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On the Schur and Singular Value Decompositions of Oscillatory Matrices

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ABSTRACT

A square matrix A is said to be *oscillatory* if it has nonnegative minors and some power A^k of A is strictly totally positive (i.e., A^k has strictly positive minors). We study the Schur and singular value decompositions of oscillatory matrices. Some applications are provided. © Elsevier Science Inc., 1997

1. INTRODUCTION

Totally positive (TP) matrices are matrices whose minors are nonnegative. These matrices have an increasing importance in approximation theory and other fields (such as statistics and economics), and for that reason linear algebraists are paying more attention to them (see [1] for a comprehensive survey from an algebraic point of view and historical references). Many classical applications of totally positive matrices and its subclasses can be found in [9]. *Strictly totally positive* (STP) matrices are matrices whose minors are strictly positive. Finally, a totally positive matrix A is said to be *oscillatory* if some power A^k of A is strictly totally positive. Oscillatory matrices were introduced by Gantmacher and Krein [5] in connection with the study of small vibrations of dynamical systems (see an example of those applications in Section 9.3 of Chapter XIII of [4]).

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In this paper we shall use the following well-known characterization of oscillatory matrices (see for instance Theorem 4.2 of [1]):

THEOREM 1.1. *A totally positive $n \times n$ matrix A is oscillatory if and only if it is nonsingular and $a_{i,i+1} > 0$, $a_{i+1,i} > 0$ for all $i \in \{1, \dots, n-1\}$.*

Oscillatory matrices were characterized in terms of their LU factorization in Section 4 of [3]. This paper is devoted to the study of another decompositions of oscillatory matrices. In Section 2 we introduce the notation and basic definitions, and in Theorem 2.5 we describe the Schur decomposition of oscillatory matrices. Section 3 deals with the singular value and polar decompositions of oscillatory matrices. We also introduce a class of rectangular matrices, which generalizes oscillatory matrices, and prove the existence and uniqueness of solution of the corresponding total least squares problem.

2. NOTATION AND SCHUR DECOMPOSITIONS

Following the notation of [1], given k, $n \in \mathbb{N}$, $1 \leqslant k \leqslant n$, $Q_{k,n}$ will denote the set of all increasing sequences of k natural numbers less than or equal to n. The dispersion $d(\alpha)$ of $\alpha \in Q_{k,n}$ is defined by $d(\alpha) := \sum_{i=1}^{k-1}(\alpha_{i+1} - \alpha_i - 1) = \alpha_k - \alpha_1 - (k-1)$, with the convention $d(\alpha) = 0$ for $\alpha \in Q_{1,n}$. Let A be a real square matrix of order n. For $k \leqslant n$, $l \leqslant n$, and for any $\alpha \in Q_{k,n}$ and $\beta \in Q_{l,n}$, we denote by $A[\alpha|\beta]$ the $k \times l$ submatrix of A containing rows numbered by α and columns numbered by β.

Now, we shall introduce some special classes of matrices which will play an important role in this paper. Let A be an $n \times n$ lower (upper) triangular matrix. Following [3], the minors $A[\alpha|\beta]$ with $\alpha_k \geqslant \beta_k$ (with $\alpha_k \leqslant \beta_k$ $\forall k$) are called nontrivial minors of A. Then a matrix A is called ΔSTP if the nontrivial minors of A are all positive. In this section, L (U) represents a lower (upper) triangular, unit diagonal matrix, and D a diagonal matrix.

DEFINITION 2.1. A nonsingular matrix A is said to be *lowerly STP* if it can be decomposed in the form $A = LDU$ and LD is ΔSTP.

Observe that since L is unit diagonal, the total positivity of LD implies that L and D are totally positive. Some applications of lowerly STP matrices to approximation theory (in particular, to the study of Tchebycheff systems) can be found in [2].

The next class of matrices appeared in [7] in the characterization of the QR factorization of the strictly totally positive matrices and in the diagonalization of the oscillatory matrices.

DEFINITION 2.2. A nonsingular matrix A is said to be a *strict γ-matrix* if it is lowerly STP and, in the factorization $A = LDU$, U^{-1} is ΔSTP.

