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Noncompactness and noncompleteness in isometries of Lipschitz spaces ☆

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ABSTRACT

We solve the following three questions concerning surjective linear isometries between spaces of Lipschitz functions $\mathrm{Lip}(X, E)$ and $\mathrm{Lip}(Y, F)$, for strictly convex normed spaces E and F and metric spaces X and Y:

(i) Characterize those base spaces X and Y for which all isometries are weighted composition maps.
(ii) Give a condition independent of base spaces under which all isometries are weighted composition maps.
(iii) Provide the general form of an isometry, both when it is a weighted composition map and when it is not.

In particular, we prove that requirements of completeness on X and Y are not necessary when E and F are not complete, which is in sharp contrast with results known in the scalar context.

1. Introduction

It is well known that not all surjective (linear) isometries between spaces of Lipschitz functions on general metric spaces X and Y can be written as weighted composition maps (see for instance [22, p. 61]). Attempts to identify the isometries which can be described in that way have been done in three ways, each trying to provide an answer to one of the following questions:

(i) Characterize those base spaces X and Y for which all isometries are weighted composition maps.
(ii) Give a condition independent of base spaces under which all isometries are weighted composition maps.
(iii) Provide the general form of an isometry, both when it is a weighted composition map and when it is not.

The first question was studied by Weaver for a general metric in the *scalar-valued* setting (see [21] or [22, Section 2.6]), and the second one has been recently treated by Jiménez-Vargas and Villegas-Vallecillos in the more general setting of *vector-valued* functions (see [14]). In the latter, the Banach spaces where the functions take values are assumed to be strictly convex. This is certainly not a heavy restriction, as this type of results is known not to hold for general Banach spaces. Strict convexity is actually a very common and reasonable assumption, even if, at least in other contexts, it is not the unique

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possible (see for instance [2,4,6,12]). As for the third question, an answer was given by Mayer-Wolf for compact base spaces in the scalar context, not for a general metric d, but for powers d^α with $0 < \alpha < 1$.

Weaver proved that completeness and 1-connectedness of X and Y are sufficient conditions, and that the weighted composition isometries must have a very special form. More concretely, given complete 1-connected metric spaces X and Y with diameter at most 2, a linear bijection $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ is an isometry if and only if $Tf = \alpha \cdot f \circ h$ for every f, where $\alpha \in \mathbb{K}$, $|\alpha| = 1$, and $h : Y \to X$ is an isometry. Requirements of 1-connectedness on both X and Y (that is, they cannot be decomposed into two nonempty disjoint sets whose distance is greater than or equal to 1) cannot be dropped in general. And, obviously, $\mathrm{Lip}(X)$ and $\mathrm{Lip}(\widehat{X})$ are linearly isometric when X is not complete (where $\widehat{X}$ denotes the completion of X), so requirements of completeness cannot be dropped either.

On the other hand, Jiménez-Vargas and Villegas-Vallecillos gave a general representation in the spirit of the classical Banach–Stone Theorem (along with related results for isometries not necessarily surjective). Assumptions include compactness of base metric spaces and the fact that the isometry fixes a (nonzero) constant function. The conclusion in the surjective case is that the isometry $T : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, E)$ is of the form $Tf(y) = Jy(f(h(y)))$ for all $f \in \mathrm{Lip}(X, E)$ and $y \in Y$, where h is a bi-Lipschitz homeomorphism (that is, h and h^{-1} are Lipschitz) from Y onto X and J is a Lipschitz map from Y into the set $I(E, E)$ of all surjective linear isometries on the strictly convex Banach space E. They also proved that this result can be sharpened under stronger hypotheses, but the above assumptions remain basically the same, so that the results do not provide an "if and only if" description.

Finally, Mayer-Wolf not only characterized the family of compact spaces for which the associated Lipschitz spaces admitted isometries that were not composition operators, but also gave their general form. In principle, it is not clear whether or not his results can be extended to spaces endowed with a metric not of the form d^α. In fact, the answer, as we will see here, is not completely positive.

The aim of this paper is to give, in the *vector-valued* setting, a complete answer to questions (i), (ii) and (iii) (just assuming strict convexity of E and F). The general answer is not known even in the scalar setting, which can be included here as a special case. We also prove, on the one hand, that conditions of compactness can be replaced with just completeness on base spaces and, on the other hand, that even completeness can be dropped when the normed spaces E and F are not complete (which is in sharp contrast with the behaviour in the scalar case).

To solve (ii), we show that the condition on the preservation of a constant function (as given in [14]) can be replaced with a milder one (see Theorem 3.1). We use it to solve (iii) (see Theorem 3.4, and more in general Theorem 3.1, Corollary 3.3 and Remark 3.5). Our answer also applies to results on metrics d^α in [18], and a key to understand the generalization is Proposition 3.9. An answer to (i) is given as a direct consequence of the results concerning (ii) and (iii) (see Corollaries 3.6 and 3.7). As a special case we provide the natural counterpart of the description given in [21] (see Corollary 5.3 and Remark 5.4). We finally mention that we do not use the same techniques as in [21] nor as in [14]; instead we study *surjective linear* isometries through biseparating maps, which has proven successful in various contexts (see for instance [2,10] for recent references).

Other papers where related operators have been recently studied in similar contexts are [1,8,9,13,17] (see also [5,11,16, 18–20]).

2. Preliminaries and notation

Recall that, given metric spaces (X, d_1) and (Y, d_2), a map $f : X \to Y$ is said to be *Lipschitz* if there exists a constant $k \geqslant 0$ such that $d_2(f(x), f(y)) \leqslant k d_1(x, y)$ for each $x, y \in X$, and that the *Lipschitz number* of f is

$$L(f) := \sup\left\{\frac{d_2(f(x), f(y))}{d_1(x, y)} \colon x, y \in X,\ x \neq y\right\}.$$

Given a normed space E (over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), we denote by $\mathrm{Lip}(X, E)$ the space of all *bounded* E-valued Lipschitz functions on X. We endow $\mathrm{Lip}(X, E)$ with the norm $\|\cdot\|_L := \max\{\|\cdot\|_\infty, L(\cdot)\}$ (where $\|\cdot\|_\infty$ denotes the usual supremum norm).

As a particular case, we can consider in X a power ${d_1}^\alpha$ of the metric d_1, $0 < \alpha < 1$. The corresponding space of all bounded E-valued Lipschitz functions on X with respect to ${d_1}^\alpha$ is then denoted by $\mathrm{Lip}^\alpha(X, E)$.

Recall also that a normed space E is said to be *strictly convex* if $\|e_1 + e_2\| < 2$ whenever e_1, e_2 are different vectors of norm 1 in E or, equivalently, that $\|e_1 + e_2\| = \|e_1\| + \|e_2\|$ $(e_1, e_2 \neq 0)$ implies $e_1 = \alpha e_2$ for some $\alpha > 0$ (see [15, pp. 332–336]). From this, it follows that, given $e_1, e_2 \in E \setminus \{0\}$,

$$\|e_1\|, \|e_2\| < \max\bigl\{\|e_1 + e_2\|, \|e_1 - e_2\|\bigr\}, \tag{2.1}$$

which is an inequality we will often use. The fact that a normed space is strictly convex does not imply that its completion is. Indeed every infinite-dimensional separable Banach space can be renormed to be not strictly convex and to contain a strictly convex dense subspace of codimension one (see [7]).

From now on, unless otherwise stated, we assume that E and F are strictly convex normed spaces (including the cases $E = \mathbb{K}$, $F = \mathbb{K}$).

As we mentioned above, on our way to Theorem 3.1 we will deal with biseparating maps. Recall that separating maps are those preserving disjointness of *cozero sets* (where the cozero set of a function $f : X \to E$ is defined as $c(f) := \{x \in X\colon\ f(x) \neq 0\}$). More concretely, we will say that a linear map $T : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$ is *separating* if $c(Tf) \cap c(Tg) = \emptyset$ whenever $f, g \in \mathrm{Lip}(X, E)$ satisfy $c(f) \cap c(g) = \emptyset$. Moreover, T is said to be *biseparating* if it is bijective and both T and its inverse are separating maps.

Obviously, if $f : X \to E$ is Lipschitz and bounded, then so is the map $\|f\| : X \to \mathbb{R}$ defined by $\|f\|(x) := \|f(x)\|$ for every $x \in X$. It is also clear that $\|f\|$ can be continuously extended to a Lipschitz function $\widehat{\|f\|} : \widehat{X} \to \mathbb{R}$ defined on the completion $\widehat{X}$ of X. More in general, if $x \in \widehat{X} \setminus X$, we say that f *admits an extension to* x if it can be continuously extended to a map $\widehat{f} : X \cup \{x\} \to E$. Clearly, when E is complete and X is not, f admits a continuous extension to the whole $\widehat{X}$, and the extension $\widehat{f} : \widehat{X} \to E$ is a Lipschitz function with $\|\widehat{f}\|_\infty = \|f\|_\infty$ and $L(\widehat{f}) = L(f)$. For this reason, when E and F are complete, every surjective linear isometry $T : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$ can be associated in a canonical way to another one $\widehat{T} : \mathrm{Lip}(\widehat{X}, E) \to \mathrm{Lip}(\widehat{Y}, F)$ (which coincides with T only if X and Y are complete).

Given $R > 0$, we define in X the following equivalence relation: we put $x \sim_R y$ if there exist $x_1, \ldots, x_n \in X$ with $x = x_1$, $y = x_n$, and $d(x_i, x_{i+1}) < R$ for $i = 1, \ldots, n-1$. We call R-component each of the equivalence classes of X by $\sim_R$. The set of all R-components in X is denoted by $\mathtt{Comp}_R(X)$.

We say that a bijective map $h : Y \to X$ preserves distances less than 2 if $d_1(h(y), h(y')) = d_2(y, y')$ whenever $d_2(y, y') < 2$. We denote by $\mathrm{iso}_{<2}(Y, X)$ the set of all maps $h : Y \to X$ such that both h and h^{-1} preserve distances less than 2. Notice that every $h \in \mathrm{iso}_{<2}(Y, X)$ is a homeomorphism and that, when X is bounded, then it is also a Lipschitz map (see also Remark 3.2).

Definition 2.1. Let $I(E, F)$ be the set of all linear isometries from E onto F. We say that a map $T : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$ is a *standard* isometry if there exist $h \in \mathrm{iso}_{<2}(Y, X)$ and a map $J : Y \to I(E, F)$ constant on each 2-component of Y such that

$$Tf(y) = Jy\big(f\big(h(y)\big)\big)$$

for all $f \in \mathrm{Lip}(X, E)$ and $y \in Y$.

