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Coorbit spaces for dual pairs

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ABSTRACT

In this paper we present an abstract framework for construction of Banach spaces of distributions from group representations. This extends the theory of coorbit spaces initiated by H.G. Feichtinger and K. Gröchenig in the 1980s. The coorbit theory sets up a correspondence between spaces of distributions and reproducing kernel Banach spaces. The original theory required that the initial representation was irreducible, unitary and integrable. As a consequence not all Bergman spaces could be described as coorbits. Our approach relies on duality arguments, which are often verifiable in cases where integrability fails. Moreover it does not require the representation to be irreducible or even come from a unitary representation on a Hilbert space. This enables us to account for the full Banach-scale of Bergman spaces on the unit disk for which we also provide atomic decompositions. Replacing the integrability criteria with duality also has the advantage that the reproducing kernel need not provide a continuous projection from a larger Banach function space. We finish the article with a wavelet characterization of Besov spaces on the forward light cone.

1. Introduction

In the 1980s H.G. Feichtinger and K. Gröchenig presented a unified framework for generation of Banach spaces of distributions using group representations. Their results published in [13–15] and [19] are based on an irreducible unitary representation (π, H) of a locally compact group G with left-invariant Haar measure dx. The construction of Feichtinger and Gröchenig requires that the space of analyzing vectors

$$\mathcal{A}_w = \left\{ u \in H \;\middle|\; \int_G \left|\left(\pi(x)u, u\right)\right| w(x)\, dx < \infty \right\}$$

is non-zero for a submultiplicative weight $w : G \mapsto \mathbb{R}^+$. Here (u, v) is the inner product of $u, v \in H$. For a non-zero analyzing vector u the space

$$H^1_w = \left\{ v \in H \;\middle|\; \int_G \left|\left(\pi(x)u, v\right)\right| w(x)\, dx < \infty \right\}$$

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is a Banach space, which does not depend on the chosen $u \in \mathcal{A}_w$. Denote by $(H_w^1)^*$ the conjugate dual of H_w^1. For a left-invariant Banach function space Y on G define

$$\mathrm{Co}_{FG}\, Y = \left\{ v \in \left(H_w^1\right)^* \;\middle|\; \left(x \mapsto \langle v, \pi(x)u\rangle\right) \text{ is in } Y \right\}.$$

Feichtinger and Gröchenig show, that the coorbit space $\mathrm{Co}_{FG}\, Y$ is a π-invariant Banach space of distributions which is isometrically isomorphic to a reproducing kernel Banach subspace of Y. Further, they construct atomic decompositions and frames for these spaces in the case where the analyzing vector u is chosen such that the mapping $x \mapsto (\pi(x)u, u)$ is in a certain Wiener amalgam space (see for example Lemma 4.6(i) in [18]).

Atomic decompositions have already been established for quite general classes of Hilbert spaces for which the wavelet coefficients are not integrable (see for example [16,12] and [23]). In the article [7] we gave examples of non-trivial Banach coorbits and atomic decompositions in cases where $\mathcal{A}_w$ and $\mathrm{Co}_{FG}\, Y$ are the zero space. In the present article we propose a generalized coorbit theory, which is able to account for the examples from [7]. We have included the proofs that were left out of [7] and further expanded on the examples from that paper. The idea of the new construction is to replace the space H_w^1 with a Fréchet space S. For square integrable representations of Lie groups the space of smooth vectors is a natural choice. As an example, the smooth vectors of the discrete series representation of the group

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \;\middle|\; a > 0, b \in \mathbb{R} \right\} \subseteq \mathrm{SL}_2(\mathbb{R})$$

are used to give a complete wavelet characterization of the Bergman spaces of holomorphic functions on the unit disk. We further present a wavelet characterization of the Besov spaces on the forward light cone as defined in [1]. This example can be described by the theory of Feichtinger and Gröchenig, however we include it here as it is of interest in its own right. We expect that this characterization will generalize to other symmetric cones.

2. Coorbit spaces for dual pairs

Let S be a Fréchet space and let S^* be the conjugate linear dual equipped with the weak* topology (any reference to weak convergence in S^* will always refer to the weak* topology). We assume that S is continuously embedded and weakly dense in S^*. The conjugate dual pairing of elements $v \in S$ and $\phi \in S^*$ will be denoted by $\langle \phi, v\rangle$. Let G be a locally compact group with a fixed left Haar measure dx, and assume that (π, S) is a continuous representation of G, i.e. $x \mapsto \pi(x)v$ is continuous for all $v \in S$. A vector $v \in S$ is called *cyclic* if $\langle \phi, \pi(x)v\rangle = 0$ for all $x \in G$ means that $\phi = 0$ in S^*. As usual, define the contragradient representation (π^*, S^*) by

$$\left\langle \pi^*(x)\phi, v\right\rangle = \left\langle \phi, \pi\left(x^{-1}\right)v\right\rangle.$$

Then π^* is a continuous representation of G on S^*. For a fixed vector $u \in S$ define the linear map $W_u : S^* \to C(G)$ by

$$W_u(\phi)(x) = \left\langle \phi, \pi(x)u\right\rangle = \left\langle \pi^*\left(x^{-1}\right)\phi, u\right\rangle.$$

The map W_u is called *the voice transform* or *the wavelet transform*. If F is a function on G then define the left translation of F by an element $x \in G$ as

$$\ell_x F(y) = F\left(x^{-1}y\right).$$

A Banach space of functions Y is called left invariant if $F \in Y$ implies that $\ell_x F \in Y$ for all $x \in G$ and there is a constant C_x such that $\|\ell_x F\|_Y \leqslant C_x \|F\|_Y$ for all $F \in Y$. In the following we will always assume that the space Y of functions on G is a left invariant Banach space for which convergence implies convergence (locally) in Haar measure on G. Examples of such spaces are $L^p(G)$ for $1 \leqslant p \leqslant \infty$ and any space continuously included in an $L^p(G)$.

Assumption 2.1. Assume that there is a non-zero cyclic vector $u \in S$ satisfying the following properties:

(R1) The reproducing formula $W_u(v) * W_u(u) = W_u(v)$ is true for all $v \in S$.
(R2) The mapping $Y \ni F \mapsto \int_G F(x) W_u(u)(x^{-1})\, dx \in \mathbb{C}$ is continuous.
(R3) If $F = F * W_u(u) \in Y$, then the mapping $S \ni v \mapsto \int F(x)\langle \pi^*(x)u, v\rangle\, dx \in \mathbb{C}$ is in S^*.
(R4) The mapping $S^* \ni \phi \mapsto \int \langle \phi, \pi(x)u\rangle \langle \pi^*(x)u, u\rangle\, dx \in \mathbb{C}$ is weakly continuous.

A vector u satisfying Assumption 2.1 is called an *analyzing vector*. Note that assumption (R2) and the left invariance of Y ensure that the convolution

$$F * W_u(u)(y) = \int_G F(x) W_u(u)\left(x^{-1}y\right) dx$$

is well defined for all $F \in Y$ at every point $y \in G$. If $F = F * W_u(u) \in Y$ then (R3) implies the existence of a unique $\phi \in S^*$ such that

$$\langle \phi, v\rangle = \int_G F(x)\big\langle \pi^*(x)u, v\big\rangle\, dx.$$

This ϕ is denoted $\pi^*(F)u$. Also (R4) implies that there is an element $v \in S$ such that

$$\langle \phi, v\rangle = \int \big\langle \phi, \pi(x)u\big\rangle\big\langle \pi^*(x)u, u\big\rangle\, dx$$

for all $\phi \in S^*$. This ensures that the vector $v \in S$ can be weakly defined by

$$v = \pi\big(W_u(u)^\vee\big)u = \int_G W_u(u)^\vee(x)\pi(x)u\, dx$$

where we have used the notation $f^\vee(x) = f(x^{-1})$.

Define the subspace Y_u of Y by

$$Y_u = \big\{F \in Y \;\big|\; F = F * W_u(u)\big\},$$

then the following result holds:

Lemma 2.2. *If F and u satisfy* (R2)*, then the space Y_u is closed in Y and hence a reproducing kernel Banach space with reproducing kernel $K(x,y) = W_u(u)(x^{-1}y)$.*

Proof. Let $\{F_n\}$ be a sequence in Y_u which converges to $F \in Y$. Then, since we assumed that convergence in Y implies convergence in measure, we know that there is a subsequence F_{n_k} which converges to F almost everywhere. $F * W_u(u)(y)$ is defined for all y by assumption (R2) and we see that for a fixed y outside a null-set

$$\begin{aligned}\big|F(y) - F * W_u(u)(y)\big| &\leqslant \big|F(y) - F_{n_k}(y)\big| + \big|F_{n_k}(y) - F_{n_k} * W_u(u)(y)\big| \\ &\quad + \big|F_{n_k} * W_u(u)(y) - F * W_u(u)(y)\big|.\end{aligned}$$

The first term can be made arbitrarily small and the second term is zero. The last term can be estimated by

$$C\|\ell_{y^{-1}}F_{n_k} - \ell_{y^{-1}}F\|_Y$$

by assumption (R2) and the left invariance of Y ensures that it can be made arbitrarily small (using that F_{n_k} converges to F in norm). Therefore $F = F * W_u(u)$ almost everywhere and $F \in Y_u$. □

Define the space

$$\mathrm{Co}_S^u\, Y = \big\{\phi \in S^* \;\big|\; W_u(\phi) \in Y\big\} \tag{1}$$

equipped with the norm $\|\phi\| = \|W_u(\phi)\|_Y$. The space $\mathrm{Co}_S^u\, Y$ is called the coorbit space of Y with respect to u and S.