An algorithmic characterization of the strict γ-matrices in terms of the signs of the multipliers of their Neville elimination can be found in Section 5 of [7].

The next result can be obtained from applying Proposition 4.6 of [7] of P and to $(P^T)^{-1}$

PROPOSITION 2.3. *If P and $(P^T)^{-1}$ are lowerly STP, then P and $(P^T)^{-1}$ are strict γ-matrices. Therefore, if P is an orthogonal and lowerly STP matrix, then it is a strict γ-matrix.*

Spectral properties of oscillatory matrices are well known: the eigenvalues λ_i are all simple, real, and positive, and the corresponding eigenvectors v_i can be chosen with certain sign variations in the sequence of their components (see for instance [4, pp. 105–106]). The following result summarizes the comments of Section 6 of [7] on the diagonalization of oscillatory matrices.

THEOREM 2.4.

(i) *Let A be an oscillatory matrix. Then there exists a strict γ-matrix P such that*

$$A = P\Lambda P^{-1} \qquad \left[\textit{and} \quad A^T = (P^T)^{-1}\Lambda P^T\right], \tag{2.1}$$

where $\Lambda = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 > \cdots > \lambda_n > 0$. Furthermore, $(P^T)^{-1}$ is also a strict γ-matrix.

(ii) *If A is a nonsingular real matrix with real eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$ and there exist strict γ-matrices P, N such that*

$$A = P \cdot \operatorname{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\} \cdot P^{-1}$$

and

$$A^T = N \cdot \operatorname{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\} \cdot N^{-1},$$

then there exists a positive integer k such that A^k is strictly totally positive.

In fact, in [4, pp. 106–107] it is shown that the minors of P and $(P^T)^{-1}$ in (2.1) satisfy the properties (4.1) of [7] and so, by Proposition 4.4 of [7], these matrices are lowerly STP matrices. Hence they are strict γ-matrices by Proposition 2.3, and this gives the proof of (i) of the previous theorem. Part (ii) follows from Theorems 6.4 and 6.1 of [1] and its proof.

It is well known that a square complex matrix is similar to a triangular matrix by way of a unitary matrix of change of basis. The corresponding matrix factorization is usually known as Schur decomposition. The next theorem characterizes the Schur decomposition of oscillatory matrices.

THEOREM 2.5.

(i) *Let A be an oscillatory matrix. Then there exists an orthogonal strict γ-matrix Q such that $Q^{-1}AQ = T$ $(= (t_{ij})_{1 \leqslant i,\, j \leqslant n})$ is an upper triangular real matrix with $t_{11} > \cdots > t_{nn} > 0$. Furthermore, there exists a nonsingular upper triangular matrix R such that QR and $Q(R^{-1})^T$ are also strict γ-matrices and $T = R\Lambda R^{-1}$, where Λ is a diagonal matrix.*

(ii) *Let us assume that a matrix A admits a Schur decomposition $A = QTQ^{-1}$, where Q is an orthogonal strict γ-matrix and $T = (t_{ij})_{1 \leqslant i,\, j \leqslant n}$ is an upper triangular matrix with $t_{11} > \cdots > t_{nn} > 0$. Let us assume also that QR and $Q(R^{-1})^T$ are strict γ-matrices, where R is a nonsingular upper triangular matrix such that $T = R\Lambda R^{-1}$ $(\Lambda = \mathrm{diag}\{\lambda_1, \dots, \lambda_n\}$ is a diagonal matrix). Then there exists a positive integer k such that A^K is a strictly totally positive matrix.*

Proof. (i): By Theorem 2.4 there exists a strict γ-matrix P such that $P^{-1}AP = \Lambda$ $(= \mathrm{diag}\{\lambda_1, \dots, \lambda_n\})$ with $\lambda_1 > \cdots > \lambda_n > 0$. The Gram-Schmidt method, applied to orthogonalize the columns of P, is described by

$$PV = M, \tag{2.2}$$

where V is an upper triangular, unit diagonal matrix and M is a matrix with orthogonal columns. Then there exists a diagonal matrix D with positive diagonal entries such that $MD = Q$ is an orthogonal matrix.