Remark 2.2. Notice that a standard isometry is indeed a surjective linear isometry. Theorem 3.1 gives a condition under which both classes of operators coincide. Also, when Y is 2-connected, the map J is constant, so there exists a surjective linear isometry $\mathbf{J} : E \to F$ such that

$$Tf(y) = \mathbf{J}\big(f\big(h(y)\big)\big)$$

for all $f \in \mathrm{Lip}(X, E)$ and $y \in Y$. In particular, Corollary 5.3 roughly says that this is the only way to obtain an isometry when one of the base spaces is 1-connected (see also Remark 5.4).

In the definition of standard isometry, we see that X and Y are very much related. In particular, one is complete if and only if the other is. There are interesting cases which are *almost* standard in some sense. For instance, when E is complete, the natural inclusion $\mathbf{i}_X : \mathrm{Lip}(X, E) \to \mathrm{Lip}(\widehat{X}, E)$ is not standard if X is not complete, but we immediately obtain a standard isometry from it in a natural way.

On the other hand, when E and F are complete, every T can be written as $T = {\mathbf{i}_Y}^{-1} \circ \widehat{T} \circ \mathbf{i}_X$. In the following definition, we distinguish between this kind of isometries and *nonstandard* isometries.

Definition 2.3. We say that a surjective linear isometry $T : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$ is *nonstandard* if T and $\widehat{T}$ (if it can be defined, that is, if E and F are complete) are not standard.

Just a special family of spaces allows defining properly nonstandard isometries. We call them spaces of *type* **A**.

Definition 2.4. We say that a metric space X is of type **A** if there are a partition of X into two subsets $\mathfrak{A}$, $\mathfrak{B}$, and a map $\varphi : \mathfrak{A} \to \mathfrak{B}$ with the following properties:

(i) $d(x, z) = 1 + d(\varphi(x), z)$ whenever $x \in \mathfrak{A}$ and $z \in \mathfrak{B}$ satisfy $d(\varphi(x), z) < 1$,
(ii) $d(x, z) \geqslant 2$ whenever $x \in \mathfrak{A}$ and $z \in \mathfrak{B}$ satisfy $d(\varphi(x), z) \geqslant 1$,
(iii) $d(x_1, x_2) \geqslant 2$ whenever $x_1, x_2 \in \mathfrak{A}$ and $\varphi(x_1) \neq \varphi(x_2)$.

For each E, the operator $S_\varphi : \mathrm{Lip}(X, E) \to \mathrm{Lip}(X, E)$ defined, for each $f \in \mathrm{Lip}(X, E)$, by

$$S_\varphi f(x) := \begin{cases} f(x) & \text{if } x \in \mathfrak{B}, \\ f(\varphi(x)) - f(x) & \text{if } x \in \mathfrak{A} \end{cases}$$

is said to be the *purely nonstandard* map associated to φ.

Remark 2.5. It is easy to check that S_φ is linear and bijective. Also $\|S_\varphi(f)\| \leqslant 1$ whenever $\|f\| \leqslant 1$. Taking into account that ${S_\varphi}^{-1} = S_\varphi$, this implies that S_φ is indeed a nonstandard isometry. Theorem 3.4 and Remark 3.5 basically say that every nonstandard isometry is the composition of a standard and a purely nonstandard one.

Throughout, for each $\mathrm{e} \in E$, the constant function from X into E taking the value e will be denoted by $\tilde{\mathrm{e}}$. Also, given a set A, χ_A stands for the characteristic function on A.

As usual, if there is no confusion both the metric of X and that of Y will be denoted by d.

Given a surjective linear isometry $T : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$, we denote

$$\mathfrak{A}(T) := \left\{ y \in Y : \ T\tilde{\mathrm{e}}(y) = 0, \ \forall \mathrm{e} \in E \right\}$$

and

$$\mathfrak{B}(T) := \mathfrak{A}(T)^c.$$

The partition of Y into these two subsets will be very much used in Sections 5 and 6, and the fact that $\mathfrak{A}(T)$ is empty will turn out to be basically equivalent to T being standard. This property will receive a special name. We define Property **P** as follows:

P: For each $y \in Y$, there exists $\mathrm{e} \in E$ with $T\tilde{\mathrm{e}}(y) \neq 0$.

3. Main results

We first give some results ensuring that an isometry is standard, and then characterize spaces and describe the isometries when this is not the case. Theorem 3.1 and Corollary 3.3 are proved in Section 4, and Theorem 3.4 in Section 6.

It is obvious that, by definition, if T is *not* nonstandard, then it satisfies Property **P**. The converse is given by Theorem 3.1 and Corollary 3.3.

Theorem 3.1. *Let $T : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$ be a surjective linear isometry satisfying Property* **P**. *Then E and F are linearly isometric.*

Furthermore, if we are in any of the following two cases:

(i) *X and Y are complete,*
(ii) *E (or F) is not complete,*

then T is standard.

Remark 3.2. In Theorem 3.1, we cannot in general ensure that the map h is an isometry or that it preserves distances equal to 2. Indeed, following the same ideas as in [22, Proposition 1.7.1], if (Z, d) is a metric space with diameter $\mathrm{diam}(Z, d) > 2$, then there is a new metric $d'(\cdot,\cdot) := \min\{2, d(\cdot,\cdot)\}$ on Z with $\mathrm{diam}(Z, d') = 2$ such that $\mathrm{Lip}(Z, E)$ with respect to d and $\mathrm{Lip}(Z, E)$ with respect to d' are linearly isometric. On the other hand, notice also that, if d'_1 and d'_2 are defined in a similar way, then the map $h : (Y, d_2) \to (X, d_1)$ belongs to $\mathrm{iso}_{<2}(Y, X)$ if and only if $h : (Y, d'_2) \to (X, d'_1)$ is an isometry.

In Theorem 3.1, when (i) and (ii) do not hold, E and F are complete and X (or Y) is not. In this case, it is easy to see that in general T is not standard. Nevertheless, we have the following result.

Corollary 3.3. *Suppose that E and F are complete and X or Y is not. If $T : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$ is a surjective linear isometry satisfying Property* **P**, *then $\widehat{T} : \mathrm{Lip}(\widehat{X}, E) \to \mathrm{Lip}(\widehat{Y}, F)$ is standard.*

We next give the general form that a nonstandard isometry (or, equivalently, an isometry not satisfying Property **P**) must take.

Theorem 3.4. *Assume that we are in any of the following two cases*:

(i) *X and Y are complete,*
(ii) *E (or F) is not complete.*

Then there exists a nonstandard isometry $T : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$ if and only if the following three conditions hold simultaneously:

(i) *X and Y are of type* **A**,
(ii) *there exists $h \in \mathrm{iso}_{<2}(Y, X)$,*
(iii) *E and F are linearly isometric.*

In this case, $T = S_\varphi \circ T'$, where $T' : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$ is a standard isometry and $S_\varphi : \mathrm{Lip}(Y, F) \to \mathrm{Lip}(Y, F)$ is a purely nonstandard isometry.

Remark 3.5. In the case when E and F are complete and X or Y is not, if $T : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$ is a nonstandard isometry, then so is $\widehat{T}$, and the description given in Theorem 3.4 applies to $\widehat{T}$. In particular, $T = \mathbf{i}_Y{}^{-1} \circ S_\varphi \circ T' \circ \mathbf{i}_X$ where $T' : \mathrm{Lip}(\widehat{X}, E) \to \mathrm{Lip}(\widehat{Y}, F)$ is a standard isometry and $S_\varphi : \mathrm{Lip}(\widehat{Y}, F) \to \mathrm{Lip}(\widehat{Y}, F)$ is purely nonstandard.

A direct consequence (and easy to check) of Theorem 3.4 and Remark 3.5 is the following.

Corollary 3.6. *If E is not complete, then there exists a nonstandard isometry from $\mathrm{Lip}(X, E)$ onto itself if and only if X is of type **A**. If E is complete, then there exists a nonstandard isometry from $\mathrm{Lip}(X, E)$ onto itself if and only if $\widehat{X}$ is of type **A**.*

Theorem 3.4 says that, under some assumptions, when two spaces of Lipschitz functions are linearly isometric, there exists in fact a standard isometry between them. The following result is a simple consequence of Theorems 3.1 and 3.4, Corollary 3.3 and Remark 3.5.

Corollary 3.7. $\mathrm{Lip}(X, E)$ *and* $\mathrm{Lip}(Y, F)$ *are linearly isometric if and only if E and F are linearly isometric and*

- $\mathrm{iso}_{<2}(Y, X)$ *is nonempty (when E and F are not complete),*
- $\mathrm{iso}_{<2}(\widehat{Y}, \widehat{X})$ *is nonempty (when E and F are complete).*

We finally adapt the above results to the special case of metrics d^α, $0 < \alpha < 1$. Even if in this case we just deal with metrics and, consequently, the general form of the isometries between spaces $\mathrm{Lip}^\alpha(X, E)$ is completely given by Theorems 3.1 and 3.4, Corollary 3.3 and Remark 3.5, it is interesting to see how the condition of being of type **A** can be translated to metrics d^α. This turns out to be more restrictive, and constitutes a generalization of the scalar case on compact spaces given in [18, Theorem 3.3].

Definition 3.8. Let $0 < \alpha < 1$. We say that a metric space (X, d) is of *type* $\mathbf{A}_\alpha$ if there are a partition of X into two subsets $\mathfrak{A}$, $\mathfrak{B}$, and a map $\varphi : \mathfrak{A} \to \mathfrak{B}$ with the following properties:

(i) $d(x, \varphi(x)) = 1$ for every $x \in \mathfrak{A}$,
(ii) $d^\alpha(x, z) \geqslant 2$ whenever $x \in \mathfrak{A}$ and $z \in \mathfrak{B}$, $z \neq \varphi(x)$,
(iii) $d^\alpha(x_1, x_2) \geqslant 2$ whenever $x_1, x_2 \in \mathfrak{A}$ and $\varphi(x_1) \neq \varphi(x_2)$.

Proposition 3.9. *Let $0 < \alpha < 1$ and let (X, d) be a metric space. The following two statements are equivalent:*

(i) (X, d^α) *is of type* **A**,
(ii) (X, d) *is of type* $\mathbf{A}_\alpha$.

Proposition 3.9 will be proved in Section 6.

It is clear that, since X is of type $\mathbf{A}_\alpha$ if and only if its completion is, the statement of Corollary 3.6 is even simpler when dealing with $\mathrm{Lip}^\alpha(X, E)$.

Also, it is immediate to see that, if (X, d) is of type $\mathbf{A}_\alpha$, then it is of type **A** and of type $\mathbf{A}_\beta$ for $\alpha < \beta < 1$. Consequently, by Theorem 3.4 and Remark 3.5, we conclude the following.

Corollary 3.10. *Let $0 < \alpha < 1$. If there exists a nonstandard isometry between $\mathrm{Lip}^\alpha(X, E)$ and $\mathrm{Lip}^\alpha(Y, F)$, then there exists a nonstandard isometry between $\mathrm{Lip}(X, E)$ and $\mathrm{Lip}(Y, F)$, and between $\mathrm{Lip}^\beta(X, E)$ and $\mathrm{Lip}^\beta(Y, F)$ whenever $\alpha < \beta < 1$.*

Obviously, the converse of Corollary 3.10 is not true in general. The following example shows somehow the differences between cases.

