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An axiomatic approach of fuzzy rough sets based on residuated lattices[☆]

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ABSTRACT

Rough set theory was developed by Pawlak as a formal tool for approximate reasoning about data. Various fuzzy generalizations of rough approximations have been proposed in the literature. As a further generalization of the notion of rough sets, *L*-fuzzy rough sets were proposed by Radzikowska and Kerre. In this paper, we present an operator-oriented characterization of *L*-fuzzy rough sets, that is, *L*-fuzzy approximation operators are defined by axioms. The methods of axiomatization of *L*-fuzzy upper and *L*-fuzzy lower set-theoretic operators guarantee the existence of corresponding *L*-fuzzy relations which produce the operators. Moreover, the relationship between *L*-fuzzy rough sets and *L*-topological spaces is obtained. The sufficient and necessary condition for the conjecture that an *L*-fuzzy interior (closure) operator derived from an *L*-fuzzy topological space can associate with an *L*-fuzzy reflexive and transitive relation such that the corresponding *L*-fuzzy lower (upper) approximation operator is the *L*-fuzzy interior (closure) operator is examined.

1. Introduction

The theory of rough sets, proposed by Pawlak [1], is a formal tool for the study of intelligent systems characterized by insufficient and incomplete information. By using the concepts of lower and upper approximations in rough set theory, knowledge hidden in information systems may be unraveled and expressed in the form of decision rules.

It is well known that the most important concepts in original rough set theory are the upper and the lower approximators derived from a binary relation R on the universe of discourse. In recent years, two different basic ways have been formed for developing rough set theories, i.e., the constructive way and the axiomatic way. In the constructive approach, a binary relation R on the universe of discourse U is fixed firstly and then starting from R the upper and the lower approximators on the power set $\mathcal{P}(U)$ are defined and investigated. Diverse forms of the constructive approach have been proposed, and some of them are discussed in conjunction with additional topological or algebraic structures on U, and sometimes a collection of binary relations on U are considered (see, e.g., [2–12]). In contrast to the constructive approach, the axiomatic approach considers abstract upper and lower approximators subject to certain axioms the primitive notions, and seeks for conditions restraining the axioms to guarantee the existence of a binary relation R on U such that the abstract upper and the abstract lower approximators can be derived from R in the usual way.

The research work of the axiomatic approach has been carried out by many authors in the study of rough set theory. Zakowski [13] studied a set of axioms on approximation operators. In [14,15] Cormer investigated axioms on approximation operators in relation to cylindric algebras. Within the framework of topological spaces, Lin and Liu [16] suggested six

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axioms on a pair of abstract operators on the power set of a universe of discourse. The similar results were stated earlier by Wiweger [17]. However, all those studies are restricted to Pawlak rough set algebra defined by equivalence relations. Wybraniel–Skarodowska examined many axioms on various classes of approximation operators and proposed several constructive methods to generate them. Thiele [18] explored axiomatic characterizations of approximation operators within modal logic for a crisp diamond and box operator represented by an arbitrary binary crisp relation. The most interesting axiomatic studies on crisp rough sets were reported by Yao [19,20], who extended axiomatic approach to rough set algebras constructed from arbitrary binary relations.

The research of the axiomatic approach as well as the constructive approach have also been extended to approximation operators in fuzzy environment. Morsi and Yakout [21] studied a set of axioms on fuzzy rough sets based on a triangular norm and a residual implicator, but their studies were restricted to the fuzzy rough set algebras constructed by fuzzy equivalence relations which were equivalence crisp relations when they degenerated into crisp ones. Thiele [22–24] investigated axiomatic characterizations of fuzzy rough approximation operators and rough fuzzy approximation operators within modal logic for fuzzy diamond and box operators. In [25], based on a fuzzy similarity relation, Radzikowska and Kerre define a broad family of the so called $(\mathcal{I}, \mathcal{T})$-fuzzy rough sets which is determined by an implicator $\mathcal{I}$ and a triangular norm $\mathcal{T}$. However, the properties and axiomatic characterization of $(\mathcal{I}, \mathcal{T})$- fuzzy rough sets corresponding to an arbitrary fuzzy relation or a special fuzzy relation have not been studied. Wu et al. [26–28] examined many axioms on various classes of rough fuzzy and fuzzy rough approximation operators. Mi and Zhang [29] discussed axiomatic characterization of a pair of dual lower and upper fuzzy approximation operators based on a residual implication. Moreover, Liu [30,31] extended the axiomatic approach to generalized rough sets over fuzzy lattices, and Zhu [32,33] proposed the axiomatic system for covering based rough set model. In [34], a further generalization of the notion of rough sets, called L-fuzzy rough sets, has been proposed by Radzikowska and Kerre. It differs from fuzzy rough sets extensively investigated in [22–29,35,36] in that it takes a complete residuated lattice L as its basic structure. This is a fairly wide constructive setting because diverse residuated pairs can be chosen and, in case $L = [0, 1]$, the fuzzy rough set theory follows. However, the axiomatic characterization of L-fuzzy rough sets corresponding to an arbitrary L-fuzzy relation or a special L-fuzzy relation has not been studied, and the aim of the present paper is to investigate and to solve these questions.

In this paper, we aim to present axiomatic characterization of the theory of fuzzy rough sets based on residuated lattices (i.e., L-fuzzy rough sets). The paper is organized as follows: In Section 2, we recall some properties of residuated lattices. In Section 3, we briefly recall foundations of L-fuzzy sets and L-fuzzy relations. Further study of the properties of L-fuzzy rough sets is given in Section 4. The main results of the present paper are given in Section 5, where the axiomatic characterization of various L-fuzzy rough sets are investigated. In Section 6, relationship between L-fuzzy rough sets and L-topological spaces is established. The paper is completed with some concluding remarks.

2. Residuated lattices

The following definitions and propositions can be found in [37,38].

Definition 2.1. A residuated lattice L is a structure $(L, \otimes, \to, \vee, \wedge, 0, 1)$, where $(L, \vee, \wedge, 0, 1)$ is a bounded lattice with the greatest element 1 and the smallest element 0; $(L, \otimes, 1)$is a monoid and $(\otimes, \to)$ is an adjoint pair on L.

Given a residuated lattice $(L, \otimes, \to, \vee, \wedge, 0, 1)$, let us define a unary operator $\rightharpoondown$, referred to as the precomplement operator, by $\rightharpoondown a = a \to 0$.

In what follows, $\otimes$ is sometimes called generalized triangular norm and $\to$ is called the residuum of $\otimes$. An implicator I is called left monotonic (resp. right monotonic) iff for every $a \in L$, $I(., a)$ is decreasing (resp. $I(a, .)$ is increasing). If I is both left monotonic and right monotonic, then it is called hybrid monotonic.

From the above definition, we have the following proposition

Proposition 2.1. *Let $(L, \vee, \wedge, 0, 1)$ be a bounded lattice, then $(L, \otimes, \to, \vee, \wedge, 0, 1)$ is a residuated lattice iff the following properties hold:*

(1) *$\otimes$ is isotone in both arguments, that is, for $a \le b a \otimes c \le b \otimes c$ holds $(a, b, c \in L)$,*
(2) *$\to$ is hybrid monotonic,*
(3) *$a \otimes b \le c$ iff $a \le b \to c$,*
(4) *$\otimes$ is commutative and associative,*
(5) *$1 \otimes a = a \otimes 1 = a$.*

Proposition 2.2. *In any residuated lattice $(L, \otimes, \to, \vee, \wedge, 0, 1)$ the following properties hold for all $a, b, c \in L$.*

(6) $a \le (b \to a \otimes b)$,
(7) $a \otimes (a \to b) \le b$,
(8) $a \otimes (\vee_{i\in I} b_i) = \vee_{i\in I}(a \otimes b_i)$,
(9) $a \to \wedge_{i\in I} b_i = \wedge_{i\in I}(a \to b_i)$; $\vee_{i\in I} a_i \to b = \wedge_{i\in I}(a_i \to b)$,
(10) $a \le \rightharpoondown\rightharpoondown a$,
(11) $a \to \rightharpoondown b = \rightharpoondown (a \otimes b)$,
(12) $\rightharpoondown (\vee_{i\in I} a_i) = \wedge_{i\in I}(\rightharpoondown a_i)$,
(13) $1 \to a = a$,

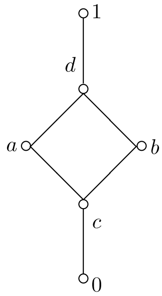

Fig. 1. The lattice L.

