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On Spectral Cantor Measures

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A probability measure in $\mathbb{R}^d$ is called a spectral measure if it has an orthonormal basis consisting of exponentials. In this paper, we study spectral Cantor measures. We establish a large class of such measures, and give a necessary and sufficient condition on the spectrum of a spectral Cantor measure. These results extend the studies by Jorgensen and Pedersen (*J. Anal. Math.* **75** (1998), 185–228) and Strichartz (*J. D'Analyse Math.* **81** (2000), 209–238).

Key Words: Spectral measure; Cantor measure; IFS; Ruelle operator

1. INTRODUCTION

It is known that certain Cantor measures in $\mathbb{R}^d$ have an orthonormal basis consisting of complex exponentials. This was first observed in [9] and studied further in [20]. Let μ be a probability measure in $\mathbb{R}^d$. We call μ a *spectral measure* if there exists a $\Lambda \subset \mathbb{R}^d$ such that the set of complex exponentials $\{e(\lambda t) : \lambda \in \Lambda\}$ forms an orthonormal basis for $L^2(\mu)$ (we use $e(t)$ to denote $e^{2\pi i t}$ throughout the paper). The set Λ is called a *spectrum* for μ; we also say that (μ, Λ) is a *spectral pair*. It should be pointed out that a spectral measure often has more than one spectrum.

In this paper, we study spectral Cantor measures in $\mathbb{R}$. Our Cantor measures are self-similar measures associated with iterated function systems

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(IFS). Consider the iterated functions system (IFS) $\{\phi_j\}_{j=1}^q$ given by

$$\phi_j(x) = \rho(x + a_j), \tag{1.1}$$

where $a_j \in \mathbb{R}$ and $|\rho| < 1$. It is well known (see, e.g. [1]) that for any given probability weights $p_1, \ldots, p_q > 0$ with $\sum_{j=1}^q p_j = 1$ there exists a unique probability measure μ satisfying

$$\mu = \sum_{j=1}^{q} p_j \mu \circ \phi_j^{-1}. \tag{1.2}$$

We ask the following question: Under what conditions is μ a spectral measure?

The familiar middle third Cantor measure given by $\rho = 1/3$ and $a_1 = 0$, $a_1 = 2$ with $p_1 = p_2 = 1/2$ is not a spectral measure, see [9]. The first known example of a spectral measure whose support has noninteger dimension was given by the same authors in that paper, who showed that the measure μ corresponding to $\rho = 1/4$, $a_1 = 0$, $a_2 = 1$ and $p_1 = p_2 = 1/2$ is spectral. A spectrum of μ is

$$\Lambda = \left\{ \sum_{k=0}^{m} d_k 4^k : m \geqslant 0,\ d_k = 0 \text{ or } 2 \right\}.$$

Strichartz [19] gave an alternative proof of this result, and found other examples of spectral Cantor measures. Later Strichartz [20] applied his method to a more general setting to show that a class of measures having a self-similar type of structure (but not necessarily self-similar in the more traditional sense) are spectral measures, provided that an implicit condition on the zero set of certain trigonometric polynomials is satisfied. This condition, however, is not necessary and is somewhat difficult to check.

Spectral measures are a natural generalization of spectral sets. A measurable set Ω in $\mathbb{R}^d$ with positive and finite measure is called *spectral* if $L^2(\Omega)$ has an orthogonal basis consisting of complex exponentials. Spectral sets have been studied rather extensively, particularly in recent years. (A partial list of these studies is in the reference of the paper.) The major unsolved problem concerning spectral sets is the following conjecture of Fuglede [2]:

Fuglede's Spectral Set Conjecture. Let Ω be a set in $\mathbb{R}^d$ with positive and finite Lebesgue measure. Then Ω is a spectral set if and only if Ω tiles $\mathbb{R}^d$ by translation.

The conjecture remains open in either direction, even in dimension one and for sets that are unions of unit intervals. As we shall see, the spectral measures-tiling connection seems to be equally compelling.

In this paper, we study self-similar measures satisfying

$$\mu = \sum_{j=1}^{q} \frac{1}{q} \mu \circ \phi_j^{-1}, \tag{1.3}$$

where $\phi_j(x) = \frac{1}{N}(x + d_j)$, $N \in \mathbb{Z}$ and $|N| > 1$, and $\mathscr{D} = \{d_j\} \subset \mathbb{Z}$. We use $\mu_{N,\mathscr{D}}$ to denote the unique probability measure satisfying (1.3). In addition, for each finite subset $\mathscr{A}$ of $\mathbb{R}$ we use $T(N, \mathscr{A})$ to denote the set

$$T(N, \mathscr{A}) := \left\{ \sum_{j=1}^{\infty} a_j N^{-j} : a_j \in \mathscr{A} \right\}$$

which is, in fact, the attractor of the IFS $\{\phi_a(x) := \frac{1}{N}(x + a) : a \in \mathscr{A}\}$, see [1]. Finally, denote

$$\Lambda(N, \mathscr{A}) := \left\{ \sum_{j=0}^{k} a_j N^j : k \geqslant 1 \text{ and } j \in \mathscr{A} \right\}.$$