Theorem 2.3. *Assume that Y and u satisfy Assumption* 2.1*, then*

(a) *$W_u(v) * W_u(u) = W_u(v)$ for $v \in \mathrm{Co}_S^u\, Y$.*
(b) *The space $\mathrm{Co}_S^u\, Y$ is a π^*-invariant Banach space.*
(c) *$W_u : \mathrm{Co}_S^u\, Y \to Y$ intertwines π^* and left translation.*
(d) *If left translation is continuous on Y, then π^* acts continuously on $\mathrm{Co}_S^u\, Y$.*
(e) *$\mathrm{Co}_S^u\, Y = \{\pi^*(F)u \mid F \in Y_u\}$.*
(f) *$W_u : \mathrm{Co}_S^u\, Y \to Y_u$ is an isometric isomorphism.*

Remark 2.4. If we replace condition (R2) by the assumption that the mapping $Y \ni F \mapsto F * W_u(u) \in Y$ is continuous, then $Y_u = Y * W_u(u)$ and the convolution operator $F \mapsto F * W_u(u)$ is a continuous projection onto the image of W_u. This is the version of the assumptions found in [7] and in [6]. However we have opted for the more general assumption which only ensures the existence of the convolution. The reason for this is that we aim at giving a wavelet characterization of Bergman spaces related to symmetric cones, in which case the projection might not be continuous (see [2,1]). Further it is often easier to show that the function $W_u(u)$ is in the dual of Y rather than $Y * W_u(u) \subseteq Y$.

Proof of Theorem 2.3. (a) We show that the reproducing formula holds for all $\phi \in S^*$. The space S is weakly dense in S^*, so choose a net v_α in S for which $v_\alpha \to \phi$ weakly in S^*. By assumption (R1) the reproducing formula $W_u(v_\alpha) * W_u(u) = W_u(v_\alpha)$ holds for each v_α. The continuity requirement (R4) gives that

$$\phi \mapsto \int \big\langle \pi^*\big(y^{-1}\big)\phi, \pi(x)u\big\rangle \big\langle \pi^*(x)u, u\big\rangle\, dx = \int \big\langle \phi, \pi(x)u\big\rangle \big\langle u, \pi\big(x^{-1}y\big)u\big\rangle\, dx$$
$$= W_u(\phi) * W_u(u)(y)$$

is weakly continuous. Therefore $W_u(v_\alpha) * W_u(u)(y) \to W_u(\phi) * W_u(u)(y)$ for every $y \in G$. By assumption $W_u(v_\alpha)(y) \to W_u(\phi)(y)$ for all $y \in G$, and we conclude that

$$W_u(\phi)(y) = W_u(\phi) * W_u(u)(y) \quad \text{for all } y \in G.$$

This reproducing formula is valid for all $\phi \in S^*$ and hence also for $\phi \in \mathrm{Co}_S^u Y \subseteq S^*$.

(b)–(d) We now show that $\|\phi\| = \|W_u(\phi)\|_Y$ is indeed a norm. The only non-obvious property is that $\|\phi\| = 0$ implies $\phi = 0$. If $\|\phi\| = 0$ then $\|W_u(\phi)\|_Y = 0$ and so $\langle \phi, \pi(x)u\rangle = 0$ for almost all x. The function $x \mapsto \langle \phi, \pi(x)u\rangle$ is continuous and thus it is zero for all x. But u is cyclic in S, so $\phi = 0$. This also proves the injectivity of W_u.

Next we prove that the space $\mathrm{Co}_S^u Y$ is complete. Assume that v_n is a Cauchy sequence in $\mathrm{Co}_S^u Y$. Then $W_u(v_n)$ is a Cauchy sequence in Y_u and $W_u(v_n)$ converges to a function $F \in Y_u$. Assumption (R3) implies that ϕ defined by

$$\langle \phi, v\rangle = \int F(x) \big\langle \pi^*(x)u, v\big\rangle\, dx$$

is in S^*, and it follows that

$$\begin{aligned} W_u(\phi)(y) &= \big\langle \phi, \pi(y)u\big\rangle \\ &= \int F(x)\big\langle \pi^*(x)u, \pi(y)u\big\rangle\, dx \\ &= \int F(x)\big\langle u, \pi\big(x^{-1}y\big)u\big\rangle\, dx \\ &= F * W_u(u)(y) \\ &= F(y). \end{aligned}$$

Thus $\phi \in \mathrm{Co}_S^u Y$.

The definition of π^* and the left invariance of Y ensure that $\mathrm{Co}_S^u Y$ is π^*-invariant and that W_u intertwines π^* and left translation: Assume that ϕ is in $\mathrm{Co}_S^u Y$, then the voice transform of $\pi^*(y)\phi$ is

$$W_u\big(\pi^*(y)\phi\big)(x) = \big\langle \pi^*(y)\phi, \pi(x)u\big\rangle = \big\langle \phi, \pi\big(y^{-1}x\big)u\big\rangle = \ell_y W_u(\phi)(x).$$

This also shows that if left translation is continuous on Y, then π^* acts continuously on $\mathrm{Co}_S^u Y$.

(f) We now show that $W_u(\mathrm{Co}_S^u Y) = Y_u$. If $\phi \in \mathrm{Co}_S^u Y$ then $W_u(\phi) \in Y$ and also $W_u(\phi) = W_u(\phi) * W_u(u) \in Y_u$. If on the other hand $F \in Y_u$ then $F = F * W_u(u)$ and assumption (R3) again tells us that there is a $\phi \in S^*$ defined by

$$\langle \phi, v\rangle = \int F(x)\big\langle \pi^*(x)u, v\big\rangle\, dx$$

for $v \in S$. Direct calculation shows that

$$W_u(\phi) = F * W_u(u) = F \in Y$$

such that $\phi \in \mathrm{Co}_S^u Y$. Therefore $W_u : \mathrm{Co}_S^u Y \to Y_u$ is surjective. That W_u is an isometry follows directly from the definition of the norm.

(e) Above we have shown that for $F \in Y_u$ there is a $\phi \in \mathrm{Co}_S^u Y$ such that $\phi = \pi^*(F)u$. If on the other hand $\phi \in \mathrm{Co}_S^u Y$, then let $F = W_u(\phi) = F * W_u(u) \in Y * W_u(u)$. Then by (R3) $\pi^*(F)u$ defines an element in S^* and

$$\begin{aligned} \big\langle \pi^*(F)u, \pi(y)u\big\rangle &= \int F(x)\big\langle \pi^*(x)u, \pi(y)u\big\rangle\, dx \\ &= F * W_u(u)(y) \\ &= F(y) \\ &= \big\langle \phi, \pi(y)u\big\rangle. \end{aligned}$$

This shows that $\pi^*(F)u$ and ϕ agree for all $\pi(y)u$, and since u is cyclic in S, it follows that $\pi^*(F)u$ and ϕ are the same element in S^*. $\square$

Corollary 2.5. *If Assumption* 2.1 *holds for u and a Banach function space Y then it also holds for any quasi-Banach space $\widetilde{Y}$ continuously included in Y. In particular* $\mathrm{Co}_S^u \widetilde{Y}$ *is a well-defined quasi-Banach space satisfying Theorem* 2.3 *and* $\mathrm{Co}_S^u \widetilde{Y}$ *is continuously included in* $\mathrm{Co}_S^u Y$.

Remark 2.6. Theorem 4.2(i) in [14] states that $\mathrm{Co}_{FG} Y$ is continuously included in $(\mathcal{H}_w^1)^*$, and Theorem 4.5.13(d) in [24] states further that $\mathcal{H}_w^1$ is continuously included in $\mathrm{Co}_{FG} Y$. In general $S \not\subseteq \mathrm{Co}_S^u Y$, since for example the coorbit space $\mathrm{Co}_{\mathcal{H}}^u L^1(G)$ for an integrable representation does not contain $\mathcal{H}$. It is an open problem if the inclusion $\mathrm{Co}_S^u Y \hookrightarrow S^*$ is continuous for the general coorbit theory.

The following theorem tells us which analyzing vectors will give the same coorbit space.

Theorem 2.7 *(Dependence on the analyzing vector). If u_1 and u_2 both satisfy Assumption* 2.1 *and for $i, j \in \{1, 2\}$ the following properties hold*

- *there are non-zero constants $c_{i,j}$ such that $W_{u_i}(v) * W_{u_j}(u_i) = c_{i,j} W_{u_j}(v)$ for all $v \in S$*
- *$Y_{u_i} \ni F \mapsto F * W_{u_j}(u_i) \in Y$ is continuous*
- *$S^* \ni \phi \mapsto \int \langle \phi, \pi(x)u_i\rangle \langle \pi^*(x)u_i, u_j\rangle\, dx \in \mathbb{C}$ is weakly continuous*

then $\mathrm{Co}_S^{u_1} Y = \mathrm{Co}_S^{u_2} Y$ *with equivalent norms.*

Proof. Assume that u_1 and u_2 are two analyzing vectors, i.e. they satisfy the properties Assumption 2.1. We claim first that

$$W_{u_1}(v) * W_{u_2}(u_1) = c_{1,2} W_{u_2}(v)$$

for all $v \in S^*$. With $v \in S$ this is true by the assumption. The space S is weakly dense in S^* and therefore the identity $W_{u_1}(v) * W_{u_2}(u_1) = c_{1,2} W_{u_2}(v)$ is true for all $v \in S^*$. This is verified by applying the third continuity condition to the integral

$$W_{u_1}(v) * W_{u_2}(u_1)(y) = \int \bigl\langle \pi^*\bigl(y^{-1}\bigr) v, \pi(x)u_1\bigr\rangle \bigl\langle \pi^*(x)u_1, u_2\bigr\rangle\, dx.$$

If $W_{u_1}(v) \in Y$ then $W_{u_1}(v) \in Y_{u_1}$ and $W_{u_1}(v) * W_{u_2}(u_1) = c_{1,2} W_{u_2}(v) \in Y$ by assumption. The continuity assumption gives the inequality

$$\bigl\| W_{u_2}(v)\bigr\|_Y = c_{1,2}^{-1} \bigl\| W_{u_1}(v) * W_{u_2}(u_1)\bigr\|_Y \leqslant C \bigl\| W_{u_1}(v)\bigr\|_Y.$$

Symmetry then gives us that $\mathrm{Co}_S^{u_1} Y = \mathrm{Co}_S^{u_2} Y$ with equivalent norms. □

In the following we will describe how the choice of the Fréchet space S affects the coorbit space. We will show that there is great freedom when choosing S.

