\), contradicting the optimality of \(A\).

Now if \(\{G_i\}_{i=1}^n\) satisfies the hypotheses of the preceding theorem, let \(G = G_1 \times \cdots \times G_n\). We define an order on \(G \times C_4^j\) by letting \(a > b\), for \(a = (a_1, \ldots, a_n, a_{i+1}, \ldots, a_j)\) and \(b = (b_1, \ldots, b_n, b_{i+1}, \ldots, b_j)\) vertices of \(G \times C_4^j\), if and only if:

(i) \(a_1 \geq b_1 \geq \cdots \geq b_n\), or

(ii) \(a_1 \geq b_1 \geq \cdots \geq b_n\) and \(a_{i+1} = b_{i+1} = \cdots = b_j\) and \(a_{i+1} > b_{i+1}\),

where \(\mathcal{L}_4^j\) is the lexicographic order on \(C_4^j\).

Notice that \(C_4^j\) is isomorphic to \(Q_{2j}\), and that by symmetry, the order defined above is \(I\)-equivalent to the order on \(G \times Q_{2j}\), i.e. for all \(t > 1\), we have \(I_G \times C_4^j(t) = I_G \times Q_{2j}(t)\).
Then we have the following corollary:

**Corollary 2.** If \( \{G_i\}_{i=1}^{n} \) is a consistent ordered family with nested optimal subsets, and \( G_i \times P_2 \) has the lexicographic order as an optimal order, then \( \{G_i\}_{i=1}^{n+j} \), where \( G_i = C_k \) for \( i > n \), is a consistent ordered family with nested optimal subsets.

Similar to the work above, if \( \{G_i\}_{i=1}^{n} \) is a consistent ordered family with nested optimal subsets and \( G = G_1 \times \cdots \times G_n \), we define an order on \( G \times C_k \) by letting \( a > b \), where \( a = (a_1, \ldots, a_n, a_{n+1}, \ldots, a_k) \) and \( b = (b_1, \ldots, b_n, b_{n+1}, \ldots, b_k) \) are vertices of \( G \times C_k \), if and only if:

(i) \( (a_1, \ldots, a_n) >_L (b_1, \ldots, b_n) \), or
(ii) \( (a_1, \ldots, a_n) = (b_1, \ldots, b_n) \) and \( (a_{n+1}, \ldots, a_k) >_L (b_{n+1}, \ldots, b_k) \),

where \( >_L \) is the lexicographic order on \( C_k \). Clearly \( \{G_i\}_{i=1}^{n+k} \), where \( G_i = C_k \) for \( i > n \), is a consistent ordered family.

**Theorem 3.** Let \( n > 0 \), \( j, k \geq 0 \). If \( \{G_i\}_{i=1}^{n} \) is a consistent ordered family with nested optimal subsets and for all \( i \) the graph \( G_i \times C_k \) has nested optimal subsets, with the lexicographic order as its optimal order, then \( \{G_i\}_{i=1}^{n+k} \), where \( G_i = C_k \) for \( n < i \), is a consistent ordered family with nested optimal subsets.

**Proof.** We proceed by induction on \( n + k \). For \( n + k = 2 \), if \( k = 0 \) or 2 the claim is obvious. If \( n = k = 1 \), then by assumption \( G_1 \times C_3 \) has nested optimal subsets.

For \( n + k \geq 3 \), if \( k = 0 \) or \( n = 0 \), the result follows from the fact that \( \{G_i\}_{i=1}^{n} \) is a consistent ordered family with nested optimal subsets and the fact that \( C_3^k \) is optimally ordered by the lexicographic order [5]. Otherwise, assume \( k \geq 1 \) and \( n \geq 1 \). Let \( G = G_1 \times \cdots \times G_n \). Let \( A \subseteq V_{G \times C_k^3} \) be a compressed optimal set, and let \( a = (a_1, \ldots, a_n, a_{n+1}, \ldots, a_k) = \max A \) and \( b = (b_1, \ldots, b_n, b_{n+1}, \ldots, b_k) = \min V_{G \times C_k^3} \setminus A \) be the least vertex not in \( A \). If \( A \) is an initial segment, we are done; otherwise \( a > b \). In this case, since \( A \) is compressed, \( a_i \neq b_i \) and \( a_i \neq b_i \) for all \( i \).

If \( (a_1, \ldots, a_k) >_L (0, \ldots, 0, 1) \), let \( (x_1, \ldots, x_k) \) be its lexicographic predecessor, and let \( (x'_1, \ldots, x'_k) \) be the lexicographic predecessor of \( (x_1, \ldots, x_k) \). Then by compression \( (a_1, \ldots, a_i + j, x'_1, \ldots, x'_k) \) and \( (a_1, \ldots, a_i + j, x'_1, \ldots, x'_k) \) are in \( A \). But \( b_k = x'_k \) or \( b_k = x_k' \), so by compression \( b \in A \), a contradiction.