Two finite sets $\mathscr{A} = \{a_j\}$ and $\mathscr{S} = \{s_j\}$ of cardinality q in $\mathbb{R}$ form a *compatible pair*, following the terminology of [20], if the matrix $M = \left[\frac{1}{\sqrt{q}} e(a_j s_k) \right]$ is a unitary matrix. In other words $(\delta_{\mathscr{A}}, \mathscr{S})$ is a spectral pair, where

$$\delta_{\mathscr{A}} := \sum_{a \in \mathscr{A}} \frac{1}{q} \delta(x - a).$$

For each finite set $\mathscr{A}$ in $\mathbb{R}$ define its *symbol* by

$$m_{\mathscr{A}}(\xi) := \frac{1}{|\mathscr{A}|} \sum_{a \in \mathscr{A}} e(-a\xi).$$

Strichartz [20] proves the following theorem:

THEOREM 1.1 (Strichartz). *Let $N \in \mathbb{Z}$ with $|N| > 1$ and $\mathscr{D}$ be a finite set of integers. Let $\mathscr{S} \subset \mathbb{Z}$ such that $0 \in \mathscr{S}$ and $\left(\frac{1}{N}\mathscr{D}, \mathscr{S}\right)$ is a compatible pair. Suppose that $m_{\mathscr{D}/N}(\xi)$ does not vanish on $T(N, \mathscr{S})$. Then the self-similar measure $\mu_{N,\mathscr{D}}$ is a spectral measure with spectrum $\Lambda(N, \mathscr{A})$.*

Unfortunately, the condition that $m_{\mathscr{D}/N}$ does not vanish on $T(N, \mathscr{S})$ is not a necessary condition, and can be very difficult to check, even when both $\mathscr{D}$ and $\mathscr{S}$ are simple. In general, we know very little about the zeros of $m_{\mathscr{D}/N}$. Our objective here is to remove the above condition. We prove that a compatible pair automatically yields a spectral measure. We also give a necessary and sufficient condition for $\Lambda(N, \mathscr{A})$ to be a spectrum.

THEOREM 1.2. *Let $N \in \mathbb{Z}$ with $|N| > 1$ and let $\mathcal{D}$ be a finite set of integers. Let $\mathcal{S} \subset \mathbb{Z}$ such that $0 \in \mathcal{S}$ and $\left(\frac{1}{N}\mathcal{D}, \mathcal{S}\right)$ is a compatible pair. Then the self-similar measure $\mu_{N,\mathcal{D}}$ is a spectral measure. If moreover* $\gcd(\mathcal{D} - \mathcal{D}) = 1$, $0 \in \mathcal{S}$ *and* $\mathcal{S} \subseteq [2 - |N|, |N| - 2]$, *then $\Lambda(N, \mathcal{S})$ is a spectrum for $\mu_{N,\mathcal{D}}$.*

In Lemma 2.2 we prove that if $\left(\frac{1}{N}\mathcal{D}, \mathcal{S}\right)$ is a compatible pair for some $\mathcal{S} \subset \mathbb{Z}$, then there is also a set $\hat{\mathcal{S}} \subset \mathbb{Z}$ satisfying the additional conditions of Theorem 1.2 (i.e. $0 \in \hat{\mathcal{S}}$ and $\hat{\mathcal{S}} \subseteq [2 - |N|, |N| - 2]$) such that $\left(\frac{1}{N}\mathcal{D}, \hat{\mathcal{S}}\right)$ is a compatible pair.

The following theorem gives a necessary and sufficient condition for $\Lambda(N, \mathcal{A})$ to be a spectrum. It also leads to a simple algorithm, see Section 3.

THEOREM 1.3. *Let $N \in \mathbb{Z}$ with $|N| > 1$ and $\mathcal{D} \subset \mathbb{Z}$ with $0 \in \mathcal{D}$ and* $\gcd(\mathcal{D}) = 1$. *Let $\mathcal{S} \subset \mathbb{Z}$ with $0 \in \mathcal{S}$ such that $\left(\frac{1}{N}\mathcal{D}, \mathcal{S}\right)$ is a compatible pair. Then $(\mu_{N,\mathcal{D}}, \Lambda(N, \mathcal{S}))$ is NOT a spectral pair if and only if there exist $s_j^* \in \mathcal{S}$ and nonzero integers η_j, $0 \leqslant j \leqslant m - 1$, such that $\eta_{j+1} = N^{-1}(\eta_j + s_j^*)$ for all $0 \leqslant j \leqslant m - 1$ (with $\eta_m := \eta_0$ and $s_m^* := s_0^*$).*

The proof of Theorem 1.3 depends on the analysis of the extreme values of the eigenfunctions of the Ruelle transfer operator. The Ruelle transfer operator was studied in [9].