Example 3.11. Let $X := \{-1\} \cup (0, 1) \subset \mathbb{R}$. X is not of type **A**, but its completion $\widehat{X} = \{-1\} \cup [0, 1]$ is. Neither X nor $\widehat{X}$ are of type A_α for $0 < \alpha < 1$. Consequently, we have:

- If E is not complete, then all linear isometries from $\mathrm{Lip}(X, E)$ onto itself are standard, but there are nonstandard isometries from $\mathrm{Lip}(\widehat{X}, E)$ onto itself.
- If E is complete, then there are nonstandard isometries from $\mathrm{Lip}(X, E)$ onto itself. Obviously, by definition of nonstandard isometry, the same holds for $\mathrm{Lip}(\widehat{X}, E)$.
- For every E and $\alpha \in (0, 1)$, all linear isometries from $\mathrm{Lip}^\alpha(X, E)$ onto itself are standard. The same holds for $\mathrm{Lip}^\alpha(\widehat{X}, E)$. In the case when E is complete and X is not, this is due to the special form of X.

4. The case when T satisfies Property P

In this section, unless otherwise stated, we assume that T is a linear isometry from $\mathrm{Lip}(X,E)$ onto $\mathrm{Lip}(Y,F)$ satisfying Property **P**.

Our first goal consists of showing that T is indeed an isometry with respect to the norm $\|\cdot\|_\infty$. The following two lemmas will be the key tools used to prove it.

Lemma 4.1. *Let $f \in \mathrm{Lip}(X,E)$ and $x_0 \in X$ be such that $f(x_0) \neq 0$. Then there exists $g \in \mathrm{Lip}(X,E)$ with $\|g(x_0)\| = \|g\|_\infty > L(g)$ such that*

$$\|g(x_0)\| + \|f(x_0)\| = \|(g+f)(x_0)\| = \|g+f\|_\infty > L(g+f).$$

Proof. We put $\mathrm{e} := f(x_0)$ and assume without loss of generality that $\|\mathrm{e}\| = 1$. We then consider $l \in \mathrm{Lip}(X,E)$ defined by $l(x) := \max\{0, 2-d(x,x_0)\}\cdot \mathrm{e}$ for each $x \in X$. Clearly l satisfies $\|l\|_L = \|l\|_\infty = \|l(x_0)\| = 2$ and $L(l) \leqslant 1$, and also $\|l(x)\| < 2$ for all $x \in X$, $x \neq x_0$.

Take $n \in \mathbb{N}$ with $n > \|f\|_L$. Firstly, it is easy to check that $\|(nl+f)(x_0)\| = 2n+1$. On the other hand, we also have that $L(nl+f) \leqslant nL(l) + L(f) < 2n$ and, for $x \in X \setminus \{x_0\}$ with $d(x,x_0) < 2$,

$$\begin{aligned}\|(nl+f)(x)\| &\leqslant n\|l(x)\| + \|f(x)\| \\ &\leqslant 2n - nd(x,x_0) + \|f(x_0)\| + L(f)d(x,x_0) \\ &< 2n+1,\end{aligned}$$

whereas if $d(x,x_0) \geqslant 2$, then $l(x) = 0$ and $\|(nl+f)(x)\| \leqslant \|f\|_L < n$.

Consequently, if we define $g := nl$, the lemma is proved. □

Remark 4.2. It is easy to see that if $f_1, f_2 \in \mathrm{Lip}(X,E)$ satisfy $\|f_1(x_0)\|, \|f_2(x_0)\| < \|f(x_0)\|$, then the proof of Lemma 4.1 can be slightly modified (by taking $n > \|f\|_L, \|f_1\|_L, \|f_2\|_L$) so that $\|g+f_i\|_\infty < \|g+f\|_\infty$ for $i = 1, 2$.

Lemma 4.3. *If $f \in \mathrm{Lip}(X,E)$ satisfies $\|Tf(y_0)\| = \|Tf\|_\infty > L(Tf)$ for some $y_0 \in Y$, then $L(f) \leqslant \|f\|_\infty$.*

Proof. Suppose that $\|f\|_\infty < L(f)$. Then, for each $\mathrm{e} \in E$, there exists $M > 0$ such that $\|f\|_\infty + M\|\mathrm{e}\| < L(f)$, so $\|f \pm M\tilde{\mathrm{e}}\|_\infty < L(f) = L(f \pm M\tilde{\mathrm{e}})$. Therefore,

$$\|f \pm M\tilde{\mathrm{e}}\|_L = L(f \pm M\tilde{\mathrm{e}}) = L(f) = \|f\|_L.$$

Since T is an isometry, $\|Tf \pm MT\tilde{\mathrm{e}}\|_L = \|Tf\|_L = \|Tf(y_0)\|$, which implies in particular that $\|Tf(y_0) \pm MT\tilde{\mathrm{e}}(y_0)\| \leqslant \|Tf(y_0)\|$ and, by inequality (2.1), that $T\tilde{\mathrm{e}}(y_0) = 0$ for every $\mathrm{e} \in E$, which goes against our hypotheses. □

Remark 4.4. Notice that, in the proof of Lemma 4.3, we just use the fact that there exists $\mathrm{e} \in E$ with $T\tilde{\mathrm{e}}(y_0) \neq 0$, and not the general assumption that Property **P** holds.

Corollary 4.5. *T is an isometry with respect to the supremum norm.*

Proof. Assume that $\|f\|_\infty < \|Tf\|_\infty$, and pick $\epsilon > 0$ and $y_0 \in Y$ such that $\|f\|_\infty + \epsilon < \|Tf(y_0)\|$. Next, by Lemma 4.1, we can take $g \in \mathrm{Lip}(Y,F)$ with $\|g(y_0)\| = \|g\|_\infty > L(g)$ and such that

$$\|g(y_0)\| + \|Tf(y_0)\| = \|(g+Tf)(y_0)\| = \|g+Tf\|_\infty > L(g+Tf).$$

Applying Lemma 4.3, we conclude both that

$$L(T^{-1}g) \leqslant \|T^{-1}g\|_\infty = \|g(y_0)\|$$

and that

$$L(T^{-1}g+f) \leqslant \|T^{-1}g+f\|_\infty = \|g+Tf\|_L = \|(g+Tf)(y_0)\|.$$

But this is impossible because

$$\|T^{-1}g+f\|_\infty < \|g(y_0)\| + \|Tf(y_0)\| - \epsilon < \|(g+Tf)(y_0)\|.$$

We conclude that $\|Tf\|_\infty \leqslant \|f\|_\infty$ for every f. We next see that T^{-1} also satisfies Property **P**, which is enough to prove the equality. By the above, given a nonzero $\mathrm{f} \in F$ we have $\|T^{-1}\tilde{\mathrm{f}}\|_\infty = \|\mathrm{f}\|$. Also, if $(T^{-1}\tilde{\mathrm{f}})(x_0) = 0$ for some $x_0 \in X$, then there exists $k \in \mathrm{Lip}(X,E)$, $k \neq 0$, with $\|k(x)\| + \|(T^{-1}\tilde{\mathrm{f}})(x)\| \leqslant \|\mathrm{f}\|$ for every $x \in X$. By inequality (2.1), $\|\mathrm{f}\| < \|\tilde{\mathrm{f}} + Tk\|_\infty$ or $\|\mathrm{f}\| < \|\tilde{\mathrm{f}} - Tk\|_\infty$, which contradicts the paragraph above. □

Remark 4.6. Notice that, in the proof of Corollary 4.5, we have seen that T^{-1} also satisfies Property **P**.

We are now ready to see that, under the assumptions we make in this section, every surjective linear isometry is biseparating.

Proposition 4.7. *T is biseparating.*

Proof. We prove that T is separating. Suppose that it is not, so there exist $f, g \in \mathrm{Lip}(X, E)$ such that $c(f) \cap c(g) = \emptyset$ but $Tf(y_0) = \mathrm{f}_1 \neq 0$ and $Tg(y_0) = \mathrm{f}_2 \neq 0$ for some $y_0 \in Y$. Taking into account inequality (2.1), we can assume without loss of generality that $\|\mathrm{f}_2\| \leqslant \|\mathrm{f}_1\| < \|\mathrm{f}_1 + \mathrm{f}_2\|$. Now, by Lemma 4.1 and Remark 4.2, there exists $k \in \mathrm{Lip}(Y, F)$ such that $\|k + Tf\|_\infty, \|k + Tg\|_\infty < \|k + Tf + Tg\|_\infty$.

On the other hand, since f and g have disjoint cozeros,

$$\|T^{-1}k + f + g\|_\infty = \max\{\|T^{-1}k + f\|_\infty, \|T^{-1}k + g\|_\infty\},$$

and consequently $\|k + Tf + Tg\|_\infty = \max\{\|k + Tf\|_\infty, \|k + Tg\|_\infty\}$, which is a contradiction.

By Remark 4.6, T^{-1} is also separating. $\square$

Remark 4.8. In [3, Theorem 3.1] (see also comments after it) a description of biseparating maps $S : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$ is given, but we cannot use it here because assumptions of completeness on X and Y are made in [3]. Under some circumstances automatic continuity of such S can be achieved and, in that case, the description goes as follows (where $\mathcal{L}(E, F)$ denotes the normed space of all linear and continuous operators from E to F): There exist a homeomorphism $k : Y \to X$ and a map $K : Y \to \mathcal{L}(E, F)$ (which is easily seen to be also Lipschitz with $L(K) \leqslant \|S\|$) such that $Sf(y) = Ky(f(k(y)))$ for all $f \in \mathrm{Lip}(X, E)$ and $y \in Y$. Also, if both X and Y are bounded, then the map k is bi-Lipschitz.

Proposition 4.9. *Given $\mathrm{e} \in E$, $T\tilde{\mathrm{e}}$ is constant on each 1-component of Y and $\|T\tilde{\mathrm{e}}(y)\| = \|\mathrm{e}\|$ for all $y \in Y$.*

Proof. Suppose that this is not the case, but there exist $\mathrm{e} \in E$, $\|\mathrm{e}\| = 1$, and $y_1, y_2 \in Y$ with $y_1 \sim_1 y_2$, such that $\mathrm{f}_1 := T\tilde{\mathrm{e}}(y_1)$ and $\mathrm{f}_2 := T\tilde{\mathrm{e}}(y_2)$ are different. Of course, we may assume without loss of generality that $D := d(y_1, y_2) < 1$ and that $\mathrm{f}_1 \neq 0$. Now, if we consider $g \in \mathrm{Lip}(Y, F)$ defined by

$$g(y) := \max\left\{0, 1 - \frac{d(y, y_1)}{D}\right\} \cdot \mathrm{f}_1$$

for all $y \in Y$, then obviously $\|g\|_L = L(g) = \|\mathrm{f}_1\|/D > \|g\|_\infty$. As a consequence, using Corollary 4.5, $\|T^{-1}g\|_L > \|T^{-1}g\|_\infty$, and we can take $M > 0$ such that $\|T^{-1}g \pm M\tilde{\mathrm{e}}\|_L = \|T^{-1}g\|_L = \|\mathrm{f}_1\|/D$.