Similarly, if \( (b_1, \ldots, b_k) <_L (0, \ldots, 0, 1) \), let \( (x''_1, \ldots, x''_k) \) be its lexicographic successor, and let \( (b'_1, \ldots, b''_k) \) be the successor of \( (b'_1, \ldots, b''_k) \). Since \( a_k = b_k' \) or \( a_k = b_k'' \), by compression \( (b_1, \ldots, b_k, b'_1, \ldots, b_k') \) is in \( A \) or \( (b_1, \ldots, b_k, b''_1, \ldots, b_k') \) is in \( A \), and in either case by compression \( b \in A \), a contradiction.

Hence, we can assume that \( a_m = x \) and \( b_m = 2 \) for \( m = 1, \ldots, k - 1 \), and \( b_k > x_k \). If \( n = 1 \) and \( a_1 > b_1 + 1 \), then \( k \geq 2 \), and

\[
a = (a_1, 0, \ldots, 0, a_k) > (a_1 - 1, 0, \ldots, 0, b_k) > (b_1, 2, \ldots, 2, b_k) = b,
\]
so by compression \( b \in A \), a contradiction. If \( a_1 = b_1 + 1 \), let \( B = (A \setminus \{ a \}) \cup \{ b \} \), and by Lemma 1
\[
|I_{G \times C_3^k}(B)| - |I_{G \times C_3^k}(A)| \geq -1 + 2(k - 1) + 1 > 0,
\]
contradicting the optimality of \( A \).

If \( n \geq 2 \), by Lemma 4, there is a vertex \( c = (c_1, \ldots, c_n) \in V_G \) with \( a > c > b \) and \( a_i = c_i \) for some \( i \), or \( \sum_{i=1}^{n} \delta_3(b_i + 1) \geq \sum_{i=1}^{n} \delta_3(a_i + 1) \). In the former case,
\[
a = (a_1, \ldots, a_n, a_1, \ldots, a_k)
\]
\[
> (c_1, \ldots, c_n, c_1, \ldots, c_k)
\]
\[
> (b_1, \ldots, b_n, b_1, \ldots, b_k) = b,
\]
so by compression \( b \in A \). In the later case, let \( B = (A \setminus \{ a \}) \cup \{ b \} \). Then by Lemma 4,
\[
|I_{G \times C_3^k}(B)| - |I_{G \times C_3^k}(A)| \geq 2k - 1 > 0,
\]
contradicting the optimality of \( A \). \( \square \)

We now turn our attention to the graph \( C_n \times C_i \) for \( i \geq 0 \), \( n > 5 \). If \( a = (a_1, \ldots, a_{i+1}) \in V_{C_n \times C_i} \) and \( (a_1, \ldots, a_{i+1}) \neq (n-1, 4, \ldots, 4) \), we define the successor of \( a \) by
\[
succ(a_1, \ldots, a_{i+1}) = \begin{cases} 
(a_1 + 1, a_2, \ldots, a_{i+1}) & \text{if } a_1 \in \{0, n-2\}, \\
(a_1 - 1, a'_2, \ldots, a'_{i+1})' & \text{if } a_1 \in \{1, n-1\} \text{ and } (a_2, \ldots, a_{i+1}) \neq (4, \ldots, 4), \\
(a_1 + 1, 0, \ldots, 0) & \text{if } a_1 \in \{1, 2\} \text{ and } (a_2, \ldots, a_{i+1}) = (4, \ldots, 4), \\
(a_2', \ldots, a'_{i+1}) & \text{if } a_1 \in \{2, \ldots, n-3\} \text{ and } (a_2, \ldots, a_{i+1}) \neq (4, \ldots, 4),
\end{cases}
\]
\[
(a_2', \ldots, a'_{i+1}) \text{ is the successor of } (a_2, \ldots, a_{i+1}) \text{ in the order } C_i \text{ whenever the latter}
\]
\[
\text{is not equal to } (4, \ldots, 4). \text{ The structure of this order is shown in Fig. 3.}
\]

With a proof nearly identical to that for \( C_3^k \), it is clear that the ordered family \( \{G_m\}_{m=1}^{i+1} \) where \( G_1 = C_n \) and \( G_m = C_3^k \) for all other \( m \) is consistent.

**Theorem 4.** The ordered family \( \{G_m\}_{m=1}^{i+1} \) where \( G_1 = C_n \) and \( G_m = C_3^k \) for all other \( m \) has nested optimal subsets.

**Proof.** This proof is very similar to the proof of Theorem 1, proceeding by induction on \( i \). When \( i = 0 \) the claim is obvious, and when \( i = 1 \), it is easy to prove. Assume \( i > 1 \), and let \( G = C_n \times C_3^k \). Let \( A \) be an optimal compressed set, and let \( a = (a_1, \ldots, a_{i+1}) \) be the greatest element in \( A \) and let \( b = (b_1, \ldots, b_{i+1}) \) be the least element not in \( A \). If \( A \) is an initial segment, the proof is done; otherwise \( a > b \). We proceed by cases, most of which correspond exactly to the cases in the proof of Theorem 1.
Case a: $a_1 - 1 > b_1$. If for some $m$ with $2 \leq m \leq n - 3$ we have $a_1 > m > b_1$, then we can use the analysis from case a1 from the proof of Theorem 1. The remaining subcases and the analysis thereof is identical to that found in Theorem 1.