Table 1
The precomplement operator $\rightharpoondown$ in L.

x	0	d	a	b	c	1
$\rightharpoondown x$	1	c	b	a	d	0

(14) $a \to (b \to a) = 1$,
(15) $a \to (b \to c) = b \to (a \to c)$,
(16) $a \to (b \to a \otimes b) = 1$.

Residuated lattices, introduced by Dilworth and Ward in [37], are a common structure among algebras associated with logical systems. As an important and ideal structure, residuated lattices play an extremely important role in modern fuzzy logic theory. Many famous logic algebras widely studied in [39–42] are all obtained from residuated lattices by adding some additional axioms.

Definition 2.2. Let $(L, \otimes, \to, \vee, \wedge, 0, 1)$ be a residuated lattice, $\rightharpoondown$ be defined as above. If $\rightharpoondown\rightharpoondown a = a$ holds for every $a \in L$, then $(L, \otimes, \to, \vee, \wedge, 0, 1)$ is called a regular residuated lattice.

The following proposition provides basic properties of regular residuated lattices.

Proposition 2.3. *Let* $(L, \otimes, \to, \vee, \wedge, 0, 1)$ *be a regular residuated lattice, then the following conditions hold:*
(1) $a \to\rightharpoondown b = b \to\rightharpoondown a, \rightharpoondown a \to b =\rightharpoondown b \to a$,
(2) $a \otimes b =\rightharpoondown (a \to\rightharpoondown b)$,
(3) $a \to b =\rightharpoondown (a\otimes \rightharpoondown b)$,
(4) $\rightharpoondown a \to (a \to b) = 1$,
(5) $\rightharpoondown \wedge_{i\in I} a_i = \vee_{i\in I} \rightharpoondown a_i$,
where $a, b, a_i \in L(\forall i \in I)$.

It follows from (2) and (3) that the operator $\otimes$ can be defined by $\to$ and vice versa, hence they are not independent.

Example 2.1. Let L be a lattice depicted in Fig. 1.

The precomplement operator $\rightharpoondown$ is given in Table 1.
The implication operator $\to$ and the generalized triangular norm $\otimes$ in L are defined as follows.
$\forall x, y \in L$,

$$x \to y = \begin{cases} 1, & x \le y, \\ \rightharpoondown x \vee y, & x \not\le y, \end{cases}$$

$$x \otimes y = \begin{cases} 0, & x \le\rightharpoondown y, \\ x \wedge y, & x \not\le\rightharpoondown y. \end{cases}$$

It can be easily verified that $(L, \otimes, \to, \vee, \wedge, 0, 1)$ is a regular residuated lattice. However, a general residuated lattice is not necessarily regular, as shown in the following example.

Example 2.2. Take $L = [0, 1]$, the adjoint pair $(\otimes, \to)$ is defined by: $\forall x, y \in L$,

$$x \otimes y = x \wedge y,$$

$$x \to y = \begin{cases} 1, & x \le y, \\ y, & x > y. \end{cases}$$

It can be easily verified that $([0, 1], \otimes, \to, \vee, \wedge, 0, 1)$ is a residuated lattice, but not regular because for any $a \in (0, 1), \rightharpoondown\rightharpoondown a = 1 \ne a$.

3. L-fuzzy sets and L-fuzzy relations

3.1. L-fuzzy sets

Let L be a complete residuated lattice. The concept of L-fuzzy set was first introduced by Goguen [43] and it was considered as a generalization of the notion of Zadeh's fuzzy sets. In what follows, we will first recall the specific definitions of L-fuzzy sets.

Definition 3.1. Let L be a complete lattice and let U be a nonempty set called the universe of discourse. A mapping $A : U \longrightarrow L$ is called an L-fuzzy set in U.

Given two L-fuzzy sets A and B, we can define new L-fuzzy sets as follows ($\forall x \in U$):

$$(A \otimes B)(x) = A(x) \otimes B(x), \tag{1}$$
$$(A \cap B)(x) = A(x) \wedge B(x), \tag{2}$$
$$(A \cup B)(x) = A(x) \vee B(x), \tag{3}$$
$$(A \rightarrow B)(x) = A(x) \rightarrow B(x), \tag{4}$$
$$(\rightharpoondown A)(x) = \rightharpoondown A(x). \tag{5}$$

Furthermore, suppose that $\hat{a}$ is the constant L- fuzzy set, i.e., $\hat{a}(x) = a$ holds for every $x \in U$. Moreover, $A \subseteq B$ means $A(x) \leq B(x)$ for every $x \in U$. Lastly, for every family $(A_i)_{i\in I}$ of L- fuzzy sets, we define the L- fuzzy sets $\cup_{i\in I} A_i$ and $\cap_{i\in I} A_i$ as follows, respectively:

$$(\cup_{i\in I} A_i)(x) = \vee_{i\in I} A_i(x), \quad x \in U, \tag{6}$$
$$(\cap_{i\in I} A_i)(x) = \wedge_{i\in I} A_i(x), \quad x \in U. \tag{7}$$

3.2. L-fuzzy relations

Definition 3.2. Let U be a nonempty universe of discourse. An L-fuzzy set $R : U \times U \longrightarrow L$ is referred to as an L-fuzzy binary relation on U. $R(x, y)$ is the degree of relation between x and y, where $(x, y) \in U \times U$.

Definition 3.3. Let R be an L-fuzzy relation on U. The relation R is serial if for all $x \in U$, $\vee_{y\in U} R(x, y) = 1$. R is reflexive if $R(x, x) = 1$ holds for all $x \in U$. R is symmetric if $R(x, y) = R(y, x)$ holds for all $x, y \in U$. R is $\mathcal{T}$ transitive if $R(x, z) \geq \vee_{y\in U} R(x, y) \otimes R(y, z)$ holds for all $x, y, z \in U$. R is a $\mathcal{T}$-similarity relation if it is reflexive, symmetric and $\mathcal{T}$-transitive.

In what follows, the set of all L-fuzzy relations on U will be denoted by $\mathcal{F}_L(U \times U)$.

4. Fuzzy rough sets based on residuated lattice

The concept of fuzzy rough sets based on residuated lattice were proposed by Radzikowska and kerre in [30]. It differs from the concept of fuzzy rough sets investigated in [22–29], because it takes a general residuated lattice instead of [0, 1] as its basic structure.

4.1. Properties of L-fuzzy approximators $\overline{R}$ and $\underline{R}$

Definition 4.1. Let $(L, \otimes, \rightarrow, \vee, \wedge, 0, 1)$ be a complete residuated lattice, U be a nonempty universe of discourse and R be an L-fuzzy binary relation on U. Then a pair (U, R) is called an L-fuzzy approximation space.