Theorem 2.8 *(Dependence on the Fréchet space). Let S and T be Fréchet spaces which are weakly dense in their conjugate duals S^* and T^* respectively. Let π and $\tilde{\pi}$ denote representations of G on S and T respectively. Assume there are vectors $u \in S$ and $\tilde{u} \in T$ such that the requirements in Assumption* 2.1 *are satisfied by both (u, S) and $(\tilde{u}, T)$. Also assume that the conjugate dual pairings of $S^* \times S$ and $T^* \times T$ satisfy $\langle u, \pi(x)u\rangle_S = \langle \tilde{u}, \tilde{\pi}(x)\tilde{u}\rangle_T$ for all $x \in G$. Then $\mathrm{Co}_S^u Y$ and $\mathrm{Co}_T^{\tilde{u}} Y$ are isometrically isomorphic. The isomorphism is given by $W_{\tilde{u}}^{-1} W_u$.*

Proof. Let $W_u(\phi)(x) = \langle \phi, \pi(x)u\rangle_S$ for $\phi \in \mathrm{Co}_u^S Y$ and $W_{\tilde{u}}(\tilde{v}')(x) = \langle \tilde{v}', \tilde{\pi}(x)\tilde{u}\rangle_T$ for $\tilde{v}' \in \mathrm{Co}_{\tilde{u}}^T Y$. Since it is assumed that $W_u(\pi(x)u) = W_{\tilde{u}}(\pi(x)\tilde{u})$ for all $x \in G$ the spaces $\mathrm{Co}_u^S Y$ and $\mathrm{Co}_{\tilde{u}}^T Y$ are both isometrically isomorphic to the space $Y_u = Y_{\tilde{u}}$. The isomorphism between $\mathrm{Co}_S^u Y$ and $\mathrm{Co}_T^{\tilde{u}} Y$ is exactly $W_{\tilde{u}}^{-1} W_u : \mathrm{Co}_u^S Y \to \mathrm{Co}_{\tilde{u}}^T Y$. □

The use of Gelfand triples has been emphasized in for example [11] and it is natural to ask how the coorbit space depends on the Gelfand triple used for its construction. Let π be a unitary irreducible representation of G on $\mathcal{H}$. Assume that the Fréchet spaces S and T are π-invariant and that $(S, \mathcal{H}, S^*)$ and $(T, \mathcal{H}, T^*)$ are Gelfand triples with the common Hilbert space $\mathcal{H}$. Then $S \cap T$ is π-invariant. If we can choose a non-zero vector $u \in S \cap T$, such that u is analyzing for both S and T, then

$$\bigl\langle u, \pi(x)u\bigr\rangle_S = \bigl(u, \pi(x)u\bigr)_{\mathcal{H}} = \bigl\langle u, \pi(x)u\bigr\rangle_T$$

and we are in the situation of the previous theorem. We summarize the statement as

Corollary 2.9. *Assume that $(S, \mathcal{H}, S^*)$ and $(T, \mathcal{H}, T^*)$ are Gelfand triples and assume there is an analyzing vector $u \in S \cap T$ such that both (u, S) and (u, T) satisfy Assumption* 2.1 *for some Banach space Y, then $\mathrm{Co}_S^u Y$ and $\mathrm{Co}_T^u Y$ are isometrically isomorphic.*

If the Fréchet space S is continuously included and dense in the Fréchet space T, then we can regard the space T^* as a subspace of S^*. With this identification the two coorbit spaces will be equal. We state the following

Theorem 2.10. *Let $(\pi, \mathcal{H})$ be a unitary representation of G, and let $(S, \mathcal{H}, S^*)$ and $(T, \mathcal{H}, T^*)$ be Gelfand triples for which (π, S) and (π, T) are representations of G. Assume that $i : S \to T$ is a continuous linear inclusion and that there is $u \in S$ such that both (u, S) and $(i(u), T)$ satisfy Assumption* 2.1*. Then the map i^* restricted to $\mathrm{Co}_T^{i(u)} Y$ is an isometric isomorphism between $\mathrm{Co}_T^{i(u)} Y$ and $\mathrm{Co}_S^u Y$.*

Proof. Since the vector $i(u)$ is assumed cyclic in T, we see that $i(S)$ is dense in T, and therefore $i^* : T^* \to S^*$ is injective. This allows us to view T^* as a subspace of S^*.

Let $W_u(\phi)(x) = \langle \phi, \pi(x)u\rangle_S$ and $W_{i(u)}(\tilde{v}') = \langle \tilde{v}', \pi(x)i(u)\rangle_T$. For $\phi \in \mathrm{Co}_S^u Y$ we have

$$\begin{aligned} W_u(\phi) * W_{i(u)}\big(i(u)\big)(x) &= \int W_u(\phi)(y)\big(i(u), \pi\big(y^{-1}x\big)i(u)\big)_{\mathcal{H}}\, dy \\ &= \int W_u(\phi)(y)\big(u, \pi\big(y^{-1}x\big)u\big)_{\mathcal{H}}\, dy \\ &= W_u(\phi) * W_u(u) \\ &= W_u(\phi) \end{aligned} \tag{2}$$

which shows that $W_u(\phi) \in Y_{i(u)}$. By (R3) there is an element $\tilde{v}' \in T^*$ such that for $\tilde{v} \in T$

$$\langle \tilde{v}', \tilde{v}\rangle_T = \int W_u(\phi)(x)\langle \pi^*(x)i(u), \tilde{v}\rangle_T\, dx.$$

Furthermore $i^*(\tilde{v}') = \phi$ in S^*, since u is cyclic and

$$\langle i^*(\tilde{v}'), \pi(x)u\rangle_S = \langle \tilde{v}', \pi(x)i(u)\rangle_T = W_u(\phi) * W_{i(u)}\big(i(u)\big)(x) = \langle \phi, \pi(x)u\rangle_S$$

for each $x \in G$. This shows that $\mathrm{Co}_S^u Y \subseteq i^*(\mathrm{Co}_T^{i(u)} Y)$.

If on the other hand $\tilde{v}' \in \mathrm{Co}_T^{i(u)} Y$, then

$$W_u\big(i^*(\tilde{v}')\big)(x) = \langle i^*(\tilde{v}'), \pi(x)u\rangle_S = \langle \tilde{v}', \pi(x)i(u)\rangle_T = W_{i(u)}(\tilde{v}')(x) \in Y$$

which shows that $i^*(\tilde{v}') \in \mathrm{Co}_S^u Y$. This implies that $i^*(\mathrm{Co}_T^{i(u)} Y) \subseteq \mathrm{Co}_S^u Y$.

That the mapping i^* is an isometry when restricted to $\mathrm{Co}_T^{i(u)} Y$ follows directly from the calculations in (2). □

Remark 2.11. If (π, S) is a representation of G and u is a cyclic vector for which it is true that $\langle \pi^*(x)u, u\rangle = \overline{\langle u, \pi(x)u\rangle}$ for all $x \in G$ and (R1) and (R4) are satisfied, then $\langle v, w\rangle$ is an inner product on S. The completion $\mathcal{H}$ of S with respect to the norm $\|v\|_{\mathcal{H}} = \sqrt{\langle v, v\rangle}$ is a Hilbert space. The representation π extends to a unitary representation on $\mathcal{H}$. This representation will be cyclic, however in general it will not be irreducible.

Remark 2.12. In general the reproducing formula (R1) does not imply unitarity as demonstrated in [29]. There Zimmermann obtains a reproducing formula from a non-unitary representation. It will be interesting to see if it is possible to construct coorbit spaces in this setting.

The following theorem is a slight generalization of [14, Theorem 4.9], which in theory enables us to apply it to more general coorbit spaces than the ones treated in [14]. The proof follows that of [14, Theorem 4.9] and is left to the reader.

Theorem 2.13. *Let Y^* be the conjugate dual space of Y and assume it is also a Banach space of functions. Assume that $u \in S$ is a vector satisfying Assumption* 2.1 *for both Y and Y^*. If the conjugate dual pairing on $Y^* \times Y$ satisfies*

$$\langle F * W_u(u), G\rangle_{Y^* \times Y} = \langle F, G * W_u(u)\rangle_{Y^* \times Y}, \tag{3}$$

then $(\mathrm{Co}_S^u Y)^* = \mathrm{Co}_S^u(Y^*)$. If Y is reflexive so is $\mathrm{Co}_S^u Y$.

If the conjugate dual pairing of Y and Y^* is the extension of an integral then property (3) is true.

The following theorem shows that besides a reproducing formula a duality requirement is sufficient for the construction of coorbit spaces. This is worth mentioning as some of the examples we treat later can be described in this manner.

Theorem 2.14. *Let π be a representation of a group G on a Fréchet space S with conjugate dual S^*. Let u be a cyclic vector in S such that $W_u(v) * W_u(u) = W_u(v)$ for all $v \in S^*$. Assume that for the Banach space Y, the mapping*

$$Y \times S \ni (F, v) \mapsto \int_G F(x) W_v(u)\big(x^{-1}\big)\,dx \in \mathbb{C}$$

is continuous, then $\mathrm{Co}_S^u Y = \{\phi \in S^* \mid W_u(\phi) \in Y\}$ *satisfies properties* (a)–(f) *of Theorem* 2.3.

Note that for $Y = L^p(G)$ the requirement is in fact a duality requirement, i.e. we require that $S \ni v \mapsto W_v(u)^\vee \in L^q(G)$ is continuous for $1/p + 1/q = 1$.

