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Grid Minors of Graphs on the Torus

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We show that any graph G embedded on the torus with face-width $r \geqslant 5$ contains the toroidal $\lfloor \frac{2}{3} r \rfloor$-grid as a minor. (The *face-width* of G is the minimum value of $|C \cap G|$, where C ranges over all homotopically nontrivial closed curves on the torus. The *toroidal k-grid* is the product $C_k \times C_k$ of two copies of a k-circuit C_k.) For each fixed $r \geqslant 5$, the value $\lfloor \frac{2}{3} r \rfloor$ is largest possible. This applies to a theorem of Robertson and Seymour showing, for each graph H embedded on any compact surface S, the existence of a number ρ_H such that every graph G embedded on S with face-width at least ρ_H contains H as a minor. Our result implies that for $H = C_k \times C_k$ embedded on torus, $\rho_H := \lceil \frac{3}{2} k \rceil$ is the smallest possible value. Our proof is based on deriving a result in the geometry of numbers. It implies that for any symmetric convex body K in $\mathbb{R}^2$ one has $\lambda_2(K) \cdot \lambda_1(K^*) \leqslant \frac{3}{2}$ and that this bound is smallest possible. (Here $\lambda_i(K)$ denotes the minimum value of λ such that $\lambda \cdot K$ contains i linearly independent integer vectors. K^* is the polar convex body.)

1. INTRODUCTION

For any graph G embedded on a surface S, the *face-width* (or *representativity*) $r(G)$ of G is the minimum of $|C \cap G|$, where C ranges over all homotopically nontrivial closed curves on S. Robertson and Seymour [1] showed:

for each graph H embedded on a compact surface S there exists an integer ρ_H so that each graph G embedded on S with $r(G) \geqslant \rho_H$ contains H as a minor. (1)

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In this paper we determine the smallest value of ρ_H for a certain class of graphs H embedded on the torus, viz. the toroidal grids. For each $k \geqslant 3$, the *toroidal k-grid* is the product $C_k \times C_k$ of two k-circuits C_k. (By definition, $C_k \times C_k$ has vertices (i, j) for $0 \leqslant i, j \leqslant k-1$, where (i, j) and (i', j') are adjacent if either $i = i'$ and $j = j \pm 1 \pmod{k}$ or $j = j'$ and $i = i' \pm 1 \pmod{k}$.)

Clearly, each toroidal k-grid can be embedded on the torus. In fact, there is a unique embedding, up to homeomorphisms (of the torus and of the grid). (If $k \geqslant 5$, this follows easily from the fact that each face of the embedded graph should be a quadrangle. For $k = 3$ and 4 this takes some elaboration.) We show

THEOREM 1. *For the toroidal k-grid $H = C_k \times C_k$ embedded on the torus, $\rho_H := \lceil \frac{3}{2} k \rceil$ is the smallest integer value one can take for ρ_H in* (1).

We derive this from

THEOREM 2. *Any graph G embedded on the torus contains the toroidal $\lfloor \frac{2}{3} r(G) \rfloor$-grid as a minor (if $r(G) \geqslant 5$). For each integer $r \geqslant 3$ there exists a graph G embedded on the torus with $r(G) = r$ and not containing the toroidal $\lfloor \frac{2}{3} r \rfloor + 1$-grid as a minor.*

Proof of the Implication Theorem 2 ⇒ Theorem 1. Choose $k \geqslant 3$. Let G be a graph with $r(G) \geqslant \lceil \frac{3}{2} k \rceil$. Since $k = \lfloor \frac{2}{3} \lceil \frac{3}{2} k \rceil \rfloor \leqslant \lfloor \frac{2}{3} r(G) \rfloor$, Theorem 2 implies that G contains the toroidal k-grid as a minor.

Let $r := \lceil \frac{3}{2} k \rceil - 1$. By Theorem 2 there exists a graph G on the torus with $r(G) = r$ and not containing the toroidal $(\lfloor \frac{2}{3} r \rfloor + 1)$-grid as a minor. Since $k = \lfloor \frac{2}{3} r \rfloor + 1$, Theorem 1 follows. ∎

To prove Theorem 2, we use some results from [2, 3]. Represent the torus as the product $S^1 \times S^1$ of two copies of the unit circle S^1 in the complex plane. For $(m, n) \in \mathbb{Z}^2$, let $C_{m,n}: S^1 \to S^1 \times S^1$ be the closed curve on the torus given by

$$C_{m,n}(z) := (z^m, z^n) \tag{2}$$

for $z \in S^1$.