Definition 4.2. Let $(L, \otimes, \rightarrow, \vee, \wedge, 0, 1)$ be a complete residuated lattice and (U, R) be an L-fuzzy approximation space. Define two mappings $\overline{R}, \underline{R} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$, called lower and upper L-fuzzy rough approximations respectively, as follows: for every $A \in \mathcal{F}_L(U)$and every $x \in U$.

$$\underline{R}(A)(x) = \wedge_{y\in U}(R(x, y) \longrightarrow A(y)), \tag{8}$$
$$\overline{R}(A)(x) = \vee_{y\in U}(R(x, y) \otimes A(y)). \tag{9}$$

$\underline{R}(A)$ (resp. $\overline{R}(A)$) is called the lower (resp. upper) L-fuzzy rough approximation of A. It can be easily checked that fuzzy rough sets defined above is a wide generalization of $(\mathcal{I}, \mathcal{T})$-fuzzy rough sets [25,28], where $\mathcal{T}$ is a t-norm and $\mathcal{I}$ is the residual implicator based on $\mathcal{T}$.

Notice that in case [0, 1] and both A and R are crisp subsets of U and $U \times U$ respectively, $\underline{R}(A)$ and $\overline{R}(A)$ will turn to be the corresponding concepts in classical rough set theory, hence Definition 4.2 is a reasonable generalization of its classical counterpart. Nevertheless, $\underline{R}(A) \subseteq \overline{R}(A)$, which is true in classical case, needs not to be true in general. Even more, the reverse conclusion $\overline{R}(A) \subseteq \underline{R}(A)$ may happen as shown in Proposition 4.3(2) below.

The following proposition in [34] provides basic properties of the lower and upper L-fuzzy rough approximation operators.

Proposition 4.1. *Let $(L, \otimes, \rightarrow, \vee, \wedge, 0, 1)$ be a complete residuated lattice. Then the L-fuzzy rough approximation operators $\overline{R}$ and $\underline{R}$ possess the following properties:*
For all $A \in \mathcal{F}_L(U)$ and $A_i \in \mathcal{F}_L(U)$ $(\forall i \in I)$

(1) $\overline{R}(\emptyset) = \emptyset$, $\underline{R}(U) = U$,
(2) *if* $A \subseteq B$, *then* $\underline{R}(A) \subseteq \underline{R}(B)$ *and* $\overline{R}(A) \subseteq \overline{R}(B)$,
(3) $\underline{R}(A) \subseteq \neg\, \overline{R}(\neg\, A)$, $\overline{R}(A) \subseteq \neg\, \underline{R}(\neg\, A)$, $\neg\, \overline{R}(A) = \underline{R}(\neg\, A)$,
(4) $\overline{R}(\cup_{i\in I} A_i) = \cup_{i\in I} \overline{R}(A_i)$, $\underline{R}(\cap_{i\in I} A_i) = \cap_{i\in I} \underline{R}(A_i)$.

As shown in Example 2.2, a general complete residuated lattice is not necessarily regular. In view of this fact and Proposition 4.1(3), we can safely conclude that L-fuzzy rough approximation operators $\underline{R}$ and $\overline{R}$ are not necessarily dual to each other. However, the following proposition shows that the rough approximators $\underline{R}$ and $\overline{R}$ are dual to each other whenever regular residuated lattices are taken as basic structures.

Proposition 4.2. *Let $(L, \otimes, \rightarrow, \vee, \wedge, 0, 1)$ be a complete regular residuated lattice. Then for every L-fuzzy approximation space (U, R) and every $A \in \mathcal{F}_L(U)$ the duality property holds: i.e., $\forall A \in \mathcal{F}_L(U)$*

$$\overline{R}(A) = \neg\, \underline{R}(\neg\, A), \qquad \underline{R}(A) = \neg\, \overline{R}(\neg\, A). \tag{10}$$

Proof. $\overline{R}(A) = \neg\, \underline{R}(\neg\, A)$ follows immediately from the third part of Proposition 4.1(3) and the regularity of residuated lattices.

$\underline{R}(A) = \neg\, \overline{R}(\neg\, A)$ is proved as follows.

First, we have from the first part of (3) in Proposition 4.3 that $\underline{R}(A) \subseteq \neg\, \overline{R}(\neg\, A)$.

Next, $\forall x \in U$, we have

$$\begin{aligned}(\neg\, \overline{R}(\neg\, A))(x) &= \neg\, \overline{R}(\neg\, A)(x)\\ &= \neg \vee_{y\in U}(R(x,y) \otimes \neg\, A(y))\\ &= \wedge_{y\in U} \neg\, (R(x,y) \otimes \neg\, A(y))\\ &= \wedge_{y\in U}(R(x,y) \rightarrow A(y))\\ &= \underline{R}(A)(x).\end{aligned}$$

This shows $\underline{R}(A) \supseteq \neg\, \overline{R}(\neg\, A)$. Hence $\underline{R}(A) = \neg\, \overline{R}(\neg\, A)$. □

Remark 4.1. Regular residuated lattices are a class of mathematical structures widely applied in modern fuzzy logic theory, especially in the fuzzy logic system with the negative connective. Many famous logic algebras such as MV algebra, R_0 algebra and IMTL algebra [39–42] are all obtained by adding some additional axioms. Hence, if we take these algebras as the basic structure, the corresponding lower and upper approximators are dual to each other.

In addition to the properties stated in the Propositions in [34], the properties listed below also hold for the L-fuzzy lower and upper approximators in every L-fuzzy approximation space (U, R).

Proposition 4.3. *Let (U, R) be a L-fuzzy approximation space. Then the L-fuzzy lower and L-fuzzy upper approximators $\underline{R}$ and $\overline{R}$ have the following properties:*

(1) $\overline{R}(\hat{a} \otimes A) = \hat{a} \otimes \overline{R}(A)$, $\underline{R}(\hat{a} \rightarrow A) = \hat{a} \rightarrow \underline{R}(A)$,
(2) $\overline{R}(\hat{a}) \subseteq \hat{a}$, $\hat{a} \subseteq \underline{R}(\hat{a})$,
(3) $\overline{R}(\hat{a}) = \hat{a} \Leftrightarrow \overline{R}(U) = U$, $\underline{R}(\hat{a}) = \hat{a} \Leftrightarrow \underline{R}(\emptyset) = \emptyset$,
(4) $\overline{R}(1_y \otimes \hat{a})(x) = R(x, y) \otimes a$, $\underline{R}(1_y \rightarrow \hat{a})(x) = R(x, y) \rightarrow a$,
(5) $\overline{R}(1_y)(x) = R(x, y)$, $\underline{R}(1_{U-\{y\}})(x) = \neg\, R(x, y) = R(x, y) \rightarrow 0$.

Proof. (1) $\forall x \in U$, we have

$$\begin{aligned}\overline{R}(\hat{a} \otimes A)(x) &= \vee_{y\in U}(R(x,y) \otimes (\hat{a} \otimes A)(y))\\ &= \vee_{y\in U}(R(x,y) \otimes (a \otimes A(y)))\\ &= a \otimes \vee_{y\in U}(R(x,y) \otimes A(y))\\ &= a \otimes \overline{R}(A)(x) = (\hat{a} \otimes \overline{R}(A))(x).\end{aligned}$$

Hence $\overline{R}(\hat{a} \otimes A) = \hat{a} \otimes \overline{R}(A)$.

$\underline{R}(\hat{a} \rightarrow A) = \hat{a} \rightarrow \underline{R}(A)$ can be proved in a similar way.