Since P is a strict γ-matrix, it is in particular lowerly STP and, by Proposition 4.4 of [7], it satisfies, for each $k \in \{1, 2, \dots, n\}$,

$$\det P[\alpha | 1, 2, \dots, k] > 0 \qquad \forall \alpha \in Q_{k,n} \text{ with } d(\alpha) = 0. \tag{2.3}$$

From (2.2), (2.3), and the Cauchy-Binet formula we deduce that similar inequalities to (2.3) hold for M and Q too. So these matrices are also lowerly

STP. Then, by Proposition 2.3, Q is a strict γ-matrix. On the other hand, from (2.2) we get

$$P = QD^{-1}V^{-1} \tag{2.4}$$

and the matrix $R := D^{-1}V^{-1}$ is upper triangular with positive diagonal. If we define now $T := R\Lambda R^{-1}$, we obtain an upper triangular matrix satisfying the required properties.

On the other hand, by (2.4), $P = QR$ and so $(P^{-1})^T = Q(R^{-1})^T$, which is a strict γ-matrix by Theorem 2.4(i).

(ii): Let us observe that the facts $T = R\Lambda R^{-1}$ and $t_{11} > \cdots > t_{nn} > 0$ imply that $\lambda_1 > \cdots > \lambda_n > 0$. Since $Q^{-1}AQ = R\Lambda R^{-1}$, we have that $P^{-1}AP = \Lambda$, where $P := QR$. By hypothesis, P and $(P^T)^{-1} = Q(R^{-1})^T$ are strict γ-matrices. Now (ii) is a consequence of Theorem 2.4(ii). ■

As well known, for symmetric matrices Schur decomposition leads to the factorization (2.1). Thus we can derive from Theorem 2.5(i) and Theorem 2.4(ii) the following corollary.

COROLLARY 2.6.

(i) *Let A be an oscillatory and symmetric matrix. Then there exists an orthogonal strict γ-matrix P such that*

$$A = P\Lambda P^{-1} \qquad \left(\text{and} \quad A^T = (P^T)^{-1}\Lambda P^T\right),$$

where $\Lambda = \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}$ *with* $\lambda_1 > \cdots > \lambda_n > 0$.

(ii) *If A is a positive definite symmetric matrix and there exists an orthogonal strict γ-matrix P such that*

$$A = P \cdot \operatorname{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \cdot P^{-1},$$

with $\lambda_1 > \cdots > \lambda_n$, then there exists a positive integer k such that A^k is strictly totally positive.

3. SINGULAR VALUE DECOMPOSITION AND APPLICATIONS

In this section we shall study the singular value decomposition of an oscillatory matrix. Singular value decomposition is a powerful tool in linear algebra. In [8] one can find many examples of the practical and theoretical value of this matricial factorization. A survey of the early history of this decomposition can be found in [10]. Polar decomposition is another important matricial factorization and is closely related with the singular value decomposition. The next result describes the singular value and polar decompositions of an oscillatory matrix.

THEOREM 3.1.

(i) *Let A be an oscillatory matrix. Then there exist orthogonal strict γ-matrices Q_1, Q_2 such that*

$$A = Q_1 \Sigma Q_2^T, \tag{3.1}$$

where $\Sigma = \operatorname{diag}\{\sigma_1, \dots, \sigma_n\}$ *with* $\sigma_1 > \cdots > \sigma_n > 0$.

(ii) *If A is an oscillatory matrix, then it can be decomposed in the form $A = QS$, where Q is orthogonal and S is a positive definite symmetrix matrix such that there exists a positive integer k with S^k strictly totally positive.*

Proof. (i): Let us observe that, by Corollary 4.3 of [1], A^TA and AA^T are also oscillatory matrices. Then, by Corollary 2.6(i), there exist orthogonal strict γ-matrices Q_1, Q_2 such that

$$AA^T = Q_1 \Lambda Q_1^T, \qquad A^TA = Q_2 \Lambda Q_2^T,$$

where $\Lambda = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$ with $\lambda_1 > \cdots > \lambda_n > 0$. Let $\Sigma = \operatorname{diag}\{\sigma_1, \dots, \sigma_n\}$ with $\sigma_i := \lambda_i^{1/2}$, $i = 1, \dots, n$. Now it is well known that the singular value decomposition of A is of the form (3.1), and (i) follows.