There appears to be a strong link between compatible pairs, tiling of integers and Fuglede's conjecture. All examples suggest that if $\mathcal{D}$ is a finite set of integers and is part of a compatible pair then $\mathcal{D}$ tiles $\mathbb{Z}$. A finite set $\mathcal{D} \subset \mathbb{Z}$ is called a *complementing set* (mod N) if there exists a $\mathcal{E} \subset \mathbb{Z}$ such that $\mathcal{D} \oplus \mathcal{E}$ is a complete residue system (mod N). It is known that $\mathcal{D}$ tiles $\mathbb{Z}$ if and only if it is a complementing set (mod N) for some N. We prove:

THEOREM 1.4. *Let $\mathcal{D} \subset \mathbb{Z}$ be a complementing set* (mod N) *with $|N| > 1$. Suppose that $|\mathcal{D}|$ has no more than two distinct prime factors. Then $\mu_{N,\mathcal{D}}$ is a spectral measure.*

In the next section, we shall prove the results just stated. Later in Section 3, we give some examples and state some open problems.

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2. PROOFS OF THEOREMS

We first state several lemmas, many of which have been proved in [9] or [20].

LEMMA 2.1. *Let $\mathscr{A}, \mathscr{B} \subset \mathbb{R}$ be finite sets of the same cardinality. Then the following are equivalent*:

(a) *$(\mathscr{A}, \mathscr{B})$ is a compatible pair.*
(b) *$m_{\mathscr{A}}(b_1 - b_2) = 0$ for any distinct $b_1, b_2 \in \mathscr{B}$.*
(c) *$\sum_{b \in \mathscr{B}} |m_{\mathscr{A}}(\xi + b)|^2 \equiv 1$.*

Proof. Note that condition (b) says precisely that the rows of the matrix $M = \left[\frac{1}{\sqrt{|\mathscr{A}|}} e(a_j b_k)\right]$ are orthonormal. So (a) and (b) are clearly equivalent. To see (a) and (c) are equivalent, let $\delta_{\mathscr{A}} = \frac{1}{|\mathscr{A}|} \sum_{a \in \mathscr{A}} \delta(x - a)$. Then $\delta_{\mathscr{A}}$ is a probability measure with $\hat{\delta}_{\mathscr{A}}(\xi) = m_{\mathscr{A}}(\xi)$. Furthermore, $(\mathscr{A}, \mathscr{B})$ is a compatible pair if and only if $(\delta_{\mathscr{A}}, \mathscr{B})$ is a spectral pair, see [20]. The equivalence of (a) and (c) follows immediately from Lemma 2.3 of [20]. ∎

LEMMA 2.2. *Let $\mathscr{D}, \mathscr{S} \subset \mathbb{Z}$ and $N \in \mathbb{Z}$, $|N| > 1$ such that $\left(\frac{1}{N}\mathscr{D}, \mathscr{S}\right)$ is a compatible pair. Then*

(a) *$\left(\frac{1}{N}\mathscr{D} + a, \mathscr{S} + b\right)$ is a compatible pair for any $a, b \in \mathbb{R}$.*
(b) *Suppose that $\hat{S} \subset \mathbb{Z}$ such that $\hat{S} \equiv S \pmod N$. Then $\left(\frac{1}{N}\mathscr{D}, \hat{S}\right)$ is a compatible pair.*
(c) *The elements in both $\mathscr{D}$ and $\mathscr{S}$ are distinct modulo N.*
(d) *Suppose that $|N| > 2$. Then there exists an $\hat{\mathscr{S}}$ with $0 \in \hat{\mathscr{S}}$ and $\hat{\mathscr{S}} \subseteq [2 - |N|, |N| - 2]$ such that $\left(\frac{1}{N}\mathscr{D}, \hat{\mathscr{S}}\right)$ is a compatible pair.*
(e) *Denote $\mathscr{D}_k = \mathscr{D} + N\mathscr{D} + \cdots + N^{k-1}\mathscr{D}$ and $\mathscr{S}_k = \mathscr{S} + N\mathscr{S} + \cdots + N^{k-1}\mathscr{S}$. Then $\left(\frac{1}{N^k}\mathscr{D}_k, \mathscr{S}_k\right)$ is a compatible pair.*

Proof. Condition (a) is essentially trivial from Lemma 2.1. It is also well known from the fact that any translate of a spectrum is also a spectrum, and any translate of a spectral measure is also a spectral measure with the same spectra.

For (b), observe that if $s \equiv \hat{s} \pmod N$ then $m_{\mathscr{D}/N}(\xi + s) = m_{\mathscr{D}/N}(\xi + \hat{s})$. Therefore,

$$\sum_{\hat{s} \in \hat{\mathscr{S}}} |m_{\mathscr{D}/N}(\xi + \hat{s})|^2 = \sum_{s \in \mathscr{S}} |m_{\mathscr{D}/N}(\xi + s)|^2 = 1.$$

This proves (b).