(2) $\forall x \in U$, we have

$$\begin{aligned}\overline{R}(\hat{a})(x) &= \vee_{y\in U}(R(x,y)\otimes \hat{a}(y))\\ &= \vee_{y\in U}(R(x,y)\otimes a)\\ &\le \vee_{y\in U}(1\otimes a) = a.\end{aligned}$$

Therefore, $\overline{R}(\hat{a}) \le \hat{a}$. The proof of $\hat{a} \le \underline{R}(\hat{a})$ can be given in a similar way as above.
(3) ($\Rightarrow$) It can be easily obtained by taking $a = 1$.
($\Leftarrow$) $\forall x \in U$, we have

$$\begin{aligned}\overline{R}(\hat{a})(x) &= \vee_{y\in U}(R(x,y)\otimes \hat{a}(x))\\ &= a\otimes \vee_{y\in U} R(x,y)\\ &= a\otimes 1\\ &= a = \hat{a}(x).\end{aligned}$$

The other can be proved similarly.
(4) $\forall x \in U$, we have

$$\begin{aligned}\overline{R}(1_y\otimes \hat{a})(x) &= \vee_{z\in U}(R(x,z)\otimes (1_y\otimes \hat{a})(z))\\ &= \vee_{z\in U, z\neq y}(R(x,z)\otimes (1_y\otimes \hat{a})(z)) \vee (R(x,y)\otimes (1_y\otimes \hat{a})(y))\\ &= 0\vee (R(x,y)\otimes a)\\ &= R(x,y)\otimes a.\end{aligned}$$

Similarly, we can prove $\underline{R}(1_y \to \hat{a})(x) = R(x,y) \to a$.
(5) They can be obtained by taking $a = 1$ and $a = 0$ in (4), respectively. □

4.2. Selected classes of fuzzy rough sets based on residuated lattices

In this subsection, several classes of L-fuzzy rough sets are considered and it will be shown that properties of some special fuzzy relations can be characterized by L-fuzzy approximation operators.

Proposition 4.4. *Let L be a complete MTL algebra and (U, R) be an L-fuzzy approximation space, then the following conditions are equivalent:*

(1) *R is serial,*
(2) $\underline{R}(A) \subseteq \overline{R}(A)$, $\forall A \in \mathcal{F}_L(U)$.

The proof of Proposition 4.4 can be found in [34]. For general residuated lattices, we can obtain the following.

Proposition 4.5. *For a complete residuated lattice L and an L-fuzzy approximation space (U, R), R is serial if and only if one of the following properties holds:*

(1) $\overline{R}(\hat{a}) = \hat{a}$, $\forall a \in L$,
(2) $\overline{R}(U) = U$.

Proof. We see from Proposition 4.3(3) that $\overline{R}(\hat{a}) = \hat{a} \Leftrightarrow \overline{R}(U) = U$. In the following we only need to show that R is serial iff $\overline{R}(\hat{a}) = \hat{a}$ holds.
($\Rightarrow$) $\forall a \in L, x \in U$,

$$\begin{aligned}\overline{R}(\hat{a})(x) &= \vee_{y\in U}(R(x,y)\otimes \hat{a}(y))\\ &= \vee_{y\in U}(R(x,y)\otimes a)\\ &= a\otimes \vee_{y\in U} R(x,y)\\ &= a\otimes 1 = a.\end{aligned}$$

($\Leftarrow$) Take $a = 1$, then it follows from the proof of necessity and $\overline{R}(\hat{1}) = \hat{1}$ that $\vee_{y\in U} R(x,y) = 1$ holds for every $x \in U$. Hence R is serial. □

In addition to the properties stated in the above proposition, we have the following

Proposition 4.6. *For every complete residuated lattice L and L-fuzzy approximation space (U, R), R is serial iff $\underline{R}(\hat{a}) = \hat{a}$, $\forall a \in L$.*

Proof. ($\Rightarrow$) $\forall a \in L, x \in U$,

$$\begin{aligned}\underline{R}(\hat{a})(x) &= \wedge_{y\in U}(R(x,y)\to \hat{a}(y))\\ &= \wedge_{y\in U}(R(x,y)\to a)\\ &= \vee_{y\in U} R(x,y)\to a\\ &= 1\to a = a.\end{aligned}$$

($\Leftarrow$) Conversely, if $\underline{R}(\hat{a}) = \hat{a}$ ($a \in L$) holds, then it follows from the proof of the necessity that $\vee_{y\in U} R(x, y) \rightarrow a = a$. Then $\vee_{y\in U} R(x, y) = 1$ holds for every $x \in U$. In fact, if there exists some $x \in U$ satisfying $\vee_{y\in U} R(x, y) < 1$, then for $a = \vee_{y\in U} R(x, y)$ we have $1 = \vee_{y\in U} R(x, y) \rightarrow a = a$. This is a contradiction. □

Remark 4.2. If L is a complete regular residuated lattice, then by duality of $\underline{R}$ and $\overline{R}$ we know that R is serial iff $\underline{R}(\emptyset) = \emptyset$. However, this does not necessarily hold when L is not regular.

Example 4.1. Let $U = \{x_1, x_2\}$, and $L = ([0, 1], \otimes, \rightarrow, \vee, \wedge, 0, 1)$ be a residuated lattice, where $\otimes$ is the *Gödel*-t norm (that is, $a \otimes b = a \wedge b$ for all $a, b \in U$) and $\rightarrow$ is the residual implication of $\otimes$. R is defined as follows:

$$R = \begin{bmatrix} 0.2 & 0.3 \\ 0.5 & 0.6 \end{bmatrix}.$$

It can be easily checked that $\underline{R}(\emptyset)(x_i) = 0$ $(i = 1, 2)$, and hence $\underline{R}(\emptyset) = \emptyset$. However, R is not serial.

The following propositions characterize the relationship between the special L-fuzzy relations and the special properties possessed by the corresponding L-fuzzy rough approximators. Their proof can be found in [34].

Proposition 4.7. *Let L be a complete residuated lattice, (U, R) be an L-fuzzy approximation space, then R is reflexive iff $\underline{R}(A) \subseteq A$ iff $A \subseteq \overline{R}(A)$ for every $A \in \mathcal{F}_L(U)$.*

Proposition 4.8. *Let $(L, \otimes, \rightarrow, \vee, \wedge, 0, 1)$ be a complete residuated lattice, (U, R) be an L-fuzzy approximation space, then R is symmetric iff $\overline{R}(\underline{R}(A)) \subseteq A$ iff $A \subseteq \underline{R}(\overline{R}(A))$ for every $A \in \mathcal{F}_L(U)$.*

Furthermore, we have from Proposition 4.3 the following.

Proposition 4.9. *Let $(L, \otimes, \rightarrow, \vee, \wedge, 0, 1)$ be a complete residuated lattice, (U, R) be an L-fuzzy approximation space, then R is symmetric iff $\overline{R}(1_y)(x) = \overline{R}(1_x)(y)$ iff $\underline{R}(1_x \rightarrow \hat{a})(y) = \underline{R}(1_y \rightarrow \hat{a})(x)$ for all $x, y \in U$ and $a \in L$.*

Proof. By Proposition 4.3, we only need to show that R is symmetric iff $\underline{R}(1_x \rightarrow \hat{a})(y) = \underline{R}(1_y \rightarrow \hat{a})(x)$ for all $x, y \in U$ and $a \in L$.

($\Rightarrow$) It follows immediately from Proposition 4.3(4).

($\Leftarrow$) $\underline{R}(1_x \rightarrow \hat{a})(y) = \underline{R}(1_y \rightarrow \hat{a})(x)$ ($\forall x, y \in U, a \in L$) means $R(x, y) \rightarrow a = R(y, x) \rightarrow a$ for all $a \in L$. By taking $a = R(x, y)$ and $a = R(y, x)$ respectively, $R(x, y) \leq R(y, x)$ and $R(y, x) \leq R(x, y)$ immediately follow. Hence $R(x, y) = R(y, x)$. □

Remark 4.3. If L is a regular residuated lattice, then it can be easily checked that R is symmetric iff $\underline{R}(1_{U-\{y\}})(x) = \underline{R}(1_{U-\{x\}})(y)$, $\forall x, y \in U$. But this property does not hold generally for the case of complete residuated lattice (see Example 4.1 above).