Notice also that, as F is strictly convex, either $\|\mathrm{f}_1 + M(\mathrm{f}_1 - \mathrm{f}_2)\| > \|\mathrm{f}_1\|$ or $\|\mathrm{f}_1 - M(\mathrm{f}_1 - \mathrm{f}_2)\| > \|\mathrm{f}_1\|$, that is, either

$$\frac{\|(g + MT\tilde{\mathrm{e}})(y_1) - (g + MT\tilde{\mathrm{e}})(y_2)\|}{d(y_1, y_2)} > \frac{\|\mathrm{f}_1\|}{D}$$

or

$$\frac{\|(g - MT\tilde{\mathrm{e}})(y_1) - (g - MT\tilde{\mathrm{e}})(y_2)\|}{d(y_1, y_2)} > \frac{\|\mathrm{f}_1\|}{D},$$

which implies that either $\|g + MT\tilde{\mathrm{e}}\|_L > \|\mathrm{f}_1\|/D$ or $\|g - MT\tilde{\mathrm{e}}\|_L > \|\mathrm{f}_1\|/D$, yielding a contradiction.

Finally suppose that $T\tilde{\mathrm{e}}(y) = \mathrm{f}$ for all y in $B \in \mathrm{Comp}_1(Y)$. Since by Proposition 4.7 T^{-1} is separating, $c(T^{-1}(\chi_B \cdot T\tilde{\mathrm{e}})) \cap c(T^{-1}(\chi_{Y \setminus B} \cdot T\tilde{\mathrm{e}})) = \emptyset$. This implies that e is the only nonzero value taken by $T^{-1}(\chi_B \cdot T\tilde{\mathrm{e}})$ and, since T^{-1} is an isometry with respect to $\|\cdot\|_\infty$, we have that $\|\mathrm{e}\| = \|\mathrm{f}\|$. $\square$

Lemma 4.10. *There exists a bijection $H : \mathrm{Comp}_1(X) \to \mathrm{Comp}_1(Y)$ and, for each $A \in \mathrm{Comp}_1(X)$, a surjective linear isometry $\mathbf{J}_A : E \to F$ with the property that $T(\chi_A \cdot \tilde{\mathrm{e}}) = \chi_{H(A)} \cdot \widetilde{\mathbf{J}_A(\mathrm{e})} = \chi_{H(A)} \cdot T\tilde{\mathrm{e}}$ for every $\mathrm{e} \in E$.*

Proof. Fix $A \in \mathrm{Comp}_1(X)$ and $\mathrm{e} \in E$ with $\|\mathrm{e}\| = 1$, and take $f := \chi_A \cdot \tilde{\mathrm{e}}$, $g := \chi_{X \setminus A} \cdot \tilde{\mathrm{e}}$ in $\mathrm{Lip}(X, E)$. We have that $c(f) \cap c(g) = \emptyset$, so by Proposition 4.7 Tf and Tg have disjoint cozeros. Then, by Proposition 4.9, $Tf(y), Tg(y) \in \{0, T\tilde{\mathrm{e}}(y)\}$ for all $y \in Y$. Now, suppose that $y \sim_1 y'$, and that $Tf(y) \neq 0$ and $Tf(y') = 0$. We can assume that $d(y, y') < 1$. Since $\|Tf(y)\| = 1$, we deduce $L(Tf) \geqslant 1/d(y, y') > 1$, which is impossible because $\|f\|_L = 1$.

Reasoning similarly with T^{-1}, $Tf = \chi_B \cdot \tilde{\mathrm{f}}$ for some 1-component B in Y and some norm-one vector $\mathrm{f} \in F$. The conclusion is now easy. $\square$

Lemma 4.11. *Given $A, B \in \mathrm{Comp}_1(X)$, if $\min\{d(A, B), d(H(A), H(B))\} < 2$, then $d(A, B) = d(H(A), H(B))$ and $\mathbf{J}_A = \mathbf{J}_B$.*

Proof. Put $D_1 := d(A, B)$, $D_2 := d(H(A), H(B))$. Due to the symmetric rôles of H and H^{-1} with respect to T and T^{-1}, we can assume without loss of generality that $D_1 \leqslant D_2$. Pick $\mathrm{e} \in E$ with $\|\mathrm{e}\| = 1$, and define $f := (\chi_A - \chi_B) \cdot \tilde{\mathrm{e}} \in \mathrm{Lip}(X, E)$. We easily see that $\|f\|_L = L(f) = 2/D_1$ and, since $L(Tf) = \|\mathbf{J}_A(\mathrm{e}) + \mathbf{J}_B(\mathrm{e})\|/D_2 \leqslant 2/D_2$, we necessarily have $D_1 = D_2$ and $\|\mathbf{J}_A(\mathrm{e}) + \mathbf{J}_B(\mathrm{e})\| = 2$, so $\mathbf{J}_A(\mathrm{e}) = \mathbf{J}_B(\mathrm{e})$ because F is strictly convex. □

Corollary 4.12. *There exists a map $J : Y \to I(E, F)$ which is constant on each 2-component of Y and such that $T\tilde{\mathrm{e}}(y) = Jy(\mathrm{e})$ for all $\mathrm{e} \in E$ and $y \in Y$.*

Proof. We define $Jy := \mathbf{J}_A$ if $y \in H(A)$ and $A \in \mathtt{Comp}_1(X)$. Applying Lemma 4.11, the result follows. □

Lemma 4.13. *Let (y_n) be a sequence in Y which is not a Cauchy sequence and such that all y_n are pairwise different. Then there exist infinite subsets A_1 and A_2 of $\{y_n\colon n \in \mathbb{N}\}$ with $d(A_1, A_2) > 0$.*

Proof. Taking a subsequence if necessary, we have that there exists $\epsilon > 0$ such that $d(y_{2n}, y_{2n-1}) \geqslant 3\epsilon$ for all $n \in \mathbb{N}$. Let $A := \{y_n\colon n \in \mathbb{N}\}$. Now we have two possibilities: either there exists n_0 such that $B(y_{n_0}, \epsilon)$ contains infinitely many y_n or $A \cap B(y_k, \epsilon)$ is finite for every k. In the first case, it is clear that $A_1 := A \cap B(y_{n_0}, \epsilon)$ and $A_2 := \{y_{2n}\colon y_{2n-1} \in A_1\} \cup \{y_{2n-1}\colon y_{2n} \in A_1\}$ satisfy $d(A_1, A_2) \geqslant \epsilon$. In the second case, we can find a subsequence (y_{n_k}) with $d(y_{n_k}, y_{n_l}) > \epsilon$ when $k \neq l$, and the result follows easily. □

In Lemma 4.14 and Corollary 4.15 we do not necessarily assume that base spaces are not complete, so it could be the case that $\widehat{X} = X$ and $\widehat{Y} = Y$.

Lemma 4.14. *Given $x_0 \in \widehat{X}$, there exists $y_0 \in \widehat{Y}$ such that $\widehat{Tf}(y_0) = 0$ whenever $f \in \mathrm{Lip}(X, E)$ satisfies $\widehat{f}(x_0) = 0$.*

Proof. Fix $\mathrm{e} \in E$ with $\|\mathrm{e}\| = 1$, and let

$$A := \Big\{ f \cdot \tilde{\mathrm{e}}\colon\ f \in \mathrm{Lip}(\widehat{X}),\ f(x_0) = 1,\ \forall \epsilon > 0,\ \sup_{d(x,x_0) \geqslant \epsilon} \big|f(x)\big| < 1 \Big\}.$$

We will see that there exists a unique point $y_0 \in \widehat{Y}$ such that $\widehat{\|Tf\|}(y_0) = 1$ for every $f \in A$.

Fix $f_0 \in A$. By Corollary 4.5, taking into account that $\|f_0\|_\infty = 1$, there exists a sequence (y_n) in Y such that $\|Tf_0(y_n)\| \geqslant 1 - 1/n$ for each $n \in \mathbb{N}$. Let us see that it is a Cauchy sequence. Suppose that this is not the case. Either if all y_n are pairwise different (by using Lemma 4.13) or not, we see that there exist subsets A_1, A_2 of $\{y_n\colon n \in \mathbb{N}\}$ such that $d(A_1, A_2) > 0$ and $\sup_{y_n \in A_i} \|Tf_0(y_n)\| = 1$, $i = 1, 2$. Then we take $g_1, g_2 \in \mathrm{Lip}(\widehat{Y})$ with $0 \leqslant g_1, g_2 \leqslant 1$ such that $g_1(A_1) = 1$, $g_2(A_2) = 1$, and $g_1 g_2 \equiv 0$. It is immediate that $\|Tf_0 + g_i Tf_0\|_\infty = 2$ for $i = 1, 2$. Since, again by Corollary 4.5, $\|T^{-1}(g_i Tf_0)\|_\infty = 1$, we deduce that $\widehat{\|T^{-1}(g_i Tf_0)\|}(x_0) = 1$ for $i = 1, 2$, which goes against the fact that T^{-1} is separating. Consequently (y_n) is a Cauchy sequence and converges to a point $y_0 \in \widehat{Y}$, which obviously satisfies $\widehat{\|Tf_0\|}(y_0) = 1$. Now it is straightforward to see that $\widehat{\|Tf\|}(y_0) = 1$ for every $f \in A$.