For (c), assume that $\mathscr{S} = \{s_j\}$ has $s_1 \equiv s_2 \pmod N$. Then we may replace s_2 by s_1 in $\mathscr{S}$ and still have a compatible pair by (b). This means the matrix M used for defining compatible pairs has two identical columns, so it cannot be unitary, a contradiction. So elements in $\mathscr{S}$ are distinct modulo N. Similarly, M will have two identical rows if elements in $\mathscr{D}$ are not distinct modulo N, again a contradiction.

To prove (d), we first translate $\mathscr{S}$ so that $0 \in \mathscr{S}_1 := \mathscr{S} + a$ for some $a \in \mathbb{Z}$. $\left(\frac{1}{N}\mathscr{D}, \mathscr{S}_1\right)$ is still a compatible pair. Now $[2-|N|, |N|-2]$ contains a complete set of residues (mod N) because it contains at least $|N|$ consecutive integers. Choose $\hat{\mathscr{S}} \subseteq [2-|N|, |N|-2]$ so that $0 \in \hat{\mathscr{S}}$ and $\hat{\mathscr{S}} \equiv \mathscr{S}_1 (\text{mod } N)$. Then $\left(\frac{1}{N}\mathscr{D}, \hat{\mathscr{S}}\right)$ is a compatible pair.

Finally (e) is a special case of Lemma 2.5 in [20]. ∎

LEMMA 2.3. *Under the assumptions of Theorem* 1.2, *let* $Q(\xi) := \sum_{\lambda \in \Lambda(N,\mathscr{S})} |\hat{\mu}(\xi+\lambda)|^2$. *Then*

(a) *The set of exponentials* $\{e(\lambda\xi) : \lambda \in \Lambda(N, \mathscr{S})\}$ *is orthonormal in* $L^2(\mu)$.

(b) $Q(\xi) \leqslant 1$ *for all* $\xi \in \mathbb{R}$ *and* $\{e(\lambda\xi) : \lambda \in \Lambda(N, \mathscr{S})\}$ *is an orthonormal basis for* $L^2(\mu)$ *if and only if* $Q(\xi) \equiv 1$.

(c) *The function* $Q(\xi)$ *is the restriction of an entire function of exponential type to the real line. Furthermore, it satisfies*

$$Q(\xi) = \sum_{s \in \mathscr{S}} |m_{\mathscr{D}}(N^{-1}(\xi+s))|^2 Q(N^{-1}(\xi+s)). \tag{2.1}$$

Proof. See [9]. The right-hand side of (2.1) is known as the *Ruelle transfer operator* (operated on Q). ∎

Proof of Theorem 1.3. Denote $\Lambda := \Lambda(N, \mathscr{S})$ and $\mu := \mu_{N,\mathscr{D}}$.

($\Leftarrow$) We prove that (μ, Λ) is not a spectral pair by proving that $Q(\xi) \not\equiv 1$, where $Q(\xi)$ is defined in Lemma 2.3. In fact, we prove that $Q(\eta_0) = 0$.

Observe that $m_{\mathscr{D}}(\eta_j) = 1$ because $\eta_j \in \mathbb{Z}$. Now $\sum_{s \in \mathscr{S}} |m_{\mathscr{D}}(N^{-1}(\eta_j + s))|^2 = 1$ by (c) of Lemma 2.1. Since $m_{\mathscr{D}}(N^{-1}(\eta_j + s_j^*)) = m_{\mathscr{D}}(\eta_{j+1}) = 1$, it follows that $m_{\mathscr{D}}(N^{-1}(\eta_j + s)) = 0$ for all $s \neq s_j^*$ in $\mathscr{S}$.

Take any $\lambda \in \Lambda$ and write $\lambda = \sum_{k=0}^{\infty} s_k N^k$ where $s_k \in \mathscr{S}$ and of course only finitely many $s_k \neq 0$. We have

$$\hat{\mu}(\eta_0 + \lambda) = \prod_{j=1}^{\infty} m_{\mathscr{D}}(N^{-j}(\eta_0 + \lambda)).$$

Note that

$$m_{\mathscr{D}}(N^{-1}(\eta_0+\lambda)) = m_{\mathscr{D}}\left(N^{-1}(\eta_0 + s_0) + \sum_{k=0}^{\infty} s_{k+1} N^k\right) = m_{\mathscr{D}}(N^{-1}(\eta_0 + s_0)).$$

Hence $m_{\mathscr{D}}(N^{-1}(\eta_0+\lambda)) = 1$ for $s_0 = s_0^*$ and $m_{\mathscr{D}}(N^{-1}(\eta_0+\lambda)) = 0$ otherwise. Suppose $s_0 = s_0^*$. Then the same argument together with the fact $\eta_1 = N^{-1} \times (\eta_0 + s_0^*)$ yield $m_{\mathscr{D}}(N^{-2}(\eta_0+\lambda)) = 1$ for $s_1 = s_1^*$ and $m_{\mathscr{D}}(N^{-2}(\eta_0+\lambda)) = 0$

otherwise. By induction, we easily obtain $m_{\mathscr{D}}(N^{-j}(\eta_0+\lambda))\neq 0$ if and only if $s_j = s^*_{j(\mathrm{mod}\, m)}$. Therefore, $\hat{\mu}(\eta_0+\lambda)\neq 0$ only if $s_j = s^*_{j(\mathrm{mod}\, m)}$ for all $j\geqslant 0$. But this is impossible since $s_j = 0$ for all sufficiently large j. Thus, $\hat{\mu}(\eta_0+\lambda) = 0$ and $Q(\eta_0) = 0$.