(ii): By (i) we can write $A = Q_1 \Sigma Q_2^T = (Q_1 Q_2^T)(Q_2 \Sigma Q_2^T)$, where Q_1 and Q_2 are orthogonal strict γ-matrices and $\Sigma = \operatorname{diag}\{\sigma_1, \dots, \sigma_n\}$ with $\sigma_1 > \cdots > \sigma_n > 0$. Now let us take $Q := Q_1 Q_2^T$, which is an orthogonal matrix, and $S := Q_2 \Sigma Q_2^T$, which is a positive definite symmetric matrix. Let us observe that $S^2 = A^TA$ and, since A^TA is oscillatory (by Corollary 4.3 of [1]), there exists a positive integer k such that $(S^2)^k$ is a strictly totally positive matrix. ∎

Finally, we shall consider a total least squares problem (see Section 12.3 of [8] or Chapters 1 and 2 of [11]), which is closely related with oscillatory matrices. Let $Ax = b$ be an overdetermined linear system of m equations and n unknowns ($m > n$). Least squares problems deal with the minimization of $\|Ax - b\|_2$. In these problems it is assumed that the errors come from the vector b. If the error also comes from the matrix A, then it may be more natural to consider the following problem (which is called the *total least squares problem*):

$$\text{minimize} \quad \left\| [A; b] - \left[\hat{A}; \hat{b}\right] \right\|_F \qquad \text{subject to} \quad \hat{b} \in R(\hat{A}) \tag{3.2}$$

[where $\|M\|_F = \sqrt{\operatorname{tr}(M^T M)}$ is the Frobenius norm of a matrix M, and $R(M)$ is the column space of M]. Once a minimizing $[\hat{A}; \hat{b}]$ is found, then any x satisfying $\hat{A}x = \hat{b}$ is called a *total least squares solution.*

Let us assume that the singular value decomposition of A is given by

$$A = U'\Sigma'V', \qquad \Sigma' = \operatorname{diag}\{\sigma_1', \dots, \sigma_n'\}, \quad \sigma_1' \geqslant \sigma_2' \geqslant \cdots \geqslant \sigma_n' \geqslant 0, \tag{3.3}$$

where $U' = (u_{ij}')_{1 \leqslant i, j \leqslant m}$ and $V' = (v_{ij}')_{1 \leqslant i, j \leqslant n}$ are orthogonal matrices. Analogously, let us assume that the singular value decomposition of $[A; b]$ is given by

$$[A; b] = U\Sigma V, \qquad \Sigma = \operatorname{diag}\{\sigma_1, \dots, \sigma_{n+1}\},$$
$$\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n \geqslant \sigma_{n+1} \geqslant 0, \tag{3.4}$$

where $U = (u_{ij})_{1 \leqslant i, j \leqslant m}$ and $V = (v_{ij})_{1 \leqslant i, j \leqslant n+1}$ are orthogonal matrices.

The next result guarantees the existence and uniqueness of solution of the total least squares problem for a class of rectangular matrices which generalize the oscillatory matrices.

PROPOSITION 3.2. *Let $Ax = b$ be a linear system, and assume that $[A; b] = (c_{ij})_{1 \leqslant i \leqslant m;\, 1 \leqslant j \leqslant n+1}$ $(m \geqslant n+1)$ is a totally positive matrix of rank $n+1$ such that $c_{ii} > 0$ $\forall i = 1, \dots, n+1$ and*

$$\textit{either} \quad c_{i,i-1} > 0 \quad \forall i = 2, \dots, n+1 \quad \textit{or} \quad c_{i,i+1} > 0 \quad \forall i = 1, \dots, n. \tag{3.5}$$

Then, if (3.4) *is the singular value decomposition of* $[A; b]$, *the corresponding total least squares problem* (3.2) *has the solution*

$$\left[\hat{A}; \hat{b}\right] = U\hat{\Sigma}V, \qquad \hat{\Sigma} = \operatorname{diag}\{\sigma_1, \dots, \sigma_n, 0\},$$

and $\hat{x} = (1/\sigma_{n+1})(v_{1,n+1}, \dots, v_{n,n+1})$ *is the unique solution to* $\hat{A}x = \hat{b}$.