Let G be a graph embedded on the torus. Define $\varphi_G: \mathbb{Z}^2 \to \mathbb{Z}$ by

$$\varphi_G(m, n) := \min_{C \sim C_{m,n}} \operatorname{cr}(C, G), \tag{3}$$

where $C \sim C'$ means that C is a closed curve freely homotopic to closed curve C' and where $\operatorname{cr}(C, G)$ denotes the number of intersections of C and

G, counting multiplicities. So $r(G)$ is equal to the minimum value of $\varphi_G(m, n)$ over all vectors $(m, n) \neq (0, 0)$ in $\mathbb{Z}^2$.

Let $P(G)$ be the following set in $\mathbb{R}^2$:

$$P(G) := \{(x, y) \in \mathbb{R}^2 \mid mx + ny \leqslant \varphi_G(m, n) \text{ for all } (m, n) \in \mathbb{Z}^2\}. \qquad (4)$$

Then $P(G)$ is a symmetric integer polygon (i.e., $P(G) = -P(G)$ and it is a polygon with all vertices having integer coordinates only). Define the *height* height(K) of a polygon K by

$$\text{height}(K) := \min_{(m, n) \in \mathbb{Z}^2,\, (m, n) \neq (0, 0)} \max\{mx + ny \mid (x, y) \in K\}. \qquad (5)$$

As $\varphi_G(m, n) = \max\{mx + ny \mid (x, y) \in P(G)\}$ (cf. [2]), we have

$$r(G) = \text{height}(P(G)). \qquad (6)$$

The following was shown in [3]:

let $k \geqslant 3$; a graph G embedded on the torus contains a toroidal k-grid as a minor, if and only if $(1/k)\, P(G)$ contains two linearly independent integer vectors. (7)

Assertions (6) and (7) imply that to prove Theorem 2, it suffices to show

THEOREM 3. *Let $r \geqslant 3$. Then for each symmetric integer polygon K of height r, the polygon $\lfloor \frac{2}{3} r \rfloor^{-1} K$ contains two linearly independent integer vectors. Here $\lfloor \frac{2}{3} r \rfloor$ cannot be replaced by any larger integer.*

We show Theorem 3 in Section 2. We first note that it implies the following bound in the geometry of numbers. Let K be a symmetric convex body in $\mathbb{R}^2$ (i.e., K is a compact full-dimensional convex set with $K = -K$). Let $\lambda_1(K)$ denote the minimum value of λ so that $\lambda \cdot K$ contains a nonzero integer vector. Let $\lambda_2(K)$ denote the minimum value of λ so that $\lambda \cdot K$ contains two linearly independent integer vectors. Let K^* denote the *polar* convex body:

$$K^* := \{ y \in \mathbb{R}^2 \mid x^T y \leqslant 1 \text{ for all } x \in K\}. \qquad (8)$$

Then

COROLLARY 3a. *For each symmetric convex body K in $\mathbb{R}^2$ one has $\lambda_2(K) \cdot \lambda_1(K^*) \leqslant \frac{3}{2}$. The bound $\frac{3}{2}$ is smallest possible.*

Proof. It suffices to show the corollary for symmetric integer polygons K with $r := \text{height}(K)$ being a multiple of three. Now $r := \lambda_1(K^*)$, while by Theorem 3, $\lambda_2(K) \leqslant (\frac{2}{3} r)^{-1}$. So $\lambda_2(K) \cdot \lambda_1(K^*) \leqslant \frac{3}{2}$. Similarly, any better value in the corollary would imply a better factor in Theorem 3. ∎

2. Proof of Theorem 3

Call a symmetric integer polygon K *r-minimal*, if height$(K) \geqslant r$ while height$(K') < r$ for each symmetric integer polygon $K' \neq K$ contained in K. So Theorem 3 follows from:

> let $r \geqslant 2$; then for each r-minimal symmetric integer polygon K, the polygon $(3/2r)\,K$ contains two linearly independent integer vectors; moreover, there exists an r-minimal symmetric integer polygon K so that $(\lfloor 2r/3 \rfloor + 1)^{-1} \cdot K$ does not contain two linearly independent integer vectors. (9)