Next suppose that $f \in \mathrm{Lip}(X, E)$ satisfies $\widehat{f}(x_0) = 0$. Then, given $\epsilon > 0$, there exists $f_\epsilon \in \mathrm{Lip}(X, E)$ such that $\widehat{f_\epsilon} \equiv 0$ on a neighborhood of x_0 and $\|f - f_\epsilon\|_\infty \leqslant \epsilon$. We can take $f'_\epsilon \in A$ with $c(f'_\epsilon) \cap c(f_\epsilon) = \emptyset$, and we deduce from the paragraph above that $\widehat{\|Tf_\epsilon\|} \equiv 0$ on a neighborhood of y_0 in $\widehat{Y}$; in particular $\widehat{\|Tf_\epsilon\|}(y_0) = 0$. Since $\|Tf - Tf_\epsilon\|_\infty \leqslant \epsilon$ (by Corollary 4.5), we conclude that $\widehat{\|Tf\|}(y_0) \leqslant \epsilon$, and we are done. □

Corollary 4.15. *There exists a bijective map $h : \widehat{Y} \to \widehat{X}$ such that $Tf(y) = Jy(\widehat{f}(h(y)))$ whenever $y \in Y$ and $f \in \mathrm{Lip}(X, E)$ admits a continuous extension to $h(y)$.*

Proof. Let x_0 and y_0 be as in Lemma 4.14. Since T^{-1} is also biseparating, there exists $x_1 \in \widehat{X}$ such that $\widehat{f}(x_1) = 0$ whenever $\widehat{Tf}(y_0) = 0$ and, in particular, whenever $\widehat{f}(x_0) = 0$. Now, as $\mathrm{Lip}(\widehat{X}, E)$ separates points in $\widehat{X}$, we deduce that $x_1 = x_0$. As a consequence, it is straightforward to see that Lemma 4.14 gives us a bijective map between $\widehat{X}$ and $\widehat{Y}$, which we denote by $h : \widehat{Y} \to \widehat{X}$, satisfying $\widehat{Tf}(y) = 0$ if and only if $\widehat{f}(h(y)) = 0$. Finally, if $f \in \mathrm{Lip}(X, E)$ can be continuously extended to $h(y)$, say $\widehat{f}(h(y)) = \mathrm{e} \in E$, then $(\widehat{f - \tilde{\mathrm{e}}})(h(y)) = 0$, and the representation follows from Corollary 4.12. □

Remark 4.16. As in the proof of Corollary 4.15, the bijection $k : \widehat{X} \to \widehat{Y}$ associated to T^{-1} satisfies $\widehat{T^{-1}g}(x) = 0$ if and only if $\widehat{g}(k(x)) = 0$, $g \in \mathrm{Lip}(Y, F)$. This implies that $k = h^{-1}$.

Lemma 4.17. *If E is not complete, then there exists a sequence (e_n) in E with $\|\mathrm{e}_n\| \leqslant 1/4^n$ such that $\sum_{n=1}^\infty \mathrm{e}_n$ does not converge in E.*

Proof. Clearly, there exists a nonconvergent sequence (u_n) in E satisfying $\|\mathrm{u}_n - \mathrm{u}_{n+1}\| \leqslant 1/4^n$ for every $n \in \mathbb{N}$. It is then easy to check that it is enough to define $\mathrm{e}_n := \mathrm{u}_n - \mathrm{u}_{n+1}$ for each n. □

Corollary 4.18. *If E is not complete, then the map h given in Corollary* 4.15 *is a bijection from Y onto X.*

Proof. We will prove first that $h(y) \in X$ whenever $y \in Y$. If this is not the case, then take $y \in Y$ with $h(y) \in \widehat{X} \setminus X$. For each $n \in \mathbb{N}$, let

$$f_n(x) := \max\{0, 1 - 2^n d(x, h(y))\}$$

for all $x \in X$. It is clear that each f_n belongs to $\mathrm{Lip}(X)$ and that $L(f_n) \leqslant 2^n$. It is easy to see that, since $\mathrm{Lip}(X, \widehat{E})$ is complete, if we take (e_n) in E as in Lemma 4.17, then $f := \sum_{n=1}^{\infty} f_n \cdot \widetilde{\mathrm{e}_n}$ belongs to $\mathrm{Lip}(X, \widehat{E})$, and since all values are taken in E, to $\mathrm{Lip}(X, E)$. Thus, by Corollary 4.5,

$$\lim_{k\to\infty} \left\| Tf - \sum_{n=1}^{k} T(f_n \cdot \widetilde{\mathrm{e}_n}) \right\|_{\infty} = 0.$$

Finally, by Corollary 4.15, this implies that $Tf(y) = \sum_{n=1}^{\infty} Jy(\mathrm{e}_n)$, which belongs to $\widehat{F} \setminus F$, and Tf takes values outside F, which is absurd.

We deduce from Remark 4.16 that $h(Y) = X$. □

Proof of Theorem 3.1. Taking into account Corollaries 4.12, 4.15 and 4.18, it is enough to show that $h \in \mathrm{iso}_{<2}(Y, X)$. Let $y_1, y_2 \in Y$ be such that $d(y_1, y_2) < 2$. We are going to see that $D := d(h(y_1), h(y_2)) \leqslant d(y_1, y_2)$.

Pick $\mathrm{e} \in E$ with $\|\mathrm{e}\| = 1$ and define $g \in \mathrm{Lip}(X, E)$ by

$$g(x) := \max\left\{-1, 1 - \frac{2d(x, h(y_1))}{D}\right\} \cdot \mathrm{e}$$

for every $x \in X$. We have that $\|g\|_\infty = 1$, $L(g) = 2/D$, $g(h(y_1)) = \mathrm{e}$, and $g(h(y_2)) = -\mathrm{e}$. Obviously, by Corollary 4.12, $Jy_1 = Jy_2$, and

$$1 < \frac{2}{d(y_1, y_2)} = \frac{\|Jy_1(\mathrm{e}) - Jy_2(-\mathrm{e})\|}{d(y_1, y_2)} = \frac{\|Tg(y_1) - Tg(y_2)\|}{d(y_1, y_2)} \leqslant \|Tg\|_L,$$

which implies that $\|g\|_L > 1$, and then $\|g\|_L = L(g) = 2/D$. This means that $\|Tg\|_L = 2/D$, and consequently $2/d(y_1, y_2) \leqslant 2/D$. The other inequality can be seen in a similar way working with T^{-1} (see Remark 4.16). □

Proof of Corollary 3.3. The fact that $\widehat{T}$ satisfies Property **P** follows easily from Proposition 4.9. The conclusion is then immediate by Theorem 3.1. □

5. The distance between $\mathfrak{A}(T)$ and $\mathfrak{B}(T)$

Propositions 5.1 and 5.2 will be used in Section 6.

Proposition 5.1. *Let $T : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$ be a surjective linear isometry. If $\mathfrak{A}(T) \neq \emptyset$, then $d(\mathfrak{A}(T), \mathfrak{B}(T)) \geqslant 1$.*

Proof. Obviously $\mathfrak{B}(T) \neq \emptyset$. Suppose first that $d(\mathfrak{A}(T), \mathfrak{B}(T)) < 1$, and take $y_0 \in \mathfrak{B}(T)$ and $\epsilon > 0$ with $d(y_0, \mathfrak{A}(T)) < 1 - 2\epsilon$. We then select $\mathrm{f} \in F$, $\|\mathrm{f}\| = 1$, and define $l \in \mathrm{Lip}(Y, F)$ by $l(y) := \max\{0, 2 - d(y, y_0)\} \cdot \mathrm{f}$ for every $y \in Y$. We have that $\|l\|_L = \|l\|_\infty = \|l(y_0)\| = 2$, $L(l) \leqslant 1$, and $\|l(y)\| < 2$ for all $y \in Y \setminus \{y_0\}$.

Now, by Lemma 4.3 (see also Remark 4.4), we have that $L(T^{-1}l) \leqslant \|T^{-1}l\|_\infty$. Consequently $\|T^{-1}l\|_L = \|T^{-1}l\|_\infty$, and then $\|T^{-1}l\|_\infty = 2$. Therefore, there is a point x_0 in X such that $\|T^{-1}l(x_0)\| > 2 - \epsilon$, that is, $T^{-1}l(x_0) = \alpha\mathrm{e}$ for some $\mathrm{e} \in E$, $\|\mathrm{e}\| = 1$, and $\alpha \in \mathbb{R}$, $\alpha > 2 - \epsilon$. Next, obviously

$$\|\tilde{\mathrm{e}} + T^{-1}l\|_L \geqslant \|\mathrm{e} + T^{-1}l(x_0)\| = \|(1 + \alpha)\mathrm{e}\| > 3 - \epsilon,$$

so $\|T\tilde{\mathrm{e}} + l\|_L > 3 - \epsilon$. Since $L(T\tilde{\mathrm{e}} + l) \leqslant L(T\tilde{\mathrm{e}}) + L(l) \leqslant 2$, this implies that $\|T\tilde{\mathrm{e}} + l\|_\infty > 3 - \epsilon$, and hence the set $B := \{y \in Y\colon \|(T\tilde{\mathrm{e}} + l)(y)\| > 3 - \epsilon\}$ is nonempty.

Notice that, since $\|T\tilde{\mathrm{e}}\|_\infty \leqslant 1$, all points $y \in B$ must satisfy $\|l(y)\| > 2 - \epsilon$, which is equivalent to $d(y, y_0) < \epsilon$. Thus, for some y_1 with $d(y_1, y_0) < \epsilon$, we have $\|T\tilde{\mathrm{e}}(y_1) + l(y_1)\| > 3 - \epsilon$, which implies that $\|T\tilde{\mathrm{e}}(y_1)\| > 1 - \epsilon$. On the other hand, taking into account that $d(y_0, \mathfrak{A}(T)) < 1 - 2\epsilon$, there exists $y_2 \in \mathfrak{A}(T)$ with $d(y_0, y_2) \leqslant 1 - 2\epsilon$. Finally, observe that

$$\frac{\|T\tilde{\mathrm{e}}(y_1) - T\tilde{\mathrm{e}}(y_2)\|}{d(y_1, y_2)} = \frac{\|T\tilde{\mathrm{e}}(y_1)\|}{d(y_1, y_2)} > \frac{1 - \epsilon}{1 - 2\epsilon + \epsilon} = 1,$$

which allows us to conclude that $L(T\tilde{\mathrm{e}}) > 1$, in contradiction with the fact that $\|\mathrm{e}\| = 1$ and T is an isometry. □

Proposition 5.2. *Let $T : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$ be a surjective linear isometry. If $y_0 \in \mathfrak{A}(T)$, then $d(y_0, \mathfrak{B}(T)) = 1$.*

Proof. Suppose on the contrary that there exists $s \in (0, 1)$ such that $d(B(y_0, s), \mathfrak{B}(T)) > 1 + s$. Take $f \in \mathrm{Lip}(Y)$ with $c(f) \subset B(y_0, s)$ and such that $0 \leqslant f \leqslant s$, $f(y_0) = s$, and $L(f) \leqslant 1$. Let $\mathrm{e} \in E$ and $\mathrm{f} \in F$ have norm 1. It is easy to check that $\|f \cdot \mathrm{f} \pm T\tilde{\mathrm{e}}\|_L \leqslant 1$, whereas, since $T^{-1}(f \cdot \mathrm{f}) \neq 0$, inequality (2.1) implies that

$$\left\|T^{-1}(f \cdot \mathrm{f}) + \tilde{\mathrm{e}}\right\|_\infty > 1$$

or

$$\left\|T^{-1}(f \cdot \mathrm{f}) - \tilde{\mathrm{e}}\right\|_\infty > 1,$$

contradicting the fact that T is an isometry. □

We next see that Property **P** holds when Y is 1-connected. Obviously, the same result holds if X is 1-connected (see Remark 4.6).

Corollary 5.3. *Let Y be 1-connected and suppose that $\mathrm{Lip}(X, E)$ and $\mathrm{Lip}(Y, F)$ are linearly isometric. Then X is also 1-connected and every surjective linear isometry $T : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$ satisfies Property* **P**.