($\Rightarrow$) Assume that (μ, Λ) is NOT a spectral pair. Then $Q(\xi)\not\equiv 1$. Note that this cannot happen if $|\mathscr{D}| = |\mathscr{S}| = 1$, in which case $\mu = \delta_0$ and $\Lambda = \{0\}$. So we may assume that $q = |\mathscr{D}| = |\mathscr{S}| > 1$. It is well known that in this case $T := T(N, \mathscr{S})$ is a compact set with infinite cardinality.

Since $Q(\xi)\not\equiv 1$, $Q(\xi)\not\equiv 1$ for $\xi \in T$, because Q is extendable to an entire function on the complex plane and T is an infinite compact set. Denote $X^- := \{\xi \in T : Q(\xi) = \min_{\eta\in T} Q(\eta)\}$. It follows from $Q(0) = 1$ that $0 \notin X^-$. We apply the Ruelle transfer operator to derive a contradiction.

For any $s \in \mathbb{R}$ denote $\phi_s(\xi) = N^{-1}(\xi + s)$. Then $T = \bigcup_{s\in\mathscr{S}} \phi_s(T)$. Hence $\phi_s(\xi) \in T$ for all $\xi \in T$. Now choose any $\xi_0 \in X^-$ and set $Y_0 = \{\xi_0\}$. Define recursively

$$Y_{n+1} = \{\phi_s(\xi) : s \in \mathscr{S},\ \xi \in Y_n, \phi_s(\xi) \in X^-\} \quad \text{(counting multiplicity)}.$$

Claim 1. *We have $|Y_{n+1}|\geqslant|Y_n|$ (counting multiplicity).*

Proof. Let $\xi^* \in X^-$. By (c) of Lemma 2.3

$$\min_{\eta\in T} Q(\eta) = Q(\xi^*) = \sum_{s\in\mathscr{S}} |m_{\mathscr{D}}(\phi_s(\xi^*))|^2 Q(\phi_s(\xi^*)).$$

But $\sum_{s\in\mathscr{S}} |m_{\mathscr{D}}(\phi_s(\xi^*))|^2 = 1$ by Lemma 2.3 and $Q(\phi_s(\xi^*))\geqslant \min_{\eta\in T} Q(\eta)$. Thus, $Q(\phi_s(\xi^*)) = \min_{\eta\in T} Q(\eta)$ whenever $m_{\mathscr{D}}(\phi_s(\xi^*))\neq 0$. In other words,

$$\phi_s(\xi^*) \in X^- \quad \text{whenever } m_{\mathscr{D}}(\phi_s(\xi^*))\neq 0. \tag{2.2}$$

Hence for each $\xi \in Y_n$ there exists at least one $s \in \mathscr{S}$ such that $\phi_s(\xi) \in Y_{n+1}$, proving Claim 1. ∎

Claim 2. *The elements of Y_n in fact all have multiplicity one.*

Proof. It is easy to see that elements in Y_n have the form $\phi_{s_n} \circ \cdots \circ \phi_{s_1}(\xi_0)$. If some element in Y_n has multiplicity more than one, then there are two distinct sequences $(s_1, \ldots, s_n)$ and $(t_1, \ldots, t_n)$ in $\mathscr{S}$ such that

$$\phi_{s_n} \circ \cdots \circ \phi_{s_1}(\xi_0) = \phi_{t_n} \circ \cdots \circ \phi_{t_1}(\xi_0).$$

Expanding the two expressions yields

$$\frac{1}{N^n}(\xi_0 + s_1 + Ns_2 + \cdots + N^{n-1}s_n) = \frac{1}{N^n}(\xi_0 + t_1 + Nt_2 + \cdots + N^{n-1}t_n).$$

But this is clearly not possible, since all elements of $\mathcal{S}$ are in different residue classes (mod N). ∎