Proof. The proof follows that of Theorem 2.3. We note that the requirements (R1) and (R4) are used to prove the reproducing formula $W_u(\phi) * W_u(u) = W_u(\phi)$ for all $\phi \in S^*$ (which we instead have assumed here). The requirement (R3) is satisfied for $F = F * W_u(u) \in Y$, since it is assumed true for all $F \in Y$. □

3. Existing coorbit theories

In this section we show that the coorbit theory of Feichtinger and Gröchenig is a special case of the coorbit theory for dual pairs.

3.1. Coorbit theory by Feichtinger and Gröchenig

In the following let $(\pi, \mathcal{H})$ be an irreducible unitary square-integrable representation on a locally compact group G. Then the Duflo–Moore Theorem [9] ensures that we can choose $u \neq 0$, such that the wavelet coefficients

$$W_u(v) = \big(v, \pi(x)u\big)$$

satisfy a reproducing formula

$$W_u(v) * W_u(u) = W_u(v)$$

for all $v \in \mathcal{H}$. Let Y be a left invariant Banach function space continuously included in $L^1_{loc}(G)$. Since convergence in $L^1_{loc}(G)$ implies convergence locally in Haar measure, the same is true for Y. Define the weight

$$w(x) = \sup_{\|F\|_Y = 1} \frac{\|\ell_{x^{-1}} F\|_Y}{\|F\|_Y},$$

and assume that the space

$$\mathcal{H}^1_w = \big\{v \in \mathcal{H} \bigm| W_u(v) \in L^1_w\big\}$$

contains u (and thus is non-zero) and equip it with the norm

$$\|v\|_{\mathcal{H}^1_w} = \big\|W_u(v)\big\|_{L^1_w}.$$

Denote the conjugate dual of $\mathcal{H}^1_w$ by $(\mathcal{H}^1_w)^*$, and define the coorbit space

$$\mathrm{Co}_{FG}\, Y = \big\{\phi \in \big(\mathcal{H}^1_w\big)^* \bigm| W_u(\phi) \in Y\big\}.$$

Feichtinger and Gröchenig prove all the properties of Theorem 2.3 are satisfied for $\mathrm{Co}_{FG}\, Y$ (see Section 4 in [14]). The proofs by Feichtinger and Gröchenig rely on Wiener amalgam spaces, which we briefly introduce here. For a compact neighborhood Q of $e \in G$ let 1_Q be the characteristic function on Q and define the control function

$$K_Q(F)(x) = \big\|(\ell_x 1_Q)F\big\|_{L^\infty}.$$

Then the space $W(Y)$ defined by

$$W(Y) = \big\{F \in Y \bigm| K_Q(F) \in Y\big\}$$

with norm $\|F\|_{W(Y)} = \|K_Q(F)\|_Y$ does not depend on Q (up to norm equivalence). These spaces were used to verify properties (R2) and (R3), which are often easier to prove by duality (see Theorem 2.14) allowing us to avoid the Wiener amalgam machinery. Feichtinger and Gröchenig further use Wiener amalgam spaces to define the set of *better vectors*

$$\mathcal{B}_w = \big\{u \in H \bigm| W_u(u) \in W\big(L^1_w\big)\big\}, \tag{4}$$

and they show that a non-zero better vector will provide atomic decompositions for the coorbit space $\mathrm{Co}_{FG}\, Y$ (see Theorem 6.1 in [14]).

Remark 3.1. We expect that the coorbit theory developed by Dahlke, Steidl and Teschke in [8] can also be generalized in a manner similar to our approach presented in Section 2.

3.2. Coorbit theory for quasi-Banach spaces

In [25] Rauhut introduces coorbits for a quasi-Banach space Y. The notation in this section follows that of the coorbit theory by Feichtinger and Gröchenig. Rauhut uses Wiener amalgam spaces to define the coorbit for Y to be the space

$$C(Y) = \left\{ f \in \left(\mathcal{H}_w^1\right)^* \;\middle|\; W_u(f) \in W(Y) \right\} = \operatorname{Co} W(Y).$$

The use of $W(Y)$ ensures that the convolution by $W_u(u) \in L_w^1$ is defined, while convolutions on quasi-Banach spaces are generally not defined. In [26] is shown that $W(Y)$ is continuously included in $L^\infty_{1/w}(G)$ (which is a Banach space for which the properties in Assumption 2.1 can be verified with $S = \mathcal{H}_w^1$). By Corollary 2.5 it then follows immediately, that $C(Y)$ is a quasi-Banach space.

If w is a weight for which $W_u(u) \in L_w^1$ it is in fact possible to define coorbit spaces for any quasi-Banach space Y' continuously included in $L^\infty_{1/w}$. In particular if Y is a quasi-Banach space the space $Y' = Y \cap L^\infty_{1/w}$ can be used. In the case of the modulation spaces described in [17] it turns out that $W(Y)$, $Y \cap L^\infty_{1/w}$ (and even Y) give the same coorbits. If this is the case in general we do not know.

4. Bergman spaces on the unit disk

Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$ equipped with area measure dz. For $1 \leqslant p < \infty$ and $\sigma > 1$ the Bergman spaces are the classes of holomorphic functions

$$A^p_\sigma(\mathbb{D}) = \left\{ f \in \mathcal{O}(\mathbb{D}) \;\middle|\; \|f\|^p_{A^p_\sigma(\mathbb{D})} = \int_{\mathbb{D}} \left|f(z)\right|^p \left(1 - |z|^2\right)^{\sigma-2} dz < \infty \right\}.$$

In this section we give a wavelet characterization of these spaces.

4.1. Coorbits for discrete series

Let $G \subseteq \mathrm{SL}_2(\mathbb{R})$ be the connected subgroup of upper triangular matrices, i.e.

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \;\middle|\; a > 0, b \in \mathbb{R} \right\}$$

with left-invariant measure $\frac{da\,db}{a^2}$. Through the Cayley transform this group can be regarded as the subgroup of $\mathrm{SU}(1,1)$ consisting of matrices

$$\begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix} = \frac{1}{2} \begin{pmatrix} a + a^{-1} + ib & b + i(a - a^{-1}) \\ b - i(a - a^{-1}) & a + a^{-1} - ib \end{pmatrix}.$$

For real numbers $s > 1$ the pairing

$$(u, v)_s = \frac{s-1}{\pi} \int_{\mathbb{D}} u(z)\overline{v(z)}\left(1 - |z|^2\right)^{s-2} dz = \frac{s-1}{\pi} \int_{\mathbb{D}} u\left(re^{i\theta}\right)\overline{v\left(re^{i\theta}\right)}\left(1 - r^2\right)^{s-2} r\,dr\,d\theta$$

is an inner product on the Hilbert space

$$\mathcal{H}_s = A^2_s(\mathbb{D}) = \left\{ v \in \mathcal{O}(\mathbb{D}) \;\middle|\; (v, v)_s < \infty \right\}.$$

The discrete series representations $(\pi_s, \mathcal{H}_s)$ are defined by

$$\pi_s \begin{pmatrix} \alpha & \beta \\ \bar\beta & \bar\alpha \end{pmatrix} v(z) = (-\bar\beta z + \alpha)^{-s} v\left(\frac{\bar\alpha z - \beta}{-\bar\beta z + \alpha} \right).$$

Since G acts transitively on the disk $\mathbb{D}$ an argument by Kobayashi [21] shows that π_s is irreducible. From now on we denote by u the wavelet in $\mathcal{H}_s$ which is identically 1 on the disk

$$u(z) = 1_{\mathbb{D}}(z).$$

Then the wavelet coefficients $W^s_u(u)$ for $s > 1$ can be calculated explicitly as

$$W^s_u(u)(a, b) = \left(u, \pi_s(a, b)u\right) = 2^s\left(a + a^{-1} - ib\right)^{-s}.$$

The following basic fact is useful to us

Lemma 4.1.

$$\int \left(\left(a+a^{-1}\right)^2+b^2\right)^{-t} a^r \frac{da\,db}{a^2} < \infty$$

if and only if $2(1-t) < r < 2t$.

This shows that the representations π_s are square integrable for all $s > 1$ and integrable for $s > 2$. That π_s is not integrable for $1 < s \leqslant 2$ turns out to not matter for the construction of coorbit spaces for these representations.

Given the submultiplicative weight $w_r(a,b) = 2^r[(a+a^{-1})^2+b^2]^{r/2}$ for $r \geqslant 0$, let $L^p_r(G)$ denote the space

$$L^p_r(G) = \left\{ f \;\middle|\; \|f\|_{L^p_r} = \left(\int \left|f(a,b)w_r(a,b)\right|^p \frac{da\,db}{a^2}\right)^{1/p} < \infty \right\}.$$

We now construct coorbit spaces for the representations π_s related to the spaces $L^p_r(G)$. For this we use the smooth vectors of the representation π_s. A vector $v \in \mathcal{H}_s$ is called *a smooth vector* if the mapping $G \ni x \mapsto \pi(x)v \in \mathcal{H}_s$ is smooth. The space of smooth vectors in $\mathcal{H}_s$ will be denoted $\mathcal{H}_s^\infty$. It is a Fréchet space when equipped with the topology induced by the semi-norms $\sqrt{(\pi(D)v, \pi(D)v)_s}$, where D is in the universal enveloping algebra of the Lie algebra of G. The following characterization of $\mathcal{H}_s^\infty$ and its conjugate dual $\mathcal{H}_s^{-\infty}$ can be found in [22] and more generally in [4]. The binomial coefficients for non-integer $s > k$ are defined as

$$\binom{s}{k} = \frac{s(s-1)\cdots(s-k+1)}{k!}$$

Lemma 4.2. *The smooth vectors $\mathcal{H}_s^\infty$ for π_s are the power series $\sum_{k=0}^\infty a_k z^k$ for which there for any m exists a constant C such that*

$$|a_k|^2 \leqslant C\binom{s+k-1}{k}(1+k)^{-m}.$$

The conjugate dual $\mathcal{H}_s^{-\infty}$ of this space consists of formal power series $\sum_{k=0}^\infty b_k z^k$ for which there is an m and a constant C such that

$$|b_k|^2 \leqslant C\binom{s+k-1}{k}(1+k)^{m}.$$

By [28, p. 254] we know that $\mathcal{H}_s^\infty$ is irreducible if and only if $\mathcal{H}_s$ is, and thus u is cyclic in $\mathcal{H}_s^\infty$. Furthermore the smooth vectors $\mathcal{H}_s^\infty$ are weakly dense in the conjugate dual $\mathcal{H}_s^{-\infty}$.