Proposition 4.10. *Let $(L, \otimes, \rightarrow, \vee, \wedge, 0, 1)$ be a complete residuated lattice, (U, R) be an L-fuzzy approximation space, then the following conditions are equivalent:*

(1) *R is Euclidean,*
(2) *$\overline{R}(A) \subseteq \underline{R}(\overline{R}(A))$ for every $A \in \mathcal{F}_L(U)$,*
(3) *$\overline{R}(\underline{R}(A)) \subseteq \underline{R}(A)$ for every $A \in \mathcal{F}_L(U)$.*

Proposition 4.11. *Let $(L, \otimes, \rightarrow, \vee, \wedge, 0, 1)$ be a complete residuated lattice, (U, R) be an L-fuzzy approximation space, then the following conditions are equivalent:*

(1) *R is $\mathcal{T}$-transitive,*
(2) *$\underline{R}(A) \subseteq \underline{R}(\underline{R}(A))$ for every $A \in \mathcal{F}_L(U)$,*
(3) *$\overline{R}(\overline{R}(A)) \subseteq \overline{R}(A)$ for every $A \in \mathcal{F}_L(U)$.*

5. Axiomatic characterization of *L*-fuzzy rough approximation operators

In this section, we will study the axiomatic characterizations of various L-fuzzy rough approximation operators. It will be shown that L-fuzzy rough approximation operators can be characterized by abstract L-fuzzy set-theoretic operators, which guarantee the existence of certain types of L-fuzzy binary relations producing the same L-fuzzy approximation operators.

In this section, unless stated otherwise, L always stands for a complete residuated lattice.

Definition 5.1. $\mathcal{H} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ is called an L-fuzzy upper approximation operator iff it satisfies the following axioms:

$(h1)$ $\mathcal{H}(\hat{a} \otimes A) = \hat{a} \otimes \mathcal{H}(A)$,
$(h2)$ $\mathcal{H}(\cup_{j\in J} A_j) = \cup_{j\in J} \mathcal{H}(A_j)$.

The following lemma is needed to prove Theorem 5.3 below.

Lemma 5.1. *For every $A \in \mathcal{F}_L(U)$, $A = \cup_{y\in U}(1_y \otimes \widehat{A(y)})$ holds.*

Proof.

$$\begin{aligned}\forall x \in U, \cup_{y\in U}(1_y \otimes A(\hat{y}))(x) &= \vee_{y\in U}(1_y(x) \otimes A(\hat{y})(x))\\ &= \vee_{y\in U, y\neq x}(1_y(x) \otimes A(\hat{y})(x)) \vee (1_x(x) \otimes A(\hat{x})(x))\\ &= 0 \vee (1 \otimes A(x))\\ &= A(x).\end{aligned}$$

Hence, $A = \cup_{y\in U}(1_y \otimes A(\hat{y}))$. □

Theorem 5.1. *$\mathcal{H} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ is an L-fuzzy upper approximation operator iff there exists an L-fuzzy binary relation R on U such that $\mathcal{H} = \overline{R}$.*

Proof. (⇐) It follows immediately from Proposition 4.1(4) and Proposition 4.3(1).
(⇒) Define an L-fuzzy binary relation R as follows:

$$\forall x, y \in U, \quad R(x, y) = \mathcal{H}(1_y)(x). \tag{11}$$

In the following, we will show that $\mathcal{H}(A) = \overline{R}(A)$ holds for every $A \in \mathcal{F}_L(U)$.

$$\begin{aligned}\forall A \in \mathcal{F}_L(U), \forall x \in U, \overline{R}(A)(x) &= \vee_{y\in U}(R(x, y) \otimes A(y)) \quad \text{by (9)}\\ &= \vee_{y\in U}(\mathcal{H}(1_y)(x) \otimes A(y)) \quad \text{by (11)}\\ &= \vee_{y\in U}(\mathcal{H}(1_y)(x) \otimes A(\hat{y})(x))\\ &= \vee_{y\in U}((\mathcal{H}(1_y) \otimes A(\hat{y}))(x)) \quad \text{by (1)}\\ &= (\cup_{y\in U}\, \mathcal{H}(1_y) \otimes A(\hat{y}))(x) \quad \text{by (6)}\\ &= (\cup_{y\in U}\, \mathcal{H}((1_y) \otimes A(\hat{y})))(x) \quad \text{by } (h1)\\ &= \mathcal{H}(\cup_{y\in U}(1_y \otimes A(\hat{y})))(x) \quad \text{by } (h1) \text{ and } (h2)\\ &= \mathcal{H}(A)(x). \quad \text{by Lemma 5.1} \quad \square\end{aligned}$$

We know from Theorem 5.1 that L-fuzzy upper approximation operator can be characterized by the axioms $(h1)$ and $(h2)$. The following theorem gives corresponding axiomatic characterization of L-fuzzy lower approximation operator (see Definition 5.2).

Definition 5.2. $\mathcal{L} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ is called an L-fuzzy lower approximation operator iff it satisfies the following axioms:
$(l1)$ $\mathcal{L}(\hat{a} \to A) = \hat{a} \to \mathcal{L}(A)$,
$(l2)$ $\mathcal{L}(\cap_{j\in J} A_j) = \cap_{j\in J}\, \mathcal{L}(A_j)$.

Theorem 5.2. *Suppose that L is a complete regular residuated lattice, then $\mathcal{L} : \mathcal{F}_L(U) \to \mathcal{F}_L(U)$ is an L-fuzzy lower approximation operator iff there exists an L-fuzzy binary relation R such that $\underline{R} = \mathcal{L}$.*

Proof. (⇐) It follows immediately from Proposition 4.1(4) and Proposition 4.3(1).
(⇒) Define an L-fuzzy binary relation R as follows: $\forall x, y \in U$

$$R(x, y) = \neg\, \mathcal{L}(1_{U-\{y\}})(x). \tag{12}$$

Then it follows immediately from the regularity of residuated lattice L that

$$R(x, y) \to 0 = \mathcal{L}(1_{U-\{y\}})(x). \tag{13}$$

Furthermore, it is not difficult to check

$$A = \cap_{y\in U}(1_{U-\{y\}} \cup A(\hat{y})) \tag{14}$$

and

$$\neg\, A(\hat{y}) \to 1_{U-\{y\}} = A(\hat{y}) \cup 1_{U-\{y\}}. \tag{15}$$

$\forall A \in \mathcal{F}_L(U),\ x \in U$, we have

$$\begin{aligned}\underline{R}(A)(x) &= \wedge_{y\in U}(R(x, y) \to A(y)) \quad \text{by (8)}\\ &= \wedge_{y\in U}(R(x, y) \to (\neg\, A(y) \to 0)) \quad \text{by the regularity of } L\\ &= \wedge_{y\in U}(\neg\, A(y) \to (R(x, y) \to 0)) \quad \text{by Proposition 2.2}\\ &= \wedge_{y\in U}(\neg\, A(y) \to \mathcal{L}(1_{U-\{y\}})(x)) \quad \text{by (13)}\end{aligned}$$

$$= \wedge_{y\in U}(\rightharpoondown \hat{A}(y)(x) \rightarrow \mathcal{L}(1_{U-\{y\}})(x))$$
$$= \wedge_{y\in U}(\rightharpoondown \hat{A}(y) \rightarrow \mathcal{L}(1_{U-\{y\}}))(x) \quad \text{by (4)}$$
$$= \wedge_{y\in U}\mathcal{L}(\rightharpoondown \hat{A}(y) \rightarrow 1_{U-\{y\}})(x) \quad \text{by}(l1)$$
$$= \wedge_{y\in U}\mathcal{L}(\hat{A(y)} \cup 1_{U-\{y\}})(x) \quad \text{by (15)}$$
$$= (\cap_{y\in U}\mathcal{L}(\hat{A(y)} \cup 1_{U-\{y\}}))(x) \quad \text{by (7)}$$
$$= \mathcal{L}(\cap_{y\in U}(\hat{A(y)} \cup 1_{U-\{y\}}))(x) \quad \text{by}(l2)$$
$$= \mathcal{L}(A)(x). \quad \text{by (14)}$$

This proves that $\underline{R}(A) = \mathcal{L}(A)$ and the proof of Theorem 5.2 is completed. □

We see from Theorem 5.2 that the L-fuzzy lower approximation operator can be characterized by axioms ($l1$) and ($l2$) if we take regular residuated lattice as the basic structure. It is still an open problem to decide whether the L-fuzzy lower approximation operator can be axiomatized for the case of general residuated lattice.