Now X^- is finite. It follows that all Y_n have the same cardinality for sufficiently large n, say $n \geqslant n_0$. Therefore, for $n \geqslant n_0$ any $\xi^* \in Y_n$ has a unique offspring $\phi_s(\xi^*) \in Y_{n+1}$. Furthermore, for any $t \in \mathcal{S}$ and $t \neq s$ we must have $m_{\mathcal{D}}(\phi_t(\xi^*)) = 0$, because otherwise by (2.2) we would have $\phi_t(\xi^*) \in Y_{n+1}$ as another offspring, a contradiction. Thus, starting with a $\xi_{n_0} \in Y_{n_0}$, we obtain a sequence $\{\xi_n\}_{n \geqslant n_0}$ in which $\xi_n \in Y_n$ is the unique offspring of $\xi_{n-1} \in Y_{n-1}$, $\xi_n = \phi_{s_{n-1}}(\xi_{n-1})$ for some $s_{n-1} \in \mathcal{S}$. It follows from the finiteness of X^- that there exist $k > n_0$ and $m > 0$ such that

$$\xi_k = \xi_{k+m} = \frac{1}{N^m}(\xi_k + s_k + s_{k+1}N + \cdots + s_{k+m}N^{m-1}).$$

In particular, $\xi_k \in \mathbb{Q}$. Set $\eta_j = \xi_{j+k}$ and $s_j^* = s_{j+k}$ for $0 \leqslant j \leqslant m-1$ ($\eta_m := \eta_0$ and $s_m^* := s_0^*$. Only the case $j = m-1$ needs to be checked). Then $\eta_j \in \mathbb{Q}$ for all j. Furthermore, $\eta_{j+1} = \phi_{s_j^*}(\eta_j) = N^{-1}(\eta_j + s_j^*)$ for all $0 \leqslant j \leqslant m-1$.

Note that $m_{\mathcal{D}}(\phi_s(\eta_j)) = 0$ for all $0 \leqslant j \leqslant m-1$ and $s \neq s_j^*$, as shown above. This yields

$$|m_{\mathcal{D}}(\eta_j)|^2 = \sum_{s \in \mathcal{S}} |m_{\mathcal{D}}(\phi_{s_{j-1}^*}(\eta_{j-1}))|^2 = 1, \qquad 1 \leqslant j \leqslant m.$$

But $m_{\mathcal{D}}(\eta_j) = \frac{1}{|\mathcal{D}|}\sum_{d \in \mathcal{D}} e(d\eta_j)$ and $0 \in \mathcal{D}$. The only way $|m_{\mathcal{D}}(\eta_j)| = 1$ can hold is $e(d \cdot \eta_j) = 1$ for all $d \in \mathcal{D}$. Hence $\eta_j \cdot d \in \mathbb{Z}$ for all $d \in \mathcal{D}$. Since all $\eta_j \neq 0$ and $\gcd(\mathcal{D}) = 1$, it follows that all $\eta_j \in \mathbb{Z}$. The theorem is proved. ∎

Proof of Theorem 1.2. Suppose that $\mu_{N,\mathcal{D}}$ is a spectral measure with spectrum Λ then $\mu_{N,a\mathcal{D}}$ is a spectral measure with spectrum $a^{-1}\Lambda$. Therefore, to prove that $\mu_{N,\mathcal{D}}$ is spectral we may without loss of generality assume that $0 \in \mathcal{D}$ and $\gcd(\mathcal{D}) = 1$.

If $|N| = 2$ then $\mathcal{D} = \{0, 1\}$ or $\mathcal{D} = \{0, -1\}$. The corresponding self-similar measure $\mu_{N,\mathcal{D}}$ is the Lebesgue measure supported on a unit interval. Clearly, in this case $\mu_{N,\mathcal{D}}$ is a spectral measure with spectrum $\mathbb{Z}$.

For $|N| \geqslant 3$, by Lemma 2.2 we may replace $\mathcal{S}$ by $\hat{\mathcal{S}} \subseteq [2-|N|, |N|-2]$ such that $0 \in \hat{\mathcal{S}}$ and $\left(\frac{1}{N}\mathcal{D}, \hat{\mathcal{S}}\right)$ is a compatible pair. Now for any such $\hat{\mathcal{S}}$, we have $T(N, \hat{\mathcal{S}}) \subseteq [-\frac{N-2}{N-1}, \frac{N-2}{N-1}]$ for $N > 0$ or $T(N, \hat{\mathcal{S}}) \subseteq [-\frac{N^2+N-2}{N^2-1}, \frac{N^2+N-2}{N^2-1}]$ for $N < 0$. In either case $T(N, \hat{\mathcal{S}})$ contains no integer other than 0. Now suppose that $(\mu_{N,\mathcal{D}}, \Lambda(N, \hat{\mathcal{S}}))$ is not a spectral pair. Then there exist s_j^* in $\mathcal{S}$ and nonzero integers η_j satisfying the condition of Theorem 1.3. Starting with $T_0 = \{\eta_j\}$ we see that $T_0 \subseteq \bigcup_{s \in \mathcal{S}} \phi_s(T_0)$. This yields $T_0 \subseteq T(N, \mathcal{S})$. But this is a contradiction, since $T(N, \hat{\mathscr{S}})$ contains no integer other than 0. Therefore $(\mu_{N,\mathscr{D}}, \Lambda(N, \hat{\mathscr{S}}))$ is a spectral pair by Theorem 1.3. ∎

Proof of Theorem 1.4. It is known that if $\mathscr{D}$ is a complementing set (mod N) then there exists an $L|N$ whose prime factors are precisely those of $|\mathscr{D}|$ such that $\mathscr{D}$ is a complementing set (mod L). We prove that there exists an $\mathscr{S} \in \mathbb{Z}$ such that $\left(\frac{1}{N}\mathscr{D}, \mathscr{S}\right)$ is a compatible pair, using a theorem of Coven and Meyerowitz [3]. The argument below is essentially a repetition of the proof of Theorem 1.5 (i) in [13] specialized to the case of two prime factors.