Proof. By Proposition 5.1, Property **P** holds when Y is 1-connected.

The fact that X is 1-connected can be easily deduced from the representation of T in Theorem 3.1 or that of $\widehat{T}$ in Corollary 3.3 (taking into account that a metric space is 1-connected if and only if so is its completion). □

Remark 5.4. An immediate consequence of Corollary 5.3 is that, when X (or Y) is 1-connected, every surjective linear isometry $T : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$ is standard in any of the cases (i), (ii), given in Theorem 3.1.

6. The case when *T* does not satisfy Property P

In this section, unless otherwise stated, we assume that T is a linear isometry from $\mathrm{Lip}(X, E)$ onto $\mathrm{Lip}(Y, F)$ that *does not* satisfy Property **P** (that is, $\mathfrak{A}(T) \neq \emptyset$). We will make use of Theorem 3.1, so we also assume that we are in any of the following two cases:

(i) X and Y are complete,
(ii) E (or F) is not complete.

It is then clear by Proposition 5.1 that X is complete if and only if both $\mathfrak{A}(T^{-1})$ and $\mathfrak{B}(T^{-1})$ are complete.

We will introduce two isometries on spaces of Lipschitz functions defined on $\mathfrak{A}(T^{-1})$ and $\mathfrak{B}(T^{-1})$. The fact that these new isometries turn out to be standard will allow us to obtain a description of T.

Lemma 6.1. *Suppose that $f \in \mathrm{Lip}(X, E)$ satisfies $f \equiv 0$ on $\mathfrak{B}(T^{-1})$. Then $Tf \equiv 0$ on $\mathfrak{B}(T)$.*

Proof. Suppose on the contrary that there exists $y_0 \in \mathfrak{B}(T)$ with $Tf(y_0) \neq 0$. By Lemma 4.1, we can find $g \in \mathrm{Lip}(Y, F)$ with $\|g(y_0)\| = \|g\|_\infty > L(g)$ such that

$$\left\|g(y_0)\right\| + \left\|Tf(y_0)\right\| = \left\|(g + Tf)(y_0)\right\| = \|g + Tf\|_\infty > L(g + Tf).$$

We see that

$$\begin{aligned}\sup_{x \in \mathfrak{B}(T^{-1})} \left\|T^{-1}g(x) + f(x)\right\| &= \sup_{x \in \mathfrak{B}(T^{-1})} \left\|T^{-1}g(x)\right\| \\ &\leqslant \|g\|_L \\ &< \left\|(g + Tf)(y_0)\right\|.\end{aligned}$$

On the other hand, if we put $\mathrm{f} := (g + Tf)(y_0)$, since $T^{-1}\tilde{\mathrm{f}} \equiv 0$ on $\mathfrak{A}(T^{-1})$, there exists $n \in \mathbb{N}$ such that

$$\begin{aligned}\sup_{x \in \mathfrak{A}(T^{-1})} \left\|T^{-1}g(x) + f(x) + nT^{-1}\tilde{\mathrm{f}}(x)\right\| &< \sup_{x \in \mathfrak{B}(T^{-1})} \left\|T^{-1}g(x) + f(x) + nT^{-1}\tilde{\mathrm{f}}(x)\right\| \\ &< (n + 1)\|\mathrm{f}\|,\end{aligned}$$

so if we denote $k := T^{-1}(g + n\tilde{\mathrm{f}}) + f$, then we see that $\|k\|_\infty < \|Tk\|_L$. Consequently, $\|k\|_\infty < L(k)$ and there exists $\mathrm{e} \in E$ with $T\tilde{\mathrm{e}}(y_0) \neq 0$ such that $\|k \pm \tilde{\mathrm{e}}\|_\infty < L(k) = L(k \pm \tilde{\mathrm{e}})$.

Also $L(Tk) = L(g + Tf) < \|\mathrm{f}\|$, so if we assume n big enough, then $\|Tk\|_L = \|Tk\|_\infty$. Therefore,

$$\left\|T(k \pm \tilde{\mathrm{e}})\right\|_L = \|k \pm \tilde{\mathrm{e}}\|_L = L(k) = \|Tk\|_\infty = (n+1)\|\mathrm{f}\|.$$

This implies that

$$\left\|(n+1)\mathrm{f} \pm T\tilde{\mathrm{e}}(y_0)\right\| = \left\|T(k \pm \tilde{\mathrm{e}})(y_0)\right\| \leqslant (n+1)\|\mathrm{f}\|,$$

which goes against inequality (2.1). □

Using Proposition 5.1, we see that the subspace

$$\mathrm{Lip}_{\mathfrak{B}}(X, E) := \left\{f \in \mathrm{Lip}(X, E):\ f\left(\mathfrak{A}\left(T^{-1}\right)\right) \equiv 0\right\}$$

is isometrically isomorphic to $\mathrm{Lip}(\mathfrak{B}(T^{-1}), E)$, via the restriction map. In the same way,

$$\mathrm{Lip}_{\mathfrak{A}}(X, E) := \left\{f \in \mathrm{Lip}(X, E):\ f\left(\mathfrak{B}\left(T^{-1}\right)\right) \equiv 0\right\}$$

and $\mathrm{Lip}(\mathfrak{A}(T^{-1}), E)$ are isometrically isomorphic. Let denote by $I_{\mathfrak{B}(T^{-1})} : \mathrm{Lip}(\mathfrak{B}(T^{-1}), E) \to \mathrm{Lip}_{\mathfrak{B}}(X, E)$ and $I_{\mathfrak{A}(T^{-1})} : \mathrm{Lip}(\mathfrak{A}(T^{-1}), E) \to \mathrm{Lip}_{\mathfrak{A}}(X, E)$, respectively, the corresponding natural isometries. In particular we can write in a natural way

$$\mathrm{Lip}(X, E) = \mathrm{Lip}_{\mathfrak{A}}(X, E) \oplus \mathrm{Lip}_{\mathfrak{B}}(X, E) = \mathrm{Lip}\left(\mathfrak{A}\left(T^{-1}\right), E\right) \oplus \mathrm{Lip}\left(\mathfrak{B}\left(T^{-1}\right), E\right),$$

where this equality has to be seen as a direct sum just in the *linear* sense.

Next, let $R_{\mathfrak{B}(T)} : \mathrm{Lip}(Y, F) \to \mathrm{Lip}(\mathfrak{B}(T), F)$ be the operator sending each function to its restriction.

Lemma 6.2. *The map*

$$T_{\mathfrak{B}} := R_{\mathfrak{B}(T)} \circ T \circ I_{\mathfrak{B}(T^{-1})} : \mathrm{Lip}\left(\mathfrak{B}\left(T^{-1}\right), E\right) \to \mathrm{Lip}\left(\mathfrak{B}(T), F\right)$$

is a surjective linear isometry.

Proof. Notice first that if $f \in \mathrm{Lip}_{\mathfrak{B}}(X, E)$ and $g \in \mathrm{Lip}(X, E)$ satisfy $f \equiv g$ on $\mathfrak{B}(T^{-1})$, then $\|f\|_L \leqslant \|g\|_L$.

$T_{\mathfrak{B}}$ is linear and, by Lemma 6.1, it is easy to check that it is surjective. We next see that it is an isometry. Of course this is equivalent to show that $\|R_{\mathfrak{B}(T)} \circ T(f)\|_L = \|T(f)\|_L$ for every $f \in \mathrm{Lip}_{\mathfrak{B}}(X, E)$, and it is clear that $\|R_{\mathfrak{B}(T)}(T(f))\|_L \leqslant \|T(f)\|_L$. Since $\|R_{\mathfrak{B}(T)}(T(f))\|_L = \|I_{\mathfrak{B}(T)}(R_{\mathfrak{B}(T)}(T(f)))\|_L$, the fact that $\|R_{\mathfrak{B}(T)}(T(f))\|_L < \|T(f)\|_L$ is equivalent to that

$$\left\|T^{-1}\left(I_{\mathfrak{B}(T)}\left(R_{\mathfrak{B}(T)}\left(T(f)\right)\right)\right)\right\|_L < \|f\|_L,$$

which goes against the first comment in this proof. □

Lemma 6.3. $T_{\mathfrak{A}} := {I_{\mathfrak{A}(T)}}^{-1} \circ T \circ I_{\mathfrak{A}(T^{-1})} : \mathrm{Lip}(\mathfrak{A}(T^{-1}), E) \to \mathrm{Lip}(\mathfrak{A}(T), F)$ *is standard.*

Proof. Suppose that this is not the case. Since $\mathfrak{A}(T) = \mathfrak{A}(T_{\mathfrak{A}}) \cup \mathfrak{B}(T_{\mathfrak{A}})$, we are in fact saying that $\mathfrak{A}(T_{\mathfrak{A}}) \neq \emptyset$.

For $\mathrm{e} \in E$ with $\|\mathrm{e}\| = 1$, we have

$$\left\|T(\tilde{\mathrm{e}} + \chi_{\mathfrak{A}(T_{\mathfrak{A}}^{-1})} \cdot \mathrm{e})\right\|_L = 2.$$

Notice that both $T\tilde{\mathrm{e}} \equiv 0$ and $T(\chi_{\mathfrak{A}(T^{-1})} \cdot \mathrm{e}) \equiv 0$ on $\mathfrak{A}(T_{\mathfrak{A}})$, so $T(\chi_{\mathfrak{B}(T^{-1})} \cdot \mathrm{e}) \equiv 0$ on $\mathfrak{A}(T_{\mathfrak{A}})$. On the other hand, by Lemma 6.1, $c(T(\chi_{\mathfrak{A}(T_{\mathfrak{A}}^{-1})} \cdot \mathrm{e})) \subset \mathfrak{A}(T_{\mathfrak{A}})$, and consequently, since

$$\|T\tilde{\mathrm{e}}\|_L = \left\|T(\chi_{\mathfrak{A}(T_{\mathfrak{A}}^{-1})} \cdot \mathrm{e})\right\|_L = 1 = \left\|T(\tilde{\mathrm{e}} + \chi_{\mathfrak{A}(T_{\mathfrak{A}}^{-1})} \cdot \mathrm{e})\right\|_\infty,$$

there are sequences (y_n) in $\mathfrak{A}(T_{\mathfrak{A}})$ and (z_n) in $\mathfrak{B}(T)$ with

$$\begin{aligned} 2 &= \lim_{n\to\infty} \frac{\|T(\tilde{\mathrm{e}} + \chi_{\mathfrak{A}(T_{\mathfrak{A}}^{-1})} \cdot \mathrm{e})(y_n) - T(\tilde{\mathrm{e}} + \chi_{\mathfrak{A}(T_{\mathfrak{A}}^{-1})} \cdot \mathrm{e})(z_n)\|}{d(y_n, z_n)} \\ &= \lim_{n\to\infty} \frac{\|T(\chi_{\mathfrak{A}(T_{\mathfrak{A}}^{-1})} \cdot \mathrm{e})(y_n) - T(\chi_{\mathfrak{B}(T^{-1})} \cdot \mathrm{e})(z_n)\|}{d(y_n, z_n)} \end{aligned}$$

$$= \lim_{n\to\infty} \frac{\|T(\chi_{\mathfrak{B}(T^{-1})}\cdot \mathrm{e} + \chi_{\mathfrak{A}(T_{\mathfrak{A}}^{-1})}\cdot \mathrm{e})(y_n) - T(\chi_{\mathfrak{B}(T^{-1})}\cdot \mathrm{e} + \chi_{\mathfrak{A}(T_{\mathfrak{A}}^{-1})}\cdot \mathrm{e})(z_n)\|}{d(y_n, z_n)}$$
$$\leqslant \big\|T\big((\chi_{\mathfrak{B}(T^{-1})} + \chi_{\mathfrak{A}(T_{\mathfrak{A}}^{-1})})\cdot \mathrm{e}\big)\big\|_L$$
$$\leqslant 1.$$

We conclude that $\mathfrak{A}(T_{\mathfrak{A}})$ is empty. □

It is easy to check that $T_{\mathfrak{B}}$ satisfies Property **P**, so by Theorem 3.1, it is standard. We deduce the following result, which allows us to give the values on $\mathfrak{B}(T)$ and on $\mathfrak{A}(T)$ of the images of all functions in $\mathrm{Lip}(X,E)$ and $\mathrm{Lip}_{\mathfrak{A}}(X,E)$, respectively.