Theorem 4.3. *The spaces $\mathrm{Co}^u_{\mathcal{H}_s^\infty} L^p_r$ are non-zero π_s-invariant Banach spaces when $2-s < r+2/p < s$.*

Proof. It is a simple matter to show that the mapping

$$L^p_r(G) \ni f \mapsto \int_G \left|f(a,b)\right|\left|W^s_u(u)(a,b)\right| \frac{da\,db}{a^2} \in \mathbb{R}$$

is continuous for $2-s < r+2/p$. We now show that for a given $f \in L^p_r$ the mapping

$$\mathcal{H}_s^\infty \ni v \mapsto \iint f(a,b) W^s_u(v)(a,b) \frac{da\,db}{a^2}$$

is continuous for $2-s < r+2/p$. Let v be a smooth vector with expansion $\sum_{k=0}^\infty a_k z^k$. Since $W^s_u(z^k) = \bar{\alpha}^{-s}(\beta/\bar{\alpha})^k$ it can be shown that

$$\left|W^s_u(v)(a,b)\right| \leqslant \left|W^s_u(u)(a,b)\right| \sum_{k=0}^\infty |a_k|.$$

Therefore $|W^s_u(v)| \leqslant C_v |W^s_u(u)|$, where the constant C_v depends continuously on v. Thus we only need to require that the integral

$$\iint \left|f(a,b) W^s_u(u)(a,b)\right| \frac{da\,db}{a^2}$$

is finite, which we have proven above.

Lastly, by Lemma 4.1 we see, that u is in the coorbit space for $r+2/p < s$. Thus the coorbit space $\mathrm{Co}^u_{\mathcal{H}_s^\infty} L^p_r$ is a non-zero Banach space when $2-s < r+2/p < s$. □

In the next section we will prove that the spaces defined in Theorem 4.3 are in fact Bergman spaces. This was mentioned in [13, Section 7], but not many details were given.

4.2. Continuous description of Bergman spaces

We start with a lemma

Lemma 4.4. *Assume that* $2 - s < r + 2/p < s$. *If* $f \in A^p_{(s-r)p/2}(\mathbb{D})$ *then* $f \in \mathcal{H}_s^{-\infty}$.

Proof. We need to estimate the coefficients b_k where $f(z) = \sum_{k=0}^{\infty} b_k z^k$. For this let us first estimate $f^{(k)}(0)$. The condition on s, r and p means in particular that $(s-r)p/2 - 1 < (s-1)p$ and we can use Theorem 1.10 in [20] to get

$$f(z) = \frac{(s-1)}{\pi} \int_{\mathbb{D}} f(w) \frac{(1-|w|^2)^{s-2}}{(1-z\bar{w})^s}\, dw.$$

Differentiate under the integral sign k times (which is allowed when for example $|z| \leqslant 1/2$)

$$f^{(k)}(z) = (s-1)s(s+1)\cdots(s+k-1) \int_{\mathbb{D}} f(w) \frac{(1-|w|^2)^{s-2}}{(1-z\bar{w})^{s+k}}\, \bar{w}^k\, dw$$

and insert $z = 0$ to get

$$f^{(k)}(0) = (s-1)s(s+1)\cdots(s+k-1) \int_{\mathbb{D}} f(w)\bigl(1-|w|^2\bigr)^{s-2} \bar{w}^k\, dw.$$

The absolute value of the integral can be estimated by

$$\begin{aligned}\int_{\mathbb{D}} \bigl|f(w)\bigr|\bigl(1-|w|^2\bigr)^{s-2}\, dw &= \int_{\mathbb{D}} \bigl|f(w)\bigr|\bigl(1-|w|^2\bigr)^{(s-r)/2-2/p}\bigl(1-|w|^2\bigr)^{(s+r)/2-2/q}\, dw \\ &\leqslant C\|f\|_{A^p_{(s-r)p/2}} \biggl(\int_{\mathbb{D}} \bigl(1-|w|^2\bigr)^{(s+r)q/2-2}\, dw\biggr)^{1/q}.\end{aligned}$$

The last integral is finite when $2 - s < r + 2/p$, and therefore the coefficients b_k can be estimated by

$$|b_k| = \frac{|f^{(k)}(0)|}{k!} \leqslant C\|f\|_{A^p_{(s-r)p/2}} \binom{s+k-1}{k}.$$

Let $\tau = \lceil s \rceil$ then we can estimate

$$\binom{s+k-1}{k} \leqslant \frac{(\tau+k-1)!}{k!} = \underbrace{(\tau+k-1)(\tau+k-2)\cdots(1+k)}_{\tau-1 \text{ terms}} \leqslant \tau^{\tau-1}(1+k)^{\tau-1},$$

and since τ is fixed there is a constant C such that

$$|b_k|^2 \leqslant C\|f\|_{A^p_{(s-r)p/2}} \binom{s+k-1}{k} (1+k)^{\tau-1}.$$

This shows that $f \in \mathcal{H}_s^{-\infty}$. □

Theorem 4.5. *For* $1 < (s-r)p/2 < (s-1)p + 1$ *the space* $A^p_{(s-r)p/2}(\mathbb{D})$ *corresponds to the coorbit* $\mathrm{Co}^u_{\mathcal{H}_s^\infty} L^p_r(G)$ *from Theorem* 4.3 *up to equivalence of norms.*

Proof. Assume that $f \in A^p_{(s-r)p/2}(\mathbb{D})$. We already know that $f \in \mathcal{H}_s^{-\infty}$, so we find that the wavelet coefficient of f is

$$W_u^s(f)(a,b) = \frac{1}{\bar{\alpha}^s} f\biggl(\frac{\beta}{\bar{\alpha}}\biggr).$$

In the last step we applied Theorem 1.10 in [20] provided that $(s-r)p/2 - 1 < (s-1)p$. Then taking $L^p_r(G)$-norm of $W_u^s(f)$ and changing to an integral over the disk we get

$$\frac{1}{2^{(s+r)p}} \int_G \left|W_u^s(f)(a,b)w_r(a,b)\right|^p \frac{da\,db}{a^2} = \|f\|^p_{A^p_{(s-r)p/2}}. \tag{5}$$

We now show that an element of the coorbit space is in the Bergman space. Since any $f \in \mathcal{H}_s^\infty$ is in $A_s^2(\mathbb{D})$ we know that (5) is valid by Proposition 1.4 in [20]. Because $\mathcal{H}_s^\infty$ is weakly dense in $\mathcal{H}_s^{-\infty}$, this equality also holds for $f \in \mathcal{H}_s^{-\infty}$. Therefore the calculations above are valid, and if $f \in \mathrm{Co}^u_{\mathcal{H}_s^\infty} L_r^p(G)$ then f is also in $A^p_{(s-r)p/2}(\mathbb{D})$. □

4.3. Discretization

In this section we obtain sampling theorems and atomic decompositions for the Bergman spaces by use of the wavelet transform. We point out that these results include the non-integrable representations which cannot be described by the work of Feichtinger and Gröchenig.

Lemma 4.6. *The mappings $f \mapsto f * W_u^s(u)$ and $f \mapsto f * |W_u^s(u)|$ are continuous $L_r^p(G) \to L_r^p(G)$ for $s > r + 2/p$.*

Proof. In the following denote by F_s the absolute value of the wavelet coefficient belonging to π_s, i.e. $F_s(a,b) = |(u, \pi_s(a,b)u)|$ and notice that

$$F_s(a,b) = w_{-s}(a,b).$$

In the calculations below we make some assumptions in order for the estimates to be true. At the end of the proof we collect these assumptions.

Let $p > 1$ and assume that $f \in L_r^p(G)$. Let q such that $1/p + 1/q = 1$ and further let t be such that the following calculations hold (we will investigate this later)

$$\begin{aligned}
&\left|\iint f(a,b)F_s\big((a,b)^{-1}(a_1,b_1)\big)\frac{da\,db}{a^2}\right|^p \\
&\quad\leqslant \left(\iint \left|f(a,b)\right|\left|w_{-r-s(1/p+1/q)}\big((a,b)^{-1}(a_1,b_1)\big)\right|^{1/p+1/q} w_r\big((a,b)^{-1}(a_1,b_1)\big)a^t\frac{da\,db}{a^2}\right)^p \\
&\quad\leqslant \left(\iint \left|f(a,b)\right|w_{-rp-s}\big((a,b)^{-1}(a_1,b_1)\big)a^{-tp}\frac{da\,db}{a^2}\right) \\
&\qquad\times \left(\iint a^{tq}w_{rq-s}\big((a,b)^{-1}(a_1,b_1)\big)^q\frac{da\,db}{a^2}\right)^{p/q}.
\end{aligned}$$

