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Note

On Polynomials and Crossing Numbers of Complete Graphs

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A long-standing, unsolved problem is that of finding the minimum number of crossings of the edges in a complete graph when embedded on a surface of genus zero. It has been shown [4, 5] that the minimum crossing number, I_n, of a complete graph on n vertices for values of $n \leqslant 10$ is given by:

n	0	1	2	3	4	5	6	7	8	9	10
I_n	0	0	0	0	0	1	3	9	18	36	60

It is known that for each n the crossing number is dominated by the following quartics to which corresponds a realization in the plane:

$$I_n \leqslant \begin{cases} \dfrac{n(n-2)^2(n-4)}{64} \equiv A, & n \text{ even}, \\[2ex] \dfrac{(n-1)^2(n-3)^2}{64} \equiv B, & n \text{ odd}. \end{cases} \qquad (*)$$

The two expressions may be combined, yielding

$$\begin{aligned} I_n &\leqslant \frac{1+(-1)^n}{2} A + \frac{1-(-1)^n}{2} B \\ &= \frac{1+\cos n\pi}{2} A + \frac{1-\cos n\pi}{2} B, \qquad n = 0, 1, \ldots . \end{aligned}$$

One can also write

$$I_n \leqslant \frac{1}{4} \left[\frac{n}{2}\right] \left[\frac{n-1}{2}\right] \left[\frac{n-2}{2}\right] \left[\frac{n-3}{2}\right].$$

THEOREM. *If I_n as a function of n can be split into a unique polynomial for all even n and into another unique polynomial for all odd n, then each polynomial is at least a quartic and is identical with the corresponding expression given on the right side of* (*). *Therefore, in* (*) *the equality would hold.*

Proof. For n even, any representation of I_n as a function of n must vanish at $n = 0, 2, 4$ and hence as a polynomial must be at least a cubic of the form $an(n-2)(n-4)$ where a is a constant. Now, $n = 6$ implies $a = 1/16$ and $I_8 = 12$, a contradiction. Thus,

$$I_n = (a_1 + b_{1n})\, n(n-2)(n-4).$$

The two values $n = 6, 8$ determine a_1 and b_1 precisely as in (*).

For n odd, a similar argument shows that $(n-1)$, $(n-3)$ must be factors of a polynomial which cannot be a quadratic or a cubic because of I_5 and I_7 ; and, hence, must be a quartic of the form

$$I_n = (n-1)(n-3)(a_2 + b_2 n + c_2 n^2)$$

whose coefficients when determined from I_5 , I_7 , I_9 are again as in (*).

Remark. Guy and Kainen [2, 3] have shown that

$$\lim_{n\to\infty} I_n \sim \frac{n^4}{64}.$$

REFERENCES

1. BLAŽEK, J., AND KOMAN, M. A., A minimal problem concerning complete plane graphs, *in* "Theory of Graphs and Its Applications," *Proc. Symp. Smolenice, June, 1963*, Academic Press, New York, 1964.
2. GUY, R. K., A combinatorial problem, *Bull. Malayan Math. Soc.* 7, No. 2, 68–72 (July 1960).
3. KAINEN, PAUL, On a problem of Erdös, *J. Combinatorial Theory* **5**, No. 4, 374–377 (Dec. 1968).
4. SAATY, THOMAS L., Two theorems on the minimum number of intersections for complete graphs, *J. Combinatorial Theory* **2**, No. 4, 571–584 (June 1967).
5. SAATY, THOMAS L., Symmetry and the crossing number of complete graphs, *J. Res. Nat. Bur. Standards Sect. B.* **738**, No. 2 (April-June 1969).

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