Remark 5.1. (1) In case $L = \{0, 1\}$, i.e., L is a residuated lattice containing only two elements. Then ($h1$) is equivalent to $\mathcal{H}(\emptyset) = \emptyset$ and ($l1$) is equivalent to $\mathcal{L}(U) = U$. ($h1$) and ($h2$) form the axiomatic system for the upper approximators of the generalized crisp rough set model, whereas ($l1$) and ($l2$) provide the axiomatic characterizations for the lower approximators in the generalized crisp rough set model. In this sense, the axiomatization of generalized crisp rough set model can be seen as a special case of our present results.

(2) In case $L = [0, 1]$, $\otimes$ is a triangular norm on L and $\rightarrow$ is its residual implicator. Then ($h1$), ($h2$) form the independent axioms for $\mathcal{T}$-upper fuzzy approximation operators in [28], and so the axiomatic characterization of $\mathcal{T}$-upper fuzzy approximation operators in [28] can also be brought into the framework of our present results. In addition, if $L = ([0, 1], \otimes, \rightarrow, \vee, \wedge, 0, 1)$ is regular in the sense of Definition 2.2, then ($l1$) and ($l2$) provide the axiomatic characterization of $\mathcal{I}$-lower fuzzy operators, which was studied in detail in [28].

In what follows, we will see that different axioms of L-fuzzy upper (lower) approximation operators correspond to different types of L-fuzzy binary relations which produce the same operators.

Theorem 5.3. *Suppose that $\mathcal{H} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ is an L-fuzzy upper approximation operator, then there exists an L-fuzzy serial relation R such that $\mathcal{H} = \overline{R}$ iff $\mathcal{H}$ satisfies the following condition:*

($h3$) $\mathcal{H}(\hat{a}) = \hat{a}, \forall a \in L$.

Proof. Since $\mathcal{H}$ is an L-fuzzy upper approximation operator, then by Theorem 5.1 there exists an L-fuzzy relation R' such that $\mathcal{H} = \overline{R'}$.

($\Rightarrow$) If there exists an L-fuzzy relation R satisfying $\mathcal{H} = \overline{R}$, then it can be easily verified that $R = R'$ because $\forall x, y \in U, R(x, y) = \mathcal{H}(1_y)(x) = R'(x, y)$. Moreover, since R is serial, applying Proposition 4.5 here, we have that $\forall a \in L, \overline{R}(\hat{a}) = \hat{a}$, i.e., $\mathcal{H}(\hat{a}) = \hat{a}$.

($\Leftarrow$) $\forall a \in L, \mathcal{H}(\hat{a}) = \hat{a}$ implies that $\forall a \in L, \overline{R'}(\hat{a}) = \hat{a}$. Take $R = R'$, then we have from Proposition 4.5 that R is serial. □

Remark 5.2. On account of Proposition 4.5, we see that ($h3$) in Theorem 5.3 can be replaced by $\mathcal{H}(U) = U$.

Theorem 5.4. *Suppose that L is a complete regular residuated lattice and $\mathcal{L} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ is an L-fuzzy lower approximation operator, then there exists an L-fuzzy serial relation R such that $\mathcal{L} = \underline{R}$ iff $\mathcal{L}$ satisfies the following condition:*

($l3$) $\mathcal{L}(\hat{a}) = \hat{a}, \forall a \in L$.

Proof. It can be proved in a similar way as that of Theorem 5.3. □

Similarly, we have the following theorems.

Theorem 5.5. *Suppose that $\mathcal{H} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ is an L-fuzzy upper approximation operator, then there exists an L-fuzzy reflexive relation R such that $\mathcal{H} = \overline{R}$ iff $\mathcal{H}$ satisfies the following condition:*

($h4$) *$A \subseteq \mathcal{H}(A)$ for every $A \in \mathcal{F}_L(U)$.*

Proof. Since $\mathcal{H} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ is an L-fuzzy upper approximation operator, then by Theorem 5.1 there exists an L-fuzzy relation R' such that $\mathcal{H} = \overline{R'}$.

($\Rightarrow$) If there exists an L-fuzzy relation R satisfying $\mathcal{H} = \overline{R}$, then it can be easily verified that $R = R'$ because $\forall x, y \in U, R(x, y) = \mathcal{H}(1_y)(x) = R'(x, y)$. Moreover, since R is reflexive, applying Proposition 4.7 here, we have that $\forall A \in \mathcal{F}_L(U), A \subseteq \overline{R}(A)$, i.e., $\forall A \in \mathcal{F}_L(U), A \subseteq \mathcal{H}(A)$.

($\Leftarrow$) $\forall A \in \mathcal{F}_L(U), A \subseteq \mathcal{H}(A)$ implies that $\forall A \in \mathcal{F}_L(U), A \subseteq \overline{R'}(A)$. Take $R = R'$, then it follows from Proposition 4.7 that R is reflexive. □

Theorem 5.6. *Suppose that L is a complete regular residuated lattice and $\mathcal{L} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ is an L-fuzzy lower approximation operator, then there exists an L-fuzzy reflexive relation R such that $\mathcal{L} = \underline{R}$ iff $\mathcal{L}$ satisfies the following condition:*
(*l*4) $\mathcal{L}(A) \subseteq A$ *for every* $A \in \mathcal{F}_L(U)$.

Proof. It can be proved in an analogous way as that of Theorem 5.5. □

Theorem 5.7. *Suppose that $\mathcal{H} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ is an L-fuzzy upper approximation operator, then there exists an L-fuzzy symmetric relation R such that $\mathcal{H} = \overline{R}$ iff $\mathcal{H}$ satisfies the following condition:*
(*h*5) $\mathcal{H}(1_y)(x) = \mathcal{H}(1_x)(y), \forall x, y \in U$.

Proof. (⇒) It follows immediately from Proposition 4.9.
(⇐) It follows immediately from Theorem 5.1 and Proposition 4.9. □

Theorem 5.8. *Suppose that L is a complete regular residuated lattice and $\mathcal{L} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ is an L-fuzzy lower approximation operator, then exists an L-fuzzy symmetric relation R such that $\mathcal{L} = \underline{R}$ iff $\mathcal{L}$ satisfies the following condition:*
(*l*5) $\mathcal{L}(1_x \to \hat{a})(y) = \mathcal{L}(1_y \to \hat{a})(x), \forall x, y \in U, a \in L$.