We know that $|w_r((a,b))| = |w_r((a,b)^{-1})|$ so the second integral becomes

$$\begin{aligned}
\iint a^{tq}\left|w_{rq-s}\big((a_1,b_1)^{-1}(a,b)\big)\right|\frac{da\,db}{a^2} &= \iint (aa_1)^{tq}\left|w_{rq-s}\big((a,b)\big)\right|\frac{da\,db}{a^2} \\
&= Ca_1^{tq}
\end{aligned}$$

provided that $w_{rq-s}(a,b)a^{tq} \in L^1(G)$. Thus we get

$$\begin{aligned}
&\left|\iint f(a,b)F_s\big((a,b)^{-1}(a_1,b_1)\big)\frac{da\,db}{a^2}\right|^p \\
&\quad\leqslant Ca_1^{tp}\iint \left|f(a,b)\right|^p a^{-tp}\left|w_{-rp-s}\big((a,b)^{-1}(a_1,b_1)\big)\right|\frac{da\,db}{a^2}
\end{aligned}$$

and we can estimate the norm of $f * F_s$ using Fubini's theorem

$$\begin{aligned}
\|f * F_s\|_{L_r^p} &\leqslant C\iiiint \left|f(a,b)\right|^p a^{-tp}w_{-rp-s}\big((a,b)^{-1}(a_1,b_1)\big)\frac{da\,db}{a^2}w_{rp}(a_1,b_1)a_1^{tp}\frac{da_1\,db_1}{a_1^2} \\
&= C\iint \left|f(a,b)\right|^p a^{-tp}\iint \left|w_{-rp-s}\big((a,b)^{-1}(a_1,b_1)\big)\right|w_{rp}(a_1,b_1)a_1^{tp}\frac{da_1\,db_1}{a_1^2}\frac{da\,db}{a^2}.
\end{aligned}$$

A change of variable and using submultiplicativity of the weight w_{rp} then gives

$$= C \iint \left|f(a,b)\right|^p a^{-tp} \iint w_{-rp-s}\big((a_1,b_1)\big) w_{rp}\big((a,b)(a_1,b_1)\big)(aa_1)^{tp} \frac{da_1\,db_1}{a_1^2}\frac{da\,db}{a^2}$$
$$\leqslant C \iint \left|f(a,b)\right|^p w_{rp}(a,b)\frac{da\,db}{a^2} \iint w_{-s}\big((a_1,b_1)\big) a_1^{tp}\frac{da_1\,db_1}{a_1^2}$$
$$\leqslant C\|f\|_{L^p_r},$$

where we in the last inequality have assumed that $w_{-s}(a_1,b_1)a_1^{tp} \in L^1(G)$.

To sum up the map $f \mapsto f * F_s$ is continuous if we are able to choose a t such that both $w_{rq-s}(a,b)a^{tq}$ and $w_{-s}(a,b)a^{tp}$ are in $L^1(G)$. This is the case if both $2-s+rq < tq < s-rq$ and $2-s < tp < s$ and such a t can be shown to exist if and only if $r + 2/p < s$.

For $p = 1$ Fubini can be applied immediately and the requirement becomes that $w_{r-s}(a,b)$ is integrable. This is the case if $2 + r < s$, so the result also holds for $p = 1$. □

The key to finding atomic decompositions will be the following result which generalizes Lemma 4.3 from [7].

Lemma 4.7. *For each $\epsilon > 0$ there is a neighborhood U of the identity such that*

$$\left|\frac{W_u^s(u)((a,b)(x,y))}{W_u^s(u)(x,y)} - 1\right| < \epsilon$$

for $(a,b) \in U$.

From this result follows easily

Corollary 4.8. *There exist a neighborhood U of the identity and constants $C_1, C_2 > 0$ such that*

$$C_1\left|W_u^s(u)(x,y)\right| \leqslant \left|W_u^s(u)\big((a,b)(x,y)\big)\right| \leqslant C_2\left|W_u^s(u)(x,y)\right|$$

for all $(x,y) \in G$ with $(a,b) \in U$. These constants can be chosen arbitrarily close to 1, by choosing U small enough.

Definition 4.9. Let V be a compact neighborhood of the identity, then the points x_i are V-separated if the sets x_iV are pairwise disjoint. Let U be a compact neighborhood of the identity, then the points x_i are U-dense if the sets x_iU cover G.

Proposition 4.10. *Let $V \subseteq U$ be compact neighborhoods of the identity. Assume that the points $\{x_i\}$ are V-separated and U-dense and that U satisfies Corollary 4.8. Let $\{\psi_i\}$ be a partition of unity for which $\operatorname{supp}(\psi_i) \subseteq x_iU$. Define the sequence space*

$$\ell_r^p = \left\{(\lambda_i) \;\middle|\; \|(\lambda_i)\|_{\ell_r^p} = \left(\sum\left|\lambda_i w_r(x_i)\right|^p\right)^{1/p}\right\}.$$

Then the following is true:

(a) *The mapping $\ell_r^p \ni (\lambda_i) \mapsto \sum_i \lambda_i \ell_{x_i} W_u^s(u) \in L_r^p(G) * W_u^s(u)$ is continuous.*
(b) *The mapping $L_r^p(G) * W_u^s(u) \ni f \mapsto (f(x_i))_{i\in I} \in \ell_r^p$ is continuous.*
(c) *The mapping $L_r^p(G) * W_u^s(u) \ni f \mapsto (\int_G f(x)\psi_i(x)dx)_{i\in I} \in \ell_r^p$ is continuous.*

As in [18] sums are understood as limits of the net of partial sums over finite subsets with convergence in $L_r^p(G)$. The proof is similar to the proof of Proposition 4.6 from [7].

Finally, the following proposition gives reconstruction formulas that provide atomic decompositions for the coorbit spaces. The proof is similar to that of Proposition 4.7 in [7] so we omit it here.

Proposition 4.11. *We can choose a compact neighborhood U, U-dense points $\{x_i\}$ and a partition ψ_i of unity with $\operatorname{supp}(\psi_i) \subseteq x_iU$ such that the operators $T_1, T_2, T_3 : L_r^p(G) * W_u^s(u) \to L_r^p(G) * W_u^s(u)$ defined below are invertible with continuous inverses*

(a) *$T_1 f = \sum_i f(x_i)\psi_i * W_u^s(u)$.*
(b) *$T_2 f = \sum_i c_i f(x_i)\ell_{x_i} W_u^s(u)$ (with $c_i = \int \psi_i$).*
(c) *$T_3 f = \sum_i (\int f(x)\psi_i(x)\,dx)\ell_{x_i} W_u^s(u)$.*

Remark 4.12. Note that we have avoided the use of integrability in the results above. This allows us to treat the cases $1 < s \leqslant 2$ which could not be treated in [13]. In particular the space A_p^p is not equal to $\mathrm{Co}_{FG} L^p$ for the representation π_2. It should be mentioned that Feichtinger and Gröchenig anticipated the possibility of a coorbit construction for these non-integrable representations. However, the machinery developed in [13] did not provide a suitable approach and also could not provide atomic decompositions for these spaces.

5. A wavelet characterization of Besov spaces on the forward light cone

The classical wavelet transform is related to the group $\mathbb{R}_+ \ltimes \mathbb{R}$ and the representation $\pi(a,b)f(x) = \frac{1}{\sqrt{a}} f(a^{-1}(x-b))$. In the present section we replace $\mathbb{R}_+$ with the group $\mathbb{R}_+\mathrm{SO}_0(n-1,1)$ acting transitively on the forward light cone in $\mathbb{R}^n$. This leads to the construction of wavelets and coorbits for the group $\mathbb{R}_+\mathrm{SO}_0(n-1,1) \ltimes \mathbb{R}^n$. We show that the constructed coorbits correspond to Besov spaces for the forward light cone introduced in [1]. The representations involved are integrable and thus the theory of Feichtinger and Gröchenig is sufficient. However we find the construction interesting enough to be included here.

5.1. Wavelets and coorbits on the forward light cone

We review the wavelet transform already studied in [3] and [10], and introduce coorbit spaces related to the forward light cone.

Let $B(x,y)$ be the bilinear form on $\mathbb{R}^n$ given by

$$B(x,y) = x_n y_n - x_{n-1} y_{n-1} - \cdots - x_1 y_1$$

and let $\mathrm{SO}_0(n-1,1)$ be the closed connected subgroup of $\mathrm{GL}(n,\mathbb{R})$ which leaves B invariant. The group $\mathrm{SO}_0(n-1,1)$ has the Iwasawa decomposition ANK

$$A = \left\{ a_t = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I_{n-2} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\},$$

$$N = \left\{ n_c = \begin{pmatrix} 1 - |c|^2/2 & -c^T & |c|^2/2 \\ c & I_{n-2} & -c \\ -|c|^2/2 & -c^T & 1 + |c|^2/2 \end{pmatrix} \;\middle|\; c \in \mathbb{R}^{n-2} \right\},$$

$$K = \left\{ k_\sigma = \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix} \;\middle|\; \sigma \in \mathrm{SO}(n-1) \right\},$$

where c^T means the transpose of c.

The forward light cone is the subset Λ of $\mathbb{R}^n$ satisfying

$$\Lambda = \left\{ (x_1, \ldots, x_n) \;\middle|\; B(x,x) > 0,\ x_n > 0 \right\}$$

with determinant given by

$$\mathrm{Det}(x) = \sqrt{B(x,x)}.$$

An element $\gamma a_t n_c k_\sigma \in \mathbb{R}_+\mathrm{SO}_0(n-1,1)$ acts from the left on $x \in \Lambda$ by matrix multiplication, i.e. $\gamma a_t n_c k_\sigma x$. This action is transitive on Λ and the measure $\mathrm{Det}(x)^{-n}\,dx$, where dx is the Lebesgue measure on $\mathbb{R}^n$, is $\mathbb{R}_+\mathrm{SO}_0(n-1,1)$-invariant. The left-regular representation of $\mathbb{R}_+\mathrm{SO}_0(n-1,1)$ on $L^2(\Lambda)$ is

$$\ell(\gamma a_t n_c k_\sigma) f(x) = f\big((\gamma a_t n_c k_\sigma)^{-1} x\big).$$

The subgroup K leaves the base point $e = (0, \ldots, 0, 1)^T$ invariant and therefore the group $H = \mathbb{R}_+ AN$ acts simply transitively on the forward light cone, i.e. every $x \in \Lambda$ can be written $x = \gamma a_t n_c e$. In particular if

$$\gamma a_t n_c e = \gamma \begin{pmatrix} \sinh t + e^t |c|^2/2 \\ -c \\ \cosh t + e^t |c|^2/2 \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

then γ, t and c are determined uniquely by

$$\gamma = \mathrm{Det}(x), \qquad c = -\gamma^{-1}(x_2, \ldots, x_{n-1})^T, \qquad t = -\ln\big(\gamma^{-1}(x_n - x_1)\big).$$