Let $\Phi_n(z)$ denote the nth cyclotomic polynomial. Let also $D(z) = \sum_{d \in \mathscr{D}} z^d$ so that $m_{\mathscr{D}}(\xi) = D(e^{2\pi i \xi})$. Assume that $|\mathscr{D}| = p^\alpha q^\beta$ and $L = p^{\alpha'} q^{\beta'}$, where p, q are distinct primes. (If $|\mathscr{D}|$ is a prime power, the proof below works and is simpler.) Let

$$\mathscr{P} := \{p^k : \Phi_{p^k}(z) \mid D(z),\ k \leqslant \alpha'\}, \qquad \mathbb{Q} := \{q^k : \Phi_{q^k}(z) \mid D(z),\ k \leqslant \beta'\}.$$

Coven and Meyerowitz [3] prove that $|\mathscr{P}| = \alpha$, $|\mathbb{Q}| = \beta$, and that $\Phi_{p^k q^l}(X) \mid m_{\mathscr{D}}(z)$ for all k, l such that $p^k \in \mathscr{P}, q^l \in \mathbb{Q}$.

We construct the set $\mathscr{S}$. Write $\mathscr{P} = \{p^{k_j}\}$ and $\mathscr{Q} = \{q^{l_j}\}$, where $k_1 < k_2 < \cdots < k_\alpha$ and $l_1 < l_2 < \cdots l_\beta$. Define

$$\mathscr{E} = \left\{ \sum_{j=1}^{\alpha} a_j p^{-k_j} + \sum_{j=1}^{\beta} b_j q^{-l_j} : 0 \leqslant a_j < p,\ 0 \leqslant b_j < q \right\},$$

and let $\mathscr{S} = N\mathscr{E}$. Clearly, $\mathscr{S} \subset \mathbb{Z}$. To prove that $\left(\frac{1}{N}\mathscr{D}, \mathscr{S}\right)$ is a compatible pair it suffices to prove that $m_{\mathscr{D}/N}(s_1 - s_2) = 0$ for any distinct $s_1, s_2 \in \mathscr{S}$. Equivalently, we only need to show that $m_{\mathscr{D}}(\lambda_1 - \lambda_2) = 0$ for any distinct $\lambda_1, \lambda_2 \in \mathscr{E}$. Note that

$$\lambda_1 - \lambda_2 = \sum_{j=1}^{\alpha} c_j p^{-k_j} + \sum_{j=1}^{\beta} d_j q^{-l_j}, \qquad -p < c_j < p, \quad -q < d_j < q.$$

If all $c_j = 0$ then not all $d_j = 0$. So $\lambda_1 - \lambda_2 = r/q^{l_{j^*}}$ for some r with $\gcd(r, q) = 1$, where j^* is the largest j such that $d_j \neq 0$. Therefore, $\Phi_{q^{l_{j^*}}}(e^{2\pi i(\lambda_1 - \lambda_2)}) = 0$ and hence $m_{\mathscr{D}}(\lambda_1 - \lambda_2) = 0$. Similarly, $m_{\mathscr{D}}(\lambda_1 - \lambda_2) = 0$ if all $d_j = 0$. Finally, assume none of the above is true. Let j_1 be the largest j such that $c_j \neq 0$ and j_2 be the largest j such that $d_j \neq 0$. Then $\lambda_1 - \lambda_2 = r_1/p^{k_{j_1}} + r_2/q^{l_{j_2}}$ with $\gcd(r_1, p) = \gcd(r_2, q) = 1$. This yields

$$\lambda_1 - \lambda_2 = \frac{r_1 q^{l_{j_2}} + r_2 q^{k_{j_1}}}{p^{k_{j_1}} q^{l_{j_2}}}.$$

The numerator is clearly coprime with the denominator. Hence $\Phi_{p^{k_{j_1}}q^{l_{j_2}}}(e^{2\pi i(\lambda_1-\lambda_2)}) = 0$ and therefore $m_{\mathscr{D}}(\lambda_1 - \lambda_2) = 0$.