Proof. (⇒) It follows immediately from Proposition 4.9.
(⇐) It follows immediately from Theorem 5.2 and Proposition 4.9. □

Remark 5.3. On account of Remark 4.3, (*l*5) is equivalent to $\mathcal{L}(1_{U-\{y\}})(x)$=$\mathcal{L}(1_{U-\{x\}})(y)(\forall x, y \in U)$ for the case of regular residuated lattice.

Theorem 5.9. *Suppose that $\mathcal{H} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ is an L-fuzzy upper approximation operator, then there exists an L-fuzzy $\mathcal{T}$-transitive relation R such that $\mathcal{H} = \overline{R}$ iff $\mathcal{H}$ satisfies the following condition:*
(*h*6) $\overline{R}(\overline{R}(A)) \subseteq \overline{R}(A)$ *for every* $A \in \mathcal{F}_L(U)$.

Proof. (⇒) It follows immediately from Proposition 4.11.
(⇐) It follows immediately from Theorem 5.1 and Proposition 4.11. □

Theorem 5.10. *Suppose that L is a complete regular residuated lattice and $\mathcal{L} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ is an L-fuzzy lower approximation operator, then there exists an L-fuzzy $\mathcal{T}$-transitive relation R such that $\mathcal{L} = \underline{R}$ iff $\mathcal{L}$ satisfies the following condition:*
(*l*6) $\underline{R}(A) \subseteq \underline{R}(\underline{R}(A))$ *for every* $A \in \mathcal{F}_L(U)$.

Proof. (⇒) It follows immediately from Proposition 4.11.
(⇐) It follows immediately from Theorem 5.2 and Proposition 4.11. □

Theorem 5.11. *Suppose that $\mathcal{H} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ is an L-fuzzy upper approximation operator, then there exists an L-fuzzy $\mathcal{T}$-similarity relation R such that $\mathcal{H} = \overline{R}$ iff $\mathcal{H}$ satisfies* (*h*4), (*h*5) *and* *h*(6).

Proof. It follows immediately from Theorems 5.5, 5.7 and 5.9. □

Theorem 5.12. *Suppose that L is a complete regular residuated lattice and $\mathcal{L} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ is an L-fuzzy lower approximation operator, then there exists an L- fuzzy $\mathcal{T}$-similarity relation R such that $\mathcal{L} = \underline{R}$ iff $\mathcal{L}$ satisfies* (*l*4), (*l*5) *and* (*l*6).

Proof. It follows immediately from Theorems 5.6, 5.8 and 5.10. □

Definition 5.3. Let $\mathcal{H}, \mathcal{L} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ be two L-fuzzy set-theoretic operators, they are referred to dual operators if for all $A \in \mathcal{F}_L(U)$,
(*hl*1)$\mathcal{L}(A) = \neg\, \mathcal{H}(\neg\, A)$,
(*hl*2)$\mathcal{H}(A) = \neg\, \mathcal{L}(\neg\, A)$.
We know from Proposition 4.2 that L-fuzzy upper and L-fuzzy lower approximation operators $\overline{R}$, $\underline{R}$ in the L-fuzzy approximation space (U, R) are dual to each other when L is a regular residuated lattice.

Theorem 5.13. *Suppose that L is a complete regular residuated lattice. Let $\mathcal{H}, \mathcal{L} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ be two L-fuzzy set theoretical operators. Then there exists an L-fuzzy binary relation R on U such that $\mathcal{L} = \underline{R}$ and $\mathcal{H} = \overline{R}$ iff $\mathcal{H}, \mathcal{L} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ are a dual pair of operators such that $\mathcal{L}$ satisfies* (*l*1) *and* (*l*2), *$\mathcal{H}$ satisfies* (*h*1) *and* (*h*2).

Proof. (⇒) If there exists an L-fuzzy relation R such that $\mathcal{L} = \underline{R}$, $\mathcal{H} = \overline{R}$, then by Proposition 4.2, $\mathcal{H}, \mathcal{L} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ are dual to each other. Moreover, it follows from Theorems 5.1 and 5.3, respectively, that $\mathcal{H}$ satisfies (*h*1) and (*h*2), and $\mathcal{L}$ satisfies (*l*1) and (*l*2).
(⇐) Define an L-binary relation L as follows:

$$\forall x, y \in U, R(x, y) = \mathcal{H}(1_y)(x). \tag{16}$$

Then $\mathcal{H} = \overline{R}$ can be proved as we did in the proof of Theorem 5.1. Moreover, since L is a regular residuated lattice, $\mathcal{L} = \underline{R}$ can be derived from the duality of $\mathcal{H}$ and $\mathcal{L}$. □

Theorem 5.14. *Suppose that L is a complete regular residuated lattice. Let $\mathcal{H}, \mathcal{L} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ be two L-fuzzy set theoretical operators. Then there exists an L-fuzzy serial relation R on U such that $\mathcal{L} = \underline{R}$ and $\mathcal{H} = \overline{R}$ iff $\mathcal{H}, \mathcal{L} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ are a dual pair of operators such that $\mathcal{L}$ satisfies (l1), (l2) and (l3), $\mathcal{H}$ satisfies (h1), (h2) and (h3).*

Proof. ($\Rightarrow$) If there exists an L-fuzzy serial relation R such that $\mathcal{L} = \underline{R}$ and $\mathcal{H} = \overline{R}$, then by Theorem 5.13 $\mathcal{H}, \mathcal{L} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ are a dual pair of operators such that $\mathcal{L}$ satisfies (l1), (l2), $\mathcal{H}$ satisfies (h1), (h2). Furthermore, since R is serial, we have from Theorems 5.3 and 5.4 that $\mathcal{H}$ satisfies (h3) and $\mathcal{L}$ satisfies (l3).

($\Leftarrow$) Conversely, since $\mathcal{H}, \mathcal{L} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ are a dual pair of operators and $\mathcal{L}$ satisfies (l1), (l2), $\mathcal{H}$ satisfies (h1), (h2), then it follows from Theorem 5.13 that there exists an L-fuzzy relation R such that $\mathcal{L} = \underline{R}$ and $\mathcal{H} = \overline{R}$. Moreover, since $\mathcal{H}$ also satisfies (h3) and $\mathcal{L}$ satisfies (l3), the seriality of R follows from either of Theorems 5.3 and 5.4. □

Theorem 5.15. *Suppose that L is a complete regular residuated lattice. Let $\mathcal{H}, \mathcal{L} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ be two L-fuzzy set theoretical operators. Then there exists an L-fuzzy reflexive relation R on U such that $\mathcal{L} = \underline{R}$ and $\mathcal{H} = \overline{R}$ iff $\mathcal{H}, \mathcal{L} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ are a dual pair of operators such that $\mathcal{L}$ satisfies (l1), (l2) and (l4), $\mathcal{H}$ satisfies (h1), (h2) and (h4).*

Proof. It follows immediately from Theorems 5.5, 5.6 and 5.13. □

Theorem 5.16. *Suppose that L is a complete regular residuated lattice. Let $\mathcal{H}, \mathcal{L} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ be two L-fuzzy set theoretical operators. Then there exists an L-fuzzy symmetric relation R on U such that $\mathcal{L} = \underline{R}$ and $\mathcal{H} = \overline{R}$ iff $\mathcal{H}, \mathcal{L} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ are a dual pair of operators such that $\mathcal{L}$ satisfies (l1), (l2) and (l5), $\mathcal{H}$ satisfies (h1), (h2) and (h5).*

Proof. It follows immediately from Theorems 5.7, 5.8 and 5.13. □

Theorem 5.17. *Suppose that L is a complete regular residuated lattice. Let $\mathcal{H}, \mathcal{L} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ be two L-fuzzy set theoretical operators. Then there exists an L-fuzzy $\mathcal{T}$-transitive relation R on U such that $\mathcal{L} = \underline{R}$ and $\mathcal{H} = \overline{R}$ iff $\mathcal{H}, \mathcal{L} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ are a dual pair of operators such that $\mathcal{L}$ satisfies (l1), (l2) and (l6), $\mathcal{H}$ satisfies (h1), (h2) and (h6).*