The left invariant measure on H is given by

$$\int_H f(h)\,dh = \int_H f(\gamma a_t n_c)\frac{d\gamma\,dc\,dt}{\gamma}$$

where dt, dc and $d\gamma$ are the Lebesgue measures on $\mathbb{R}$, $\mathbb{R}^{n-2}$ and $\mathbb{R}_+$ respectively. We can pass from an integral over the cone to an integral over the group by

$$\int_\Lambda f(x)\frac{dx}{\mathrm{Det}(x)^n} = \int_H f(\gamma a_t n_c e)\frac{d\gamma\,dc\,dt}{\gamma}.$$

An integral over the light cone with respect to Lebesgue measure can therefore be written as an integral over the group in the following way

$$\int_\Lambda f(x)\,dx = \int_H f(\gamma a_t n_c e)\gamma^{n-1}\,d\gamma\,dc\,dt.$$

The right Haar measure on H is given by

$$f \mapsto \int_{\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}^{n-2}} f(\gamma a_t n_c)e^{(n-2)t}\frac{d\gamma\,dt\,dc}{\gamma}.$$

The modular function on H is then $\Delta(\lambda a_t n_c) = e^{(n-2)t}$ satisfying

$$\int_H f(\gamma a_t n_c \lambda a_{t_1} n_{c_1})\frac{d\gamma\,dt\,dc}{\gamma} = \Delta(\lambda a_{t_1} n_{c_1})\int_H f(\gamma a_t n_c)\frac{d\gamma\,dt\,dc}{\gamma}$$

and

$$\int_H f\big((\gamma a_t n_c)^{-1}\big)\frac{d\gamma\,dt\,dc}{\gamma} = \int_H f(\gamma a_t n_c)\Delta(\gamma a_t n_c)^{-1}\frac{d\gamma\,dt\,dc}{\gamma}.$$

Introduce the Fourier transform related to the bilinear form B by letting

$$\tilde{f}(w) = \widetilde{\mathcal{F}}(f)(w) = \frac{1}{\sqrt{2\pi}^n}\int_{\mathbb{R}^n} f(x)e^{-iB(x,w)}\,dx$$

for $f \in L^1(\mathbb{R}^n)$. We know that $\widetilde{\mathcal{F}}$ extends to a unitary operator on $L^2(\mathbb{R}^n)$ and is a topological isomorphism from $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n)$. It acts on convolutions like the usual Fourier transform

$$\widetilde{f * g}(w) = \sqrt{2\pi}^n\tilde{f}(w)\tilde{g}(w).$$

Denote by $\mathcal{S}(\mathbb{R}^n)$ the space of rapidly decreasing smooth functions with topology induced by the semi-norms

$$\|f\|_{k,l} = \sup_{|\alpha|\leqslant k}\sup_{x\in\mathbb{R}^n}\big|D^\alpha f(x)\big|\big(1+|x|^2\big)^l.$$

Here α is a multi-index and $k, l \geqslant 0$ are integers. Since $\widetilde{\mathcal{F}} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a topological isomorphism, we can extend the Fourier transform to tempered distributions in the usual way. In this text we work with the conjugate dual $\mathcal{S}^*(\mathbb{R}^n)$ of $\mathcal{S}(\mathbb{R}^n)$ (in order for it to resemble an inner product) and thus we define the Fourier transform $\tilde{\phi}$ for $\phi \in \mathcal{S}^*(\mathbb{R}^n)$ by

$$\langle\tilde{\phi}, \tilde{f}\rangle = \langle\phi, f\rangle.$$

The group $G = H \ltimes \mathbb{R}^n$ has a natural representation on

$$L^2_\Lambda = \big\{f \in L^2\big(\mathbb{R}^n\big) \bigm| \mathrm{supp}(\tilde{f}) \subseteq \Lambda\big\}$$

given by

$$\pi(\gamma a_t n_c, b)f(x) = \frac{1}{\gamma^{n/2}}f\big((\gamma a_t n_c)^{-1}(x-b)\big).$$

This generalizes the quasi-regular representation of the group $\mathbb{R}_+ \ltimes \mathbb{R}$ from the classical wavelet transform. In the Fourier domain this representation becomes

$$\tilde{\pi}(\gamma a_t n_c, b)\tilde{f}(w) = \gamma^{n/2}\tilde{f}\big(\gamma(a_t n_c)^{-1}w\big)\big)e^{-iB(b,w)}$$

and we recognize that it arises from the left action of H on the cone Λ, and that $\widetilde{\mathcal{F}}$ is an intertwining operator. The group G has left invariant measure given by

$$\int_G f(g)\,dg = \int f(\gamma a_t n_c, b)\frac{d\gamma\,dc\,dt\,db}{\gamma^{n+1}}.$$

The following result has a generalization to symmetric cones (see for example [10] and [3]) and ensures that wavelets for this representation exist.

Theorem 5.1. *The representation (π, L^2_Λ) is square-integrable.*

We introduce the space $\mathcal{S}_\Lambda$ of rapidly decreasing functions whose Fourier transform is supported on the closure of the cone, i.e.

$$\mathcal{S}_\Lambda = \big\{ f \in \mathcal{S}\big(\mathbb{R}^n\big) \;\big|\; \operatorname{supp}(\tilde{f}) \subseteq \overline{\Lambda} \big\}.$$

This space will be equipped with the subspace topology it inherits from $\mathcal{S}(\mathbb{R}^n)$. The representation π can be restricted to $\mathcal{S}_\Lambda$ and we denote the resulting representation by $(\pi, \mathcal{S}_\Lambda)$ or simply π.

Lemma 5.2. *Let $u \in \mathcal{S}_\Lambda$ be compactly supported such that $0 \leqslant \tilde{u} \leqslant 1$ and also $1/2 < \tilde{u} \leqslant 1$ on a neighborhood U of e. Then u is cyclic in $(\pi, \mathcal{S}_\Lambda)$. Further let*

$$C_u = \int_\Lambda \big|\tilde{u}(w)\big|^2 \operatorname{Det}(w)^{2(1-n)}(w_n - w_1)^{n-2}\,dw.$$

Then the reproducing formula

$$W_u(v) * W_u(u) = C_u W_u(v)$$

holds for $v \in \mathcal{S}_\Lambda$.

Proof. The Fourier transform $\widetilde{\mathcal{F}}$ has the same properties as the usual Fourier transform. The calculations below are immediate adaptations of results found in for example [27, Chapters 6 and 7].

Let L be in the conjugate dual of $\mathcal{S}_\Lambda$ and assume that $\langle L, \pi(\gamma a_t n_c, b)u\rangle = 0$ for all $(\gamma a_t n_c, b) \in G$. Then the Fourier transform can be used to obtain

$$\big\langle \widetilde{L}, \tilde{\pi}(\gamma a_t n_c, b)\tilde{u}\big\rangle = 0.$$

Let $e_b(w) = e^{-iB(b,w)}$. The equation above can be rewritten as

$$0 = \big\langle \widetilde{L}, e_b\tilde{\pi}(\gamma a_t n_c, 0)\tilde{u}\big\rangle = \big\langle \big(\overline{\tilde{\pi}(\gamma a_t n_c, 0)\tilde{u}}\big)\widetilde{L}, e_b\big\rangle$$

which shows that the compactly supported functional $(\overline{\tilde{\pi}(\gamma a_t n_c, 0)\tilde{u}})\widetilde{L}$ is equal to 0 (see [27, Theorem 7.23]). This means that for all $v \in \mathcal{S}_\Lambda$ for which $\tilde{v}$ has compact support $C \subseteq \Lambda$ we have the equalities

$$\big\langle \widetilde{L}, \tilde{v}\tilde{\pi}(\gamma a_t n_c, 0)\tilde{u}\big\rangle = \big\langle \overline{\tilde{\pi}(\gamma a_t n_c, 0)\tilde{u}}\widetilde{L}, \tilde{v}\big\rangle = 0.$$

We will now show that $\langle v, L\rangle$ is also 0. Since C is compact we can cover C by a finite number of translates of U

$$C \subseteq \bigcup_{i=1}^m (\gamma a_t n_c)_i U.$$

Define the function

$$\Psi = \sum_{i=1}^m \pi\big((\gamma a_t n_c)_i, 0\big)\tilde{u}$$

which has support containing C (here we use that $\tilde{u}$ is bounded away from 0 on U). Then $\tilde{v}/\Psi$ is in C_c^∞ and we see that

$$\langle \widetilde{L}, \tilde{v}\rangle = \langle \Psi\widetilde{L}, v/\Psi\rangle = \sum_{i=1}^n \big\langle \overline{\tilde{\pi}\big((\gamma a_t n_c)_i, 0\big)\tilde{u}}\widetilde{L}, v/\Psi\big\rangle = 0.$$

Lastly, any function in $\mathcal{S}_\Lambda$ can be approximated by a function whose Fourier transform is compact, and therefore $L = 0$ in the conjugate dual of $\mathcal{S}_\Lambda$. $\square$

We will need the following lemma, which corresponds to Lemma 3.11 in [1].

Lemma 5.3. *If $f \in \mathcal{S}_\Lambda$ and k, l are non-negative integers then there is a constant C_k such that*

$$\bigl|\tilde{f}(w)\bigr| \leqslant C_k \|\tilde{f}\|_{k,l} \frac{\operatorname{Det}(w)^k}{(1+|w|^2)^l}.$$

We will further need an estimate of the wavelet coefficients of Schwartz functions. The estimate actually shows that the wavelet coefficients are integrable.

Lemma 5.4. *The mapping*

$$\mathcal{S}_\Lambda \ni v \mapsto \int_G \bigl|W_u(v)(\gamma a_t n_c, b)\bigr| \gamma^r \frac{d\gamma\, dt\, dc\, db}{\gamma^{n+1}} \in \mathbb{R}^+$$

is continuous for all $r \in \mathbb{R}$.