Corollary 6.4. *There exist*

(i) $h_{\mathfrak{B}} \in \mathrm{iso}_{<2}(\mathfrak{B}(T), \mathfrak{B}(T^{-1}))$ *and* $h_{\mathfrak{A}} \in \mathrm{iso}_{<2}(\mathfrak{A}(T), \mathfrak{A}(T^{-1}))$*, and*
(ii) *maps* $J_{\mathfrak{B}} : \mathfrak{B}(T) \to I(E,F)$ *and* $J_{\mathfrak{A}} : \mathfrak{A}(T) \to I(E,F)$ *constant on each* 2*-component of* $\mathfrak{B}(T)$ *and* $\mathfrak{A}(T)$*, respectively,*

such that

(i) $Tf(y) = J_{\mathfrak{B}}y(f(h_{\mathfrak{B}}(y)))$ *for all* $f \in \mathrm{Lip}(X,E)$ *and* $y \in \mathfrak{B}(T)$*, and*
(ii) $Tf(y) = J_{\mathfrak{A}}y(f(h_{\mathfrak{A}}(y)))$ *for all* $f \in \mathrm{Lip}_{\mathfrak{A}}(X,E)$ *and* $y \in \mathfrak{A}(T)$*.*

Lemma 6.5. *Let* $y_0 \in \mathfrak{A}(T)$ *and* $A \subset \mathfrak{B}(T^{-1})$ *be such that* $d(h_{\mathfrak{A}}(y_0), A) = 1$*. If* $f \in \mathrm{Lip}_{\mathfrak{B}}(X,E)$ *satisfies* $f(A) \equiv \mathrm{e} \in E$*, then*

$$Tf(y_0) = -J_{\mathfrak{A}}y_0(\mathrm{e}).$$

Proof. Notice first that, since $y_0 \in \mathfrak{A}(T)$, $T\tilde{\mathrm{e}}(y_0) = 0$, and consequently, by Corollary 6.4, $T(\chi_{\mathfrak{B}(T^{-1})}\cdot \mathrm{e})(y_0) = -T(\chi_{\mathfrak{A}(T^{-1})}\cdot \mathrm{e})(y_0) = -J_{\mathfrak{A}}y_0(\mathrm{e})$.

Next we prove the result through several steps. We denote $\mathrm{a} := J_{\mathfrak{A}}y_0(\mathrm{e})$ for short.

Step 1. Assume that $\|\mathrm{e}\| = 1 = \|f\|_L$.

Consider $k' \in \mathrm{Lip}(X)$ defined by $k'(x) := \max\{0, 1 - d(x, h_{\mathfrak{A}}(y_0))\}$ for every $x \in X$, and $k \in \mathrm{Lip}_{\mathfrak{A}}(X,E)$ defined by $k := -k'\cdot \mathrm{e}$. It is easy to see that $(k+f)(h_{\mathfrak{A}}(y_0)) = -\mathrm{e}$ and that $(k+f)(x) = \mathrm{e}$ for every $x \in A$. As a consequence, $\|k+f\|_L = 2$.

Suppose now that $Tf(y_0) = \mathrm{f} \neq -\mathrm{a}$. By Corollary 6.4, $Tk(y_0) = -\mathrm{a}$ and, since $\|f\|_\infty = 1$, we can take $M < 2$ such that

$$\big\|T(k+f)(y_0)\big\| = \|-\mathrm{a} + \mathrm{f}\| < M.$$

Consequently there exists $0 < r < 1$ such that $\|T(k+f)(y)\| < M$ for every $y \in B(y_0, r)$. On the other hand, for $y \in \mathfrak{A}(T)$ with $d(y, y_0) \geqslant r$,

$$\begin{aligned}\big\|Tk(y)\big\| &= \big\|T\big(-k'\cdot \mathrm{e}\big)(y)\big\| \\ &= \big\|\max\big\{0, 1 - d\big(h_{\mathfrak{A}}(y), h_{\mathfrak{A}}(y_0)\big)\big\}\cdot J_{\mathfrak{A}}y(\mathrm{e})\big\| \\ &\leqslant 1 - r,\end{aligned}$$

so $\|T(k+f)(y)\| \leqslant 2 - r$. Since $\|T(k+f)(y)\| = \|Tf(y)\| \leqslant 1$ for every $y \in \mathfrak{B}(T)$, we deduce that $\|T(k+f)\|_\infty < 2 = \|T(k+f)\|_L$. Let $M > 0$ with $M + \|T(k+f)\|_\infty < 2$, and $y \in \mathfrak{B}(T)$ such that $h_{\mathfrak{B}}(y) \in A$ and $d(h_{\mathfrak{B}}(y), h_{\mathfrak{A}}(y_0)) < 1 + M/2$. Define $\mathrm{b} := MT\tilde{\mathrm{e}}(y) \in F$. By Corollary 6.4, $T^{-1}\tilde{\mathrm{b}}(h_{\mathfrak{B}}(y)) = M\mathrm{e}$, and consequently

$$\begin{aligned}2d\big(h_{\mathfrak{B}}(y), h_{\mathfrak{A}}(y_0)\big) &< 2 + M \\ &= \big\|\big(k + f + T^{-1}\tilde{\mathrm{b}}\big)\big(h_{\mathfrak{B}}(y)\big) - \big(k + f + T^{-1}\tilde{\mathrm{b}}\big)\big(h_{\mathfrak{A}}(y_0)\big)\big\|,\end{aligned}$$

against the fact that $\|k + f + T^{-1}\tilde{\mathrm{b}}\|_L = 2$.

Step 2. Assume that $\|\mathrm{e}\| = 1 = \|f\|_\infty$.

It is easy to check that if $n \geqslant L(f)$, then $\|n\chi_{\mathfrak{B}(T^{-1})}\cdot \mathrm{e}\|_\infty = \|n\chi_{\mathfrak{B}(T^{-1})}\cdot \mathrm{e}\|_L = n$ and that

$$\|f + n\chi_{\mathfrak{B}(T^{-1})}\cdot \mathrm{e}\|_\infty = \|f + n\chi_{\mathfrak{B}(T^{-1})}\cdot \mathrm{e}\|_L = n + 1.$$

Using Step 1, $T(n\chi_{\mathfrak{B}(T^{-1})}\cdot \mathrm{e})(y_0) = -n\mathrm{a}$ and $T(f + n\chi_{\mathfrak{B}(T^{-1})}\cdot \mathrm{e})(y_0) = -(n+1)\mathrm{a}$. The conclusion is easy.

Step 3. Assume that $\mathrm{e} = 0$.

Of course we must prove that $Tf(y_0) = 0$. Fix $\mathrm{d} \in E$ with norm 1. Consider $m \in \mathrm{Lip}_{\mathfrak{B}}(X, E)$ defined by $m(x) := \max\{0, 1 - d(x, A)\} \cdot \mathrm{d}$ for each $x \in X$. We easily check that $\|m\|_\infty = 1 = \|\mathrm{d}\|$, and if we assume that $\|f\|_L \leqslant 1$, then $\|f(x)\| \leqslant d(x, A)$ for every x. As in the proof of Lemma 4.1, we see that $\|m + f\|_\infty = 1 = \|\mathrm{d}\|$. The conclusion follows immediately from Step 2.

The rest of the proof is easy. □

Corollary 6.6. *Suppose that $A_1, A_2 \subset \mathfrak{B}(T^{-1})$ satisfy $d(h_{\mathfrak{A}}(y_0), A_i) = 1$ for $i = 1, 2$. Then $d(A_1, A_2) = 0$.*

Proof. Just assume that $d(A_1, A_2) > 0$ and apply Lemma 6.5 to any $f \in \mathrm{Lip}_{\mathfrak{B}}(X, E)$ such that $f(A_i) \equiv (-1)^i \mathrm{e} \neq 0$ for $i = 1, 2$. This leads to two different values for $Tf(y_0)$. □

Corollary 6.7. *Let $y_0 \in \mathfrak{A}(T)$. Then there exists exactly one point $\varphi(y_0)$ in $\mathfrak{B}(T)$ such that $d(h_{\mathfrak{B}}(\varphi(y_0)), h_{\mathfrak{A}}(y_0)) = 1$. Also,*

$$Tf(y_0) = -J_{\mathfrak{A}} y_0\big(f\big(h_{\mathfrak{B}}\big(\varphi(y_0)\big)\big)\big)$$

for every $f \in \mathrm{Lip}_{\mathfrak{B}}(X, E)$.

Proof. By Lemma 4.13 and Corollary 6.6, we deduce that if (x_n) is a sequence in X such that $d(h_{\mathfrak{A}}(y_0), x_n) \leqslant 1 + 1/n$ for each $n \in \mathbb{N}$, then it is a Cauchy sequence, so there is a limit x_0 in $\widehat{X}$, which necessarily belongs to $\widehat{\mathfrak{B}(T^{-1})}$. Obviously the point x_0 does not depend on the sequence we take.