In order to prove (9), we use the classification of r-minimal symmetric integer polygons given in [3]. Each of these polygons is a quadrangle or a hexagon. The quadrangles arise as follows. Choose integer values $0 \leqslant \alpha < r$ and $0 \leqslant \beta < r$. Let $Q_{\alpha,\beta}$ be the convex hull of the points $\pm(r, \alpha)$, $\pm(-\beta, r)$. Then $Q_{\alpha,\beta}$ is r-minimal, and all symmetric r-minimal integer polygons that are quadrangle arise in this way, up to unimodular transformations (=linear transformations of $\mathbb{R}^2$ fixing $\mathbb{Z}^2$).

The hexagons arise as follows. Choose integer values $0 < \alpha < r$, $0 < \beta < r$, and $0 < \gamma < r$. Let $H_{\alpha,\beta,\gamma}$ be the convex hull of the points $\pm(r, \alpha)$, $\pm(r-\beta, r)$, $\pm(-\gamma, r-\gamma)$. Again, $H_{\alpha,\beta,\gamma}$ is r-minimal, and all symmetric r-minimal integer polygons that are hexagons arise in this way, up to unimodular transformations. So it suffices to show the following two lemmas.

Lemma 1. *For each choice of integers $0 \leqslant \alpha < r$ and $0 \leqslant \beta < r$, we have $\lambda_2(Q_{\alpha,\beta}) \leqslant 3/2r$. For fixed r, we cannot replace $3/2r$ by k^{-1} for any integer $k > 2r/3$.*

Proof. One easily finds that $Q_{\alpha,\beta}$ is determined by the following inequalities:

$$\left|\frac{r-\alpha}{r^2+\alpha\beta}x + \frac{r+\beta}{r^2+\alpha\beta}y\right| \leqslant 1, \qquad \left|\frac{r+\alpha}{r^2+\alpha\beta}x + \frac{\beta-r}{r^2+\alpha\beta}y\right| \leqslant 1. \tag{10}$$

For each vector (x, y), let the norm $\|(x, y)\|$ be the minimum λ for which (x, y) belongs to $\lambda \cdot Q_{\alpha,\beta}$. Note that (x, y) can be easily calculated from (10):

$$\|(x, y)\| = \frac{\max\{|(r-\alpha)x + (r+\beta)y|, |(r+\alpha)x + (\beta-r)y|\}}{r^2+\alpha\beta}. \tag{11}$$

To show the first statement in the lemma, we have to find two linearly independent integer vectors each with norm at most $3/2r$. We may assume $\alpha \leqslant \beta$. Then

$$\|(1,0)\| = \frac{r+\alpha}{r^2+\alpha\beta} \leqslant \frac{r+\alpha}{r^2+\alpha^2} \leqslant \frac{3}{2r}. \tag{12}$$

(The first inequality follows from $\alpha \leqslant \beta$. The second inequality follows from the fact that $(1+x) \leqslant \frac{3}{2}(1+x^2)$ for all $x \in \mathbb{R}$.)

If $\beta < r/3$, then

$$\|(0,1)\| = \frac{r+\beta}{r^2+\alpha\beta} < \frac{r+r/3}{r^2} < \frac{3}{2r}. \tag{13}$$

If $\beta \geqslant r/3$, then

$$\begin{aligned}\|(0,1)\| + \|(1,-1)\| &= \frac{r+\beta}{r^2+\alpha\beta} + \frac{2r+\alpha-\beta}{r^2+\alpha\beta} \\ &= \frac{3r+\alpha}{r^2+\alpha\beta} \leqslant \frac{3r+3\alpha\beta/r}{r^2+\alpha\beta} = \frac{3}{r},\end{aligned} \tag{14}$$

implying that at least one of $(0, 1)$, $(1, -1)$ has norm at most $3/2r$. This shows the first statement of the lemma.

To show the second statement, choose $r \geqslant 3$. Let $k := \lfloor 2r/3 \rfloor + 1$. Let $\alpha := 0$ and $\beta := \lfloor r/2 \rfloor$. We define a norm as in (11). Let (x, y) be any integer vector with norm at most $1/k$. We show that $y = 0$, implying that there do not exist two linearly independent integer vectors each with norm at most $1/k$. We may assume $x \geqslant 0$.