Proof. Let us see first that the matrix $B := [A; b]^T[A; b] = (b_{ij})_{1 \leqslant i, j \leqslant n+1}$ is oscillatory. By Theorem 3.1 of [1], B is totally positive. Since rank$[A; b]$ is maximal, we have that B is invertible. Then, by Corollary 3.8 of [1], $b_{ii} > 0$ for all $i = 1, \dots, n+1$. Besides, one can check using (3.5) that $b_{i,i-1} > 0$ $\forall i = 2, \dots, n+1$ and that $b_{i,i+1} > 0$ $\forall i = 1, \dots, n$. Thus, by Theorem 1.1, B is an oscillatory matrix.

Since B is oscillatory, by Theorem 6.5 of [1] one has that $\lambda'_n > \lambda_{n+1}$, where λ'_n is the least eigenvalue of $A^TA = B[1, \dots, n|1, \dots, n]$ and λ_{n+1} is the least eigenvalue of B. Hence $\sigma'_n > \sigma_{n+1}$, where σ'_n is the least singular value of A and σ_{n+1} is the least singular value of $[A; b]$. Now the result follows from Theorem 2.6 of [11]. ∎

Since strictly totally positive matrices have positive elements and maximal rank, we may deduce the following consequence of the previous theorem.

COROLLARY 3.3. *Let $Ax = b$ be an overdetermined linear system, and assume that $[A; b]$ is a strictly totally positive matrix. Then, if (3.4) is the singular value decomposition of $[A; b]$, the corresponding total least squares problem (3.2) has the solution*

$$\left[\hat{A}; \hat{b}\right] = U\hat{\Sigma}V, \qquad \hat{\Sigma} = \operatorname{diag}\{\sigma_1, \dots, \sigma_n, 0\},$$

and $\hat{x} = (1/\sigma_{n+1})(v_{1,n+1}, \dots, v_{n,n+1})$ is the unique solution to $\hat{A}x = \hat{b}$.

REFERENCES

1 T. Ando, Totally positive matrices, *Linear Algebra Appl.* 90:165–219 (1987).
2 J. M. Carnicer and J. M. Peña, On transforming a Tchebycheff system into a strictly totally positive system, *J. Approx. Theory* 81:274–295 (1995).
3 C. W. Cryer, *LU*-factorization of totally positive matrices, *Linear Algebra Appl.* 7:83–92 (1973).
4 F. R. Gantmacher, *The Theory of Matrices*, Vol. 2, Chelsea, New York, 1959.
5 F. R. Gantmacher and M. G. Krein, *Oszillationsmatrizen, Oszillationskerne und Kleine Schwingungen Mechanischer Systeme*, Akademie-Verlag, Berlin, 1960.
6 M. Gasca and J. M. Peña, Total positivity and Neville elimination, *Linear Algebra Appl.* 165:25–44 (1992).
7 M. Gasca and J. M. Peña, Total positivity, *QR*-factorization and Neville elimination, *SIAM J. Matrix Anal. Appl.* 14:1132–1140 (1993).

8 G. H. Golub and C. F. Van Loan, *Matrix Computations*, John Hopkins U.P., Baltimore, 1989.
9 S. Karlin, *Total Positivity*, Vol. I, Stanford U.P., Stanford, Calif., 1968.
10 G. W. Stewart, On the early history of the singular value decomposition, *SIAM Rev.* 35:551–566 (1993).
11 S. Van Huffel and J. Vandewalle, *The Total Least Squares Problem: Computational Aspects and Analysis*, Frontiers Appl. Math. 9, SIAM, Philadelphia, 1991.

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