So $(\mathscr{D}/N, \mathscr{S})$ is a compatible pair, and hence $\mu_{N,\mathscr{D}}$ is a spectral measure. ∎

3. EXAMPLES AND OPEN QUESTIONS

Theorem 1.3 leads to an algorithm for determining whether $\Lambda(N, \mathscr{S})$ is a spectrum for $\mu_{N,\mathscr{D}}$. To find whether the integer sequence $\{\eta_j\}$ exists, we only need to check a finite number of integers. This is because we have shown in the proof of Theorem 1.3 that if the sequence $\{\eta_j\}$ exists, it must be contained in $T(N, \mathscr{S})$. However, $T(N, \mathscr{S})$ is compact. In fact $T(N, \mathscr{S}) \subseteq \left[\frac{a}{N-1}, \frac{b}{N-1}\right]$ for $N > 0$ and $T(N, \mathscr{S}) \subseteq \left[\frac{a+Nb}{N^2-1}, \frac{b+Na}{N^2-1}\right]$ for $N < 0$, where a, b are the smallest and the largest elements in $\mathscr{S}$, respectively.

EXAMPLE 3.1. Our first example addresses the condition in Theorem 1.1 by Strichartz. Let $N = 5$ and $\mathscr{D} = \{0, \pm 2, \pm 11\}$. Since $\mathscr{D}$ is a residue system (mod N), the set $T(N, \mathscr{D})$ is a fundamental domain of $\mathbb{Z}$ and $\mu_{N,\mathscr{D}}$ is simply the restriction of the Lebesgue measure to $T(N, \mathscr{D})$, see [5]. Let $\mathscr{S} = \{0, \pm 1, \pm 2\}$. Then $\Lambda(N, \mathscr{S}) = \mathbb{Z}$ is a spectrum for $\mu_{N,\mathscr{D}}$. However, $m_{\mathscr{D}}(\xi)$ has a zero in $[0, \frac{1}{4}]$, which is contained in $T(N, \mathscr{S}) = [-\frac{1}{2}, \frac{1}{2}]$.

The condition can also be hard to check. Consider the same $\mathscr{D}$ as above, let $N = 10$ and $\mathscr{S} = \{0, 2, 4, 6, 8\}$. Then $(\mathscr{D}/N, \mathscr{S})$ is a compatible pair. By Theorem 1.2 $\Lambda(N, \mathscr{S})$ is a spectrum for $\mu_{N,\mathscr{D}}$. Nevertheless, it is difficult to check whether $T(N, \mathscr{S})$ contains a zero of $m_{\mathscr{D}}(\xi)$.

EXAMPLE 3.2. Our next example illustrates that a spectral measure can have many spectra. Take $N = 6$ and $\mathscr{D} = \{0, 1, 2\}$. Then $(\mathscr{D}/N, \mathscr{S})$ are compatible pairs for both $\mathscr{S} = \{0, 2, 4\}$ or $\{0, -2, 2\}$. By Theorem 1.2 $\Lambda \times (N, \mathscr{S})$ are spectra of $\mu_{N,\mathscr{D}}$ for both $\mathscr{S}$.

A far more striking example is to take $\mathscr{S}' = \{0, 4, 8\}$. Then $(\mathscr{D}/N, \mathscr{S}')$ is a compatible pair because $\mathscr{S}' \equiv \mathscr{S} = \{0, 2, 4\} (\mathrm{mod}\, N)$. One can check using the algorithm described earlier that $\Lambda(N, \mathscr{S}')$ is indeed a spectrum of $\mu_{N,\mathscr{D}}$, as is $\Lambda(N, \mathscr{S})$. But $\Lambda(N, \mathscr{S}') = 2\Lambda(N, \mathscr{S})$! This is rather striking because $\Lambda(N, \mathscr{S}')$ is intuitively "twice as sparse as" $\Lambda(N, \mathscr{S})$.

The study in this paper also leaves several questions unanswered. For example, do Theorems 1.2 and 1.3, or something similar, hold in higher dimensions? The difficulty is that an analytic function of two or more variables may attain its infimum at infinitely many points, even on a compact set. The technique of the Ruelle transfer operator employed in this paper has its origin in the study of wavelets and self-affine tiles, see e.g. [5], in dimension 1. It was extended to higher dimensions in [17]. Could the techniques there be applied to higher dimensions to yield results on spectral measures?

Note that we have studied spectral Cantor measures in which the probability weights are equally distributed. Is this a general rule? We conclude this paper with the following conjecture:

Conjecture 3.1. Let μ be the self-similar measure associated with the IFS $\phi_j(x) = \rho(x + a_j)$, $1 \leqslant j \leqslant q$, with probability weights $p_1, \ldots, p_q > 0$, where $|\rho| < 1$. Suppose that μ is a spectral measure. Then

(a) $\rho = \frac{1}{N}$ for some $N \in \mathbb{Z}$.

(b) $p_1 = \cdots = p_q = \frac{1}{q}$.

(c) Suppose that $0 \in \mathscr{A} = \{a_j\}$. Then $\mathscr{A} = \alpha\mathscr{D}$ for some $\alpha \in \mathbb{R}$ and $\mathscr{D} \subset \mathbb{Z}$. Furthermore, $\mathscr{D}$ must be a complementing set (mod N).

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