We next assume that X is not complete and prove that $x_0 \in \mathfrak{B}(T^{-1})$. If this is not the case, for each $n \in \mathbb{N}$, let

$$f_n(x) := \max\big\{0, 1 - d\big(x, B(x_0, 1/n)\big)\big\}$$

for all $x \in X$. It is clear that each f_n belongs to $\mathrm{Lip}(X)$. Since $\mathrm{Lip}(X, \widehat{E})$ is complete, if we take (e_n) in E as in Lemma 4.17, then $f := \sum_{n=1}^\infty f_n \cdot \widetilde{\mathrm{e}_n}$ belongs to $\mathrm{Lip}(X, \widehat{E})$, and since all values are taken in E, to $\mathrm{Lip}(X, E)$, and indeed to $\mathrm{Lip}_{\mathfrak{B}}(X, E)$. Thus, since $f = \lim_{k\to\infty} \sum_{n=1}^k f_n \cdot \widetilde{\mathrm{e}_n}$, we deduce from Lemma 6.5 that

$$\begin{aligned} Tf(y_0) &= \lim_{k\to\infty} \sum_{n=1}^k T\big(f_n \cdot \widetilde{\mathrm{e}_n}(y_0)\big) \\ &= -\lim_{k\to\infty} \sum_{n=1}^k J_{\mathfrak{A}} y_0(\mathrm{e}_n) \\ &= -\sum_{n=1}^\infty J_{\mathfrak{A}} y_0(\mathrm{e}_n), \end{aligned}$$

which belongs to $\widehat{F} \setminus F$. This is absurd.

If we define $\varphi(y_0) := {h_{\mathfrak{B}}}^{-1}(x_0) \in \mathfrak{B}(T)$, then we are done. □

Proposition 6.8. *For every $y \in \mathfrak{A}(T)$, $J_{\mathfrak{A}} y = -J_{\mathfrak{B}}\varphi(y)$.*

Proof. Fix $y \in \mathfrak{A}(T)$ and $\mathrm{e} \in E$, and let $\mathrm{f} := T\tilde{\mathrm{e}}(\varphi(y))$. Then $T^{-1}\tilde{\mathrm{f}}(h_{\mathfrak{B}}(\varphi(y))) = \mathrm{e}$ and $T^{-1}\tilde{\mathrm{f}} \equiv 0$ on $\mathfrak{A}(T^{-1})$. We conclude from Corollary 6.4 that $J_{\mathfrak{B}}\varphi(y)(\mathrm{e}) = \mathrm{f}$, and from Corollary 6.7 that $-J_{\mathfrak{A}} y(\mathrm{e}) = \mathrm{f}$. □

Next result follows now easily from Corollaries 6.4 and 6.7, and Proposition 6.8.

Corollary 6.9. *For $y \in \mathfrak{A}(T)$ and $f \in \mathrm{Lip}(X, E)$,*

$$\begin{aligned} Tf(y) &= -J_{\mathfrak{A}} y\big(f\big(h_{\mathfrak{B}}\big(\varphi(y)\big)\big)\big) + J_{\mathfrak{A}} y\big(f\big(h_{\mathfrak{A}}(y)\big)\big) \\ &= J_{\mathfrak{B}}\varphi(y)\big(f\big(h_{\mathfrak{B}}\big(\varphi(y)\big)\big)\big) - J_{\mathfrak{B}}\varphi(y)\big(f\big(h_{\mathfrak{A}}(y)\big)\big). \end{aligned}$$

Corollary 6.10. *Let $y_0 \in \mathfrak{A}(T)$. If $y \in \mathfrak{B}(T)$ is such that $d(y, \varphi(y_0)) \geqslant 2$, then*

$$d(y, y_0) \geqslant 2.$$

Proof. Let $e_1, e_2 \in E$ be vectors with norm 1 and such that $J_{\mathfrak{B}}\varphi(y_0)(e_1) = J_{\mathfrak{B}}y(e_2)$. Define $f := f_1 - f_2 \in \mathrm{Lip}_{\mathfrak{B}}(X, E)$, where

$$f_1(x) := \max\{0, 1 - d(x, h_{\mathfrak{B}}(\varphi(y_0)))\} \cdot e_1$$

and

$$f_2(x) := \max\{0, 1 - d(x, h_{\mathfrak{B}}(y))\} \cdot e_2$$

for every $x \in X$. Obviously, $\|f_1\|_L = 1 = \|f_2\|_L$, so to show that $\|f\|_L = 1$, it is enough to see that, if $d(x, h_{\mathfrak{B}}(\varphi(y_0)))$, $d(z, h_{\mathfrak{B}}(y)) < 1$, then $\|f_1(x)\| + \|f_2(z)\| \leqslant d(x, z)$. Taking into account that

$$2 \leqslant d(h_{\mathfrak{B}}(y), h_{\mathfrak{B}}(\varphi(y_0))) \leqslant d(z, h_{\mathfrak{B}}(y)) + d(x, z) + d(x, h_{\mathfrak{B}}(\varphi(y_0))),$$

it follows that

$$\begin{aligned}\|f_1(x)\| + \|f_2(z)\| &= (1 - d(x, h_{\mathfrak{B}}(\varphi(y_0)))) + (1 - d(z, h_{\mathfrak{B}}(y))) \\ &\leqslant d(x, z).\end{aligned}$$

On the other hand, by Corollary 6.9, $Tf(y_0) = J_{\mathfrak{B}}\varphi(y_0)(e_1) = Tf(\varphi(y_0))$, and by the way we have taken e_1 and e_2, we have $Tf(y) = -J_{\mathfrak{B}}y(e_2) = -J_{\mathfrak{B}}\varphi(y_0)(e_1)$. We conclude that, since $\|Tf\|_L = 1$,

$$2 = \|Tf(y) - Tf(y_0)\| \leqslant d(y, y_0). \qquad \square$$

Corollary 6.11. *Let $y_0 \in \mathfrak{A}(T)$. Given $y \in \mathfrak{B}(T)$, if $0 \leqslant d(y, \varphi(y_0)) < 1$, then*

$$d(y, y_0) = 1 + d(y, \varphi(y_0)),$$

and if $1 \leqslant d(y, \varphi(y_0)) < 2$, then

$$d(y, y_0) \geqslant 2.$$

Proof. Fix $e \in E$ with norm 1 and let $f(x) := \min\{1, d(x, h_{\mathfrak{B}}(\varphi(y_0)))\} \cdot e$ for every $x \in X$. Let $y \in \mathfrak{B}(T)$ with $d(y, \varphi(y_0)) < 2$. Taking into account that $\|f\|_L = 1$, Corollaries 6.4 and 6.9 give

$$\begin{aligned}d(y, y_0) &\geqslant \|Tf(y) - Tf(y_0)\| \\ &= \|\min\{1, d(h_{\mathfrak{B}}(y), h_{\mathfrak{B}}(\varphi(y_0)))\} \cdot J_{\mathfrak{B}}y(e) + J_{\mathfrak{B}}y(e)\| \\ &\geqslant \min\{2, d(h_{\mathfrak{B}}(y), h_{\mathfrak{B}}(\varphi(y_0))) + 1\}.\end{aligned}$$

The conclusion is immediate. $\square$

Corollary 6.12. *If $y_1, y_2 \in \mathfrak{A}(T)$ satisfy $\varphi(y_1) \neq \varphi(y_2)$, then $d(y_1, y_2) \geqslant 2$.*

Proof. Suppose that $M := d(y_1, y_2)/2 < 1$, so by Corollary 6.4 $J_{\mathfrak{A}}y_1 = J_{\mathfrak{A}}y_2$. Put $N := d(h_{\mathfrak{B}}(\varphi(y_1)), h_{\mathfrak{B}}(\varphi(y_2)))$ and, for a fixed $e \in E$ with norm 1, let

$$f(x) := \max\{-M, M - d(x, h_{\mathfrak{A}}(y_1))\} \cdot e$$

and

$$g(x) := \max\{0, N - d(x, h_{\mathfrak{B}}(\varphi(y_1)))\} \cdot e$$

for every $x \in X$. If we take $A > 0$ such that $M + AN < 1$, then $k := \chi_{\mathfrak{A}(T^{-1})} f - A\chi_{\mathfrak{B}(T^{-1})} g$ has norm 1. Also, by Corollary 6.9, $Tk(y_1) = MJ_{\mathfrak{A}}y_1(e) + ANJ_{\mathfrak{A}}y_1(e)$ and $Tk(y_2) = -MJ_{\mathfrak{A}}y_2(e) = -MJ_{\mathfrak{A}}y_1(e)$. Consequently

$$\left\|\frac{Tk(y_1) - Tk(y_2)}{d(y_1, y_2)}\right\| > 1,$$

which is impossible. $\square$

Proof of Theorem 3.4. Corollaries 6.10, 6.11 and 6.12 show that Y is of type **A**. We consider the associated purely non-standard map $S_\varphi : \mathrm{Lip}(Y, F) \to \mathrm{Lip}(Y, F)$ and see that, given $e \in E$, $e \neq 0$, the composition $S_\varphi \circ T : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$ satisfies $S_\varphi \circ T(\tilde{e}) = J_{\mathfrak{B}}\varphi(y)(e) \neq 0$ if $y \in \mathfrak{A}(T)$ and $S_\varphi \circ T(\tilde{e}) = J_{\mathfrak{B}}y(e) \neq 0$ if $y \in \mathfrak{B}(T)$, that is, the composition satisfies Property **P**.

This implies that $S_\varphi \circ T$ is standard. Since $S_\varphi = {S_\varphi}^{-1}$, we have that $T = S_\varphi \circ (S_\varphi \circ T)$, and we are done. $\square$

Remark 6.13. It is easy to check that, if $h : Y \to X$ and $J : Y \to I(E, F)$ are the associated maps to $S_\varphi \circ T$ in the proof of Theorem 3.4, then $h \equiv h_{\mathfrak{A}}$ on $\mathfrak{A}(T)$ and $h \equiv h_{\mathfrak{B}}$ on $\mathfrak{B}(T)$. In the same way, $J \equiv -J_{\mathfrak{A}}$ on $\mathfrak{A}(T)$ and $J \equiv J_{\mathfrak{B}}$ on $\mathfrak{B}(T)$. Finally, it is also apparent that, given $A \in \mathtt{Comp}_2(\mathfrak{B}(T^{-1}))$, $A = A' \cap \mathfrak{B}(T^{-1})$, where $A' \in \mathtt{Comp}_2(X)$, and that a similar fact does not necessarily hold for the elements in $\mathtt{Comp}_2(\mathfrak{A}(T^{-1}))$.

Proof of Proposition 3.9. Suppose that (X, d^α) is of type **A**. Then

$$d^\alpha(z, y) = 1 + d^\alpha\big(z, \varphi(y)\big)$$

whenever $y \in \mathfrak{A}(T)$ and $z \in \mathfrak{B}(T)$ satisfy $0 < d^\alpha(z, \varphi(y)) < 1$. In such case, $d^\alpha(z, y) < d(z, y)$ and $d(z, \varphi(y)) < d^\alpha(z, \varphi(y))$, and this implies

$$d(z, y) > 1 + d\big(z, \varphi(y)\big) = d\big(y, \varphi(y)\big) + d\big(z, \varphi(y)\big),$$

which is impossible. We deduce that, if $d(z, \varphi(y)) < 1$, then $z = \varphi(y)$. The rest of the proof is easy. □

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