First let r be even. Then $\|(x,y)\| = \max\{|x+\frac{3}{2}y|, |x-\frac{1}{2}y|\}/r \leqslant 1/k$. If $x=0$ then $|\frac{3}{2}y| \leqslant r/k < \frac{3}{2}$, and hence $y=0$. If $x \geqslant 1$, $y \geqslant 1$, then $r/k \geqslant |x+\frac{3}{2}y| \geqslant \frac{5}{2} > r/k$. If $x \geqslant 1$, $y \leqslant -1$, then $r/k \geqslant |x-\frac{1}{2}y| \geqslant \frac{3}{2} > r/k$.

Next let r be odd. Then $\|(x,y)\| = \max\{|x+(\frac{3}{2}-1/2r)y|, |x-(\frac{1}{2}+1/2r)y|\}/r \leqslant 1/k$. Note that $k \geqslant \frac{2}{3}r + \frac{1}{3}$, implying $k(\frac{3}{2}-1/2r) \geqslant (2r/3+\frac{1}{3})((\frac{3}{2}-1/2r) = r + \frac{1}{6} - 1/6r > r$. If $x=0$ then $|(\frac{3}{2}-1/2r)y| \leqslant r/k < (\frac{3}{2}-1/2r)$, yielding $y=0$. If $x \geqslant 1$, $y \geqslant 1$, then $r/k \geqslant |x+(\frac{3}{2}-1/2r)y| \geqslant \frac{5}{2}-1/2r > \frac{3}{2} > r/k$. If $x \geqslant 1$, $y \leqslant -1$ then $r/k \geqslant |x-(\frac{1}{2}+1/2r)y| > \frac{3}{2} > r/k$. So also if $x \geqslant 1$ then $y=0$. ∎

LEMMA 2. *For each choice of integers* $0 < \alpha < r$, $0 < \beta < r$, *and* $0 < \gamma < r$, *we have* $\lambda_2(H_{\alpha,\beta,\gamma}) < 3/2r$.

Proof. One easily finds that $H_{\alpha,\beta,\gamma}$ is determined by the following inequalities:

$$\left|\frac{r-\alpha}{r^2+\alpha\beta-\alpha r}x+\frac{\beta}{r^2+\alpha\beta-\alpha r}y\right|\leqslant 1,$$

$$\left|\frac{\gamma}{r^2+\beta\gamma-\beta r}x+\frac{\beta-r-\gamma}{r^2+\beta\gamma-\beta r}y\right|\leqslant 1, \tag{15}$$

$$\left|\frac{\gamma-r-\alpha}{r^2+\gamma\alpha-\gamma r}x+\frac{r-\gamma}{r^2+\gamma\alpha-\gamma r}y\right|\leqslant 1.$$

For each vector (x, y), let the norm $\|(x, y)\|$ be the minimum λ for which (x, y) belongs to $\lambda \cdot H_{\alpha,\beta,\gamma}$. Again, (x, y) can be easily calculated from (15). It folows that

$$\|(1,0)\|=\frac{r+\alpha-\gamma}{r^2+\gamma\alpha-\gamma r},$$

$$\|(0,1)\|=\frac{r+\gamma-\beta}{r^2+\beta\gamma-\beta r}, \tag{16}$$

$$\|(1,1)\|=\frac{r+\beta-\alpha}{r^2+\alpha\beta-\alpha r}.$$

We show that at least two of these norms are less than $3/2r$. Suppose not. By symmetry we may assume that $\|(1,0)\|\geqslant 3/2r$ and $\|(0,1)\|\geqslant 3/2r$. As $0<\gamma<r$, the first norm in (16) is monotonically increasing in α, while the second norm is monotonically decreasing in β. So

$$\frac{r+\alpha-\gamma}{r^2+\gamma\alpha-\gamma r}<\frac{2r-\gamma}{r^2} \quad \text{and} \quad \frac{r+\gamma-\beta}{r^2+\beta\gamma-\beta r}<\frac{r+\gamma}{r^2}. \tag{17}$$

Since $2r-\gamma\leqslant\frac{3}{2}r$ or $r+\gamma\leqslant\frac{3}{2}r$ (as $(2r-\gamma)+(r+\gamma)=3r$), this contradicts our assumption. ∎

References

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