Case b: $a_1 - 1 = b_1$ and $b_1 \in \{1, \ldots, n - 3\}$. The analysis of this case is identical to that of cases a2–a4.

Cases c, d, and e are identical to the corresponding cases of the theorem for $C_5$.

Let the discrete torus $T(0, i, j, k) = C_i^5 \times C_j^4 \times C_k^3$ for $i, j, k \geq 0$. We define an order $\mathcal{T}$ on $T(0, i, j, k)$ as follows: if $a = (a_1, \ldots, a_{i+j+k+1}) \in V_T(0, i, j, k)$, $b = (b_1, \ldots, b_{i+j+k+1}) \in V_T(0, i, j, k)$, then $a > \mathcal{T} b$ if and only if:

(i) $(a_1, \ldots, a_i) >_{Q_j^4} (b_1, \ldots, b_i)$, or
(ii) $(a_1, \ldots, a_i) = (b_1, \ldots, b_i)$ and $(a_{i+1}, \ldots, a_{i+j}) >_{Q_j^4} (b_{i+1}, \ldots, b_{i+j})$, or
(iii) $(a_1, \ldots, a_{i+j}) = (b_1, \ldots, b_{i+j})$ and $(a_{i+1}, \ldots, a_k) >_{Q_j^3} (b_{i+1}, \ldots, b_k)$,

where $Q_j^4$ denotes the lexicographic order on $C_j^4$ and $Q_j^3$ denotes the lexicographic order on $C_j^3$.

Let the discrete torus $T(n, i, j, k) = C_n \times C_i^5 \times C_j^4 \times C_k^3$ for $i, j, k \geq 0$ and $n > 5$. If $a = (a_1, \ldots, a_{i+j+k+1})$, $b = (b_1, \ldots, b_{i+j+k+1}) \in V_T(n, i, j, k)$, we let $a > \mathcal{T} b$ if and only if:

(i) $(a_1, \ldots, a_{i+1}) >_{C_n \times C_i^5} (b_1, \ldots, b_{i+1})$, or
(ii) $(a_1, \ldots, a_{i+1}) = (b_1, \ldots, b_{i+1})$ and $(a_{i+2}, \ldots, a_{i+j+1}) >_{Q_j^4} (b_{i+2}, \ldots, b_{i+j+1})$, or
(iii) $(a_1, \ldots, a_{i+j+1}) = (b_1, \ldots, b_{i+j+1})$ and $(a_{i+2}, \ldots, a_{i+j+k+1}) >_{Q_j^3} (b_{i+2}, \ldots, b_{i+j+k+1})$.

This defines an order $\mathcal{T}$ on $V_T(n, i, j, k)$.

**Theorem 5.** For $n > 5$ and $i, j, k \geq 0$, the discrete tori $T(0, i, j, k)$ and $T(n, i, j, k)$ have nested optimal subsets, with the optimal order given.
Proof. This follows from the fact that \( \{C_3\}^n_{i=1} \) and \( \{C_k\} \cup \{C_3\}^n_{i=2} \) are consistent ordered families with nested optimal subsets, using Corollary 2 and Theorem 3.

5. Further conjectures and open problems

We have presented a solution for the edge-isoperimetric problem for powers of \( C_5 \), the only previously unsolved case for cycles. We have also proven that discrete tori of the forms \( C^i \times C^j \times C^k \) and \( C_n \times C^i \times C^j \times C^k \) for \( i, j, k \geq 0 \) and \( n \geq 5 \) has nested optimal subsets, and have given an optimal order for such tori, thus generalizing results from [4,5]. We have found no other discrete torus with nested optimal subsets, and this leads us to the following conjecture.

**Conjecture 1.** If a discrete torus \( T \) has nested optimal subsets, it is of one of the forms \( C^i \times C^j \times C^k \) or \( C_n \times C^i \times C^j \times C^k \).

Indeed, it seems that this conjecture should be intuitively true, since any other discrete torus would contain the product \( C_n \times C_m \) for \( n, m \geq 5 \), which does not have nested optimal subsets. However, it is an open problem when, if \( G \times H \) has nested optimal subsets, you can show that \( G \) and \( H \) have nested optimal subsets.

Note that there is nothing special about \( C_4 \), that is about even powers of \( P_2 \): from the proofs of last section it is clear that \( T(0, i, j, k) \times P_2 \) and \( T(n, i, j, k) \times P_2 \) have nested optimal subsets for \( i, j, k \geq 0 \) and \( n \geq 5 \).

It is also unclear when there is an ordered family with nested optimal subsets which is not clearly order-isomorphic, by symmetry, to a consistent ordered family with nested optimal subsets. All ordered families with nested optimal subsets presented in the literature are consistent.

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References