Proof. It follows immediately from Theorems 5.9, 5.10 and 5.13. □

Theorem 5.18. *Suppose that L is a complete regular residuated lattice. Let $\mathcal{H}, \mathcal{L} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ be two L-fuzzy set theoretical operators. Then there exists an L-fuzzy $\mathcal{T}$-similarity relation R on U such that $\mathcal{L} = \underline{R}$ and $\mathcal{H} = \overline{R}$ iff $\mathcal{H}, \mathcal{L} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ are a dual pair of operators such that $\mathcal{L}$ satisfies (l1), (l2), (l4), (l5) and (l6), $\mathcal{H}$ satisfies (h1), (h2), (h4), (h5) and (h6).*

Proof. It follows immediately from Theorems 5.11–5.13. □

6. Relationship between *L*-fuzzy rough approximation operators and *L*-fuzzy topological spaces

The relationships between rough sets and topological spaces have been studied by many authors. In [17,44–46], the relationships between crisp rough sets and crisp topological spaces were studied in detail. As to the case of fuzzy environment, the relationships between fuzzy rough sets and fuzzy topological spaces were also investigated by Boixader et al. in [47], but their studies were restricted to fuzzy T-rough sets defined by fuzzy T-similarity relations which were equivalence crisp relations when they degenerated into crisp ones. Both Wu, [48] and Qin, and Pei, [49] have extended the study of the relationship between fuzzy rough approximation operators and fuzzy topological spaces to general case. However, both their studies were restricted to fuzzy rough sets in which [0, 1] was taken as the basic structure. In this section, we aim to study the relationship between general L-fuzzy rough approximation operators and L-fuzzy topological space.

First we recall some basic definitions of L-fuzzy topology and L-fuzzy closure (respectively, interior) operator, which are detailed in [50]. In what follows, the smallest element and the largest element of $\mathcal{F}_L(U)$ will be denoted by 0 and 1, respectively.

Definition 6.1. A subset τ of $\mathcal{F}_L(U)$ is referred to as an L-fuzzy topology on U iff it satisfies

(1) $0, 1 \in \tau$,
(2) If $\mathcal{A} \subseteq \tau$, then $\cup_{A \in \mathcal{A}} A \in \tau$,
(3) If $A, B \in \tau$, then $A \cap B \in \tau$.

Definition 6.2. A mapping $\text{int} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ is referred to as an L-fuzzy interior operator iff for all $A,\ B \in \mathcal{F}_L(U)$ it satisfies:

(1) $\text{int}(A) \subseteq A$,
(2) $\text{int}\,(A \cap B) = \text{int}\,(A) \cap \text{int}\,(B)$,
(3) $\text{int}\,(\text{int}\,(A)) = \text{int}\,(A)$,
(4) $\text{int}\,(1) = 1$.

Definition 6.3 ([46]). A mapping $cl : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ is referred to as an L-fuzzy closure operator iff for all $A, B \in \mathcal{F}_L(U)$ it satisfies:

(1) $A \subseteq cl(A)$,
(2) $cl(A \cup B) = cl(A) \cup cl(B)$,
(3) $cl(cl(A)) = cl(A)$,
(4) $cl(0) = 0$.

Similar to Theorem 13 and Theorem 14 in [48], the following two theorems can be easily derived:

Theorem 6.1. *Let L be a complete residuated lattice and* (U, R) *be an L-fuzzy approximation space. Then* $\underline{R} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ *is an L-fuzzy interior operator (an L-fuzzy closure operator, respectively) iff R is an L-fuzzy reflexive and transitive relation.*

Theorem 6.2. *Let L be a complete residuated lattice and* (U, R) *be an L-fuzzy approximation space and* $cl : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ *be an L-fuzzy closure operator, then there exists an L-fuzzy reflexive and transitive relation R on U such that* $\overline{R}(A) = cl(A)$ *for all* $A \in \mathcal{F}_L(U)$ *iff* $cl : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ *satisfies the following conditions:*

(1) $cl(\cup_{j\in J} A_j) = \cup_{j\in J} cl(A_j), A_j \in \mathcal{F}_L(U), j \in J$,
(2) $cl(A \otimes \hat{a}) = cl(A) \otimes \hat{a}$.

Proof. ($\Rightarrow$) It follows immediately from Theorem 5.1 and Definition 5.1
($\Leftarrow$) If the conditions (1) and (2) are satisfied, then we know from Definition 5.1 that
$cl : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ is an L-fuzzy upper approximation operator. Moreover, since $cl : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ is an L-fuzzy closure operator, both $A \subseteq cl(A)$ and $cl(cl(A)) = cl(A)$ hold for every $A \in \mathcal{F}_L(U)$. Then, by Theorems 5.5 and 5.9, there exists an L- fuzzy reflexive and transitive relation R on U such that $\overline{R}(A) = cl(A)$. □

Theorem 6.3. *Let L be a complete regular residuated lattice,* (U, R) *be an L-fuzzy approximation space and* $\mathrm{int} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ *be an L-fuzzy interior operator, then there exists an L- fuzzy reflexive and transitive relation R on U such that* $\underline{R}(A) = \mathrm{int}\,(A)$ *for all* $A \in \mathcal{F}_L(U)$ *iff* $\mathrm{int} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ *satisfies the following conditions:*

(1) $\mathrm{int}\,(\cap_{j\in J} A_j) = \cap_{j\in J} \mathrm{int}\,(A_j), A_j \in \mathcal{F}_L(U), j \in J$,
(2) $\mathrm{int}\,(\hat{a} \to A) = \hat{a} \to \mathrm{int}\,(A)$.

Proof. ($\Rightarrow$) It follows immediately from Theorem 5.2 and Definition 5.2.
($\Leftarrow$) If the conditions (1) and (2) are satisfied, then we know from Definition 5.2 that $\mathrm{int} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ is an L-fuzzy lower approximation operator. Moreover, on account of the fact that $\mathrm{int} : \mathcal{F}_L(U) \longrightarrow \mathcal{F}_L(U)$ is an L-fuzzy interior operator, we can easily have from Definition 6.2 that both $\mathrm{int}\,(A) \subseteq A$ and $\mathrm{int}\,(\mathrm{int}\,(A)) \subseteq \mathrm{int}\,(A)$ hold. Hence, by Theorems 5.6 and 5.10 there exists an L-fuzzy reflexive and transitive relation R on U such that $\underline{R}(A) = \mathrm{int}\,(A)$ for all $A \in \mathcal{F}_L(U)$. □

7. Concluding remarks

As is well known, there are at least two approaches to the study of rough set theory, namely the constructive and axiomatic approaches. In [34], the notion of fuzzy rough sets was generalized by taking an arbitrary residuated lattice as a basic algebraic structure and several classes of L-fuzzy rough sets have been considered and their properties have been investigated as well. Hence, it seems that the study of L-rough sets in [34] has been mainly concentrated on the constructive approach. In this paper, more efforts have been made on the axiomatic approach, and the axiomatic characterizations of L-fuzzy rough sets have been obtained. Moreover, certain kinds of L-fuzzy approximation operators have been characterized by corresponding axioms. The axiomatization of L-fuzzy approximation operators guarantees the existence of corresponding L-fuzzy relations which produce the operators. Finally, the relationship between L-fuzzy rough sets and L-fuzzy topological spaces has been proposed. We hope that the axiomatic approaches presented in this paper can be used to help us to gain much more insights into the mathematical structures of L-fuzzy approximation operators.

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