Proof. First note that the wavelet coefficients can be rewritten as

$$\begin{aligned} W_u(f)(\gamma a_t n_c, b) &= \bigl(f, \pi(\gamma a_t n_c, b)u\bigr) \\ &= \gamma^{n/2} \int_\Lambda \tilde{f}(w)\tilde{u}(\gamma n_{-c} a_{-t} w) e^{-iB(w,b)}\, dw \\ &= -b_i^{-2} \gamma^{n/2} \int_\Lambda \frac{\partial^2}{\partial w_i^2}\bigl[\tilde{f}(w)\tilde{u}(\gamma n_{-c} a_{-t} w)\bigr] e^{-iB(w,b)}\, dw \end{aligned}$$

where we have used integration by parts twice. Therefore, if $L = -\sum_{k=1}^n \frac{\partial^2}{\partial w_k^2}$ is the Laplacian we obtain

$$\bigl(1+|b|^2\bigr) W_u(f)(\gamma a_t n_c, b) = \gamma^{n/2} \int_\Lambda (1+L)\bigl[\tilde{f}(w)\tilde{u}(\gamma n_{-c} a_{-t} w)\bigr] e^{-iB(w,b)}\, dw.$$

If we repeat the argument we are able to obtain

$$\bigl|W_u(f)(\gamma a_t n_c, b)\bigr| \leqslant \bigl(1+|b|^2\bigr)^{-N} \gamma^{n/2} \int_\Lambda \bigl|(1+L)^N\bigl[\tilde{f}(w)\tilde{u}(\gamma n_{-c} a_{-t} w)\bigr]\bigr|\, dw$$

for any N, thus proving that the wavelet coefficients are indeed integrable in b. We have

$$(1+L)^N\bigl[\tilde{f}(w)\tilde{u}(\gamma n_{-c} a_{-t} w)\bigr] = \sum_{|\alpha+\beta| \leqslant 2N} p_\beta(\gamma n_{-c} a_{-t}) \partial^\alpha \tilde{f}(w) \partial^\beta \tilde{u}(\gamma n_{-c} a_{-t} w)$$

where α, β are multi-indices and $p_\beta(\gamma n_{-c} a_{-t})$ are polynomials in the entries of the matrix $\gamma n_{-c} a_{-t}$ (see the form of this matrix in (8)). We thus have to show the integrability in γ, t and v of expressions of the form

$$\bigl|p_\beta(\gamma n_{-c} a_{-t})\bigr| \gamma^{n/2} \int_\Lambda \bigl|\partial^\alpha \tilde{f}(w) \partial^\beta \tilde{u}(\gamma n_{-c} a_{-t} w)\bigr|\, dw.$$

A change of variable and use of the fact that $C = \operatorname{supp}(\tilde{u})$ is compact, reduces this to show that

$$\gamma^{-n/2} \bigl|p_\beta(\gamma n_{-c} a_{-t})\bigr| \bigl\|\partial^\beta \tilde{u}\bigr\|_\infty \int_C \bigl|\partial^\alpha \tilde{f}\bigl(\gamma^{-1} a_t n_c w\bigr)\bigr|\, dw$$

is integrable. By Lemma 5.3 we can estimate any such expression by

$$C_k \gamma^{-n/2} \bigl|p_\beta(\gamma n_{-c} a_{-t})\bigr| \bigl\|\partial^\beta \tilde{u}\bigr\|_\infty \bigl\|\partial^\alpha \tilde{f}\bigr\|_{k,l} \int_C \frac{\operatorname{Det}(\gamma^{-1} w)^k}{(1+|\gamma^{-1} a_t n_c w|^2)^l} \gamma^{n/2}\, dw \tag{6}$$

for arbitrary k, l. Since the set C is compact, w is bounded away from 0 and we see that

$$C_1\big(1+\big|\gamma^{-1}a_t n_c e\big|^2\big) \leqslant 1+\big|\gamma^{-1}a_t n_c w\big|^2 \leqslant C_2\big(1+\big|\gamma^{-1}a_t n_c e\big|^2\big),$$

and we can estimate (6) by

$$C_k\big\|\partial^\beta \tilde{u}\big\|_\infty\big\|\partial^\alpha \tilde{v}\big\|_{k,l}\frac{|p_\beta(\gamma n_{-c}a_{-t})|\gamma^{-k-n/2}}{(1+|\gamma^{-1}a_t n_c e|^2)^l}.$$

All that is left now is to show that

$$\int_H \frac{|p_\beta(\gamma n_{-c}a_{-t})|\gamma^{-k-n/2}}{(1+|\gamma^{-1}a_t n_c e|^2)^l}\gamma^r\frac{d\gamma\,dt\,dc}{\gamma^{n+1}} < \infty. \tag{7}$$

We split this integral into two cases.

Case 1. $0 < \gamma \leqslant 1$.

The expression (7) can be estimated by

$$\int_{\mathbb{R}^{n-2}}\int_{\mathbb{R}}\int_0^1 \frac{|p_\beta(n_{-c}a_{-t})|\gamma^{r-k-3n/2-1}}{|\gamma^{-1}a_t n_c e|^{2l}}\,d\gamma\,dt\,dc$$
$$= \int_{\mathbb{R}^{n-2}}\int_{\mathbb{R}}\int_0^1 \frac{|p_\beta(n_{-c}a_{-t})|\gamma^{2l+r-k-3n/2-1}}{|a_t n_c e|^{2l}}\,d\gamma\,dt\,dc.$$

The integral over γ is finite for $0 < \gamma \leqslant 1$ if l is chosen large enough (and $k = 0$). Now

$$n_{-c}a_{-t} = \begin{pmatrix} \cosh t - e^t|c|^2 & c^T & -\sinh t + e^t|c|^2 \\ -e^t c & I & e^t c \\ -\sinh t - e^t|c|^2 & c^T & \cosh t + e^t|c|^2 \end{pmatrix} \tag{8}$$

and

$$a_t n_c e = \begin{pmatrix} \sinh t + e^t|c|^2 \\ -c \\ \cosh t + e^t|c|^2 \end{pmatrix},$$

so we see that $|p_\beta(n_{-c}a_{-t})|$ will be dominated by $|a_t n_c e|^{2l} \geqslant 1$ if $l > \deg(p_\beta)$. Thus choosing l large enough, the integral in (7) will be finite.

Case 2. $\gamma \geqslant 1$.

In this case (7) can be estimated by

$$\int_{\mathbb{R}^{n-2}}\int_{\mathbb{R}} \frac{|p_\beta(n_{-c}a_{-t})|}{|a_t n_c e|^{2l}}\,dt\,dc \int_1^\infty \gamma^{r+2l+\deg(p_\beta)-k-3n/2-1}\,d\gamma.$$

The first integral is finite if l is large enough, and the second integral is finite when k is chosen large enough (depending on l).

To sum up we have obtained the following estimate

$$\int_G \big|W_u(v)(\gamma a_t n_c, b)\big|\gamma^r\frac{d\gamma\,dt\,dc\,db}{\gamma^{n+1}} \leqslant C\sum_{|\alpha+\beta|\leqslant 2N}\big\|\partial^\beta\tilde{u}\big\|_\infty\big\|\partial^\alpha\tilde{v}\big\|_{k,l},$$

which shows the continuous dependence on v. □

Denote by $L_s^{p,q}(G)$ the space of measurable functions f on the group for which

$$\|f\|_{L_s^{p,q}} = \left(\int_H \left(\int_{\mathbb{R}^n} \left|f(\gamma a_t n_c, b)\right|^p db\right)^{q/p} \gamma^s \frac{d\gamma\, dt\, dc}{\gamma^{n+1}}\right)^{1/q} < \infty$$

then the integrability of $W_u(v)$ shows that

Lemma 5.5. *For $u, v \in \mathcal{S}_\Lambda$ it holds that $L_s^{p,q} * W_u(v) \subseteq L_s^{p,q}$ and*

$$L_s^{p,q} \ni F \mapsto F * W_u(v) \in L_s^{p,q}$$

is continuous.

Further the integrability also shows that for $1/p + 1/p' = 1$ and $1/q + 1/q' = 1$ the wavelet coefficient $W_u(v)$ is in $L_{1/s}^{p',q'}$ and therefore

Lemma 5.6. *The mapping*

$$\mathcal{S}_\Lambda \ni v \mapsto \int_G \left|F(\gamma a_t n_c, b) W_u(v)(\gamma a_t n_c, b)\right| \frac{d\gamma\, dt\, dc}{\gamma^{n+1}} \in \mathbb{C}$$

is continuous for all $F \in L_s^{p,q}$.

This verifies the assumptions for construction of coorbit spaces for the spaces $L_s^{p,q}$ and therefore we can define

$$\mathrm{Co}_{\mathcal{S}_\Lambda}^u L_s^{p,q} = \left\{\Phi \in \mathcal{S}'_\Lambda \;\middle|\; W_u(\Phi) \in L_s^{p,q}\right\}.$$

Furthermore, Lemma 5.5 shows that this space is independent on the wavelet u.

Remark 5.7 *(Discretization).* The representation used for this construction is integrable (as we have shown) and therefore the discretization procedure by Feichtinger and Gröchenig can be used directly. For the quasi-regular representation in question the Fréchet space $\mathcal{S}_\Lambda$ is contained in the set of better vectors $\mathcal{B}_w$ defined in (4) for weights of the type

$$w(\gamma a_t n_c, b) = \gamma^s.$$

This can be shown by direct calculations. Another approach builds on smoothness of the representation coefficients, and this will be explored in [5].

5.2. Besov spaces as coorbits

In this section we introduce Littlewood–Paley decompositions and a family of Besov spaces related to these decompositions. The construction has been carried out for all symmetric cones in [1] and we refer to this article for proofs. The last result of this paper is a wavelet description of the Besov spaces. In particular we show that the Besov spaces are the coorbit spaces defined in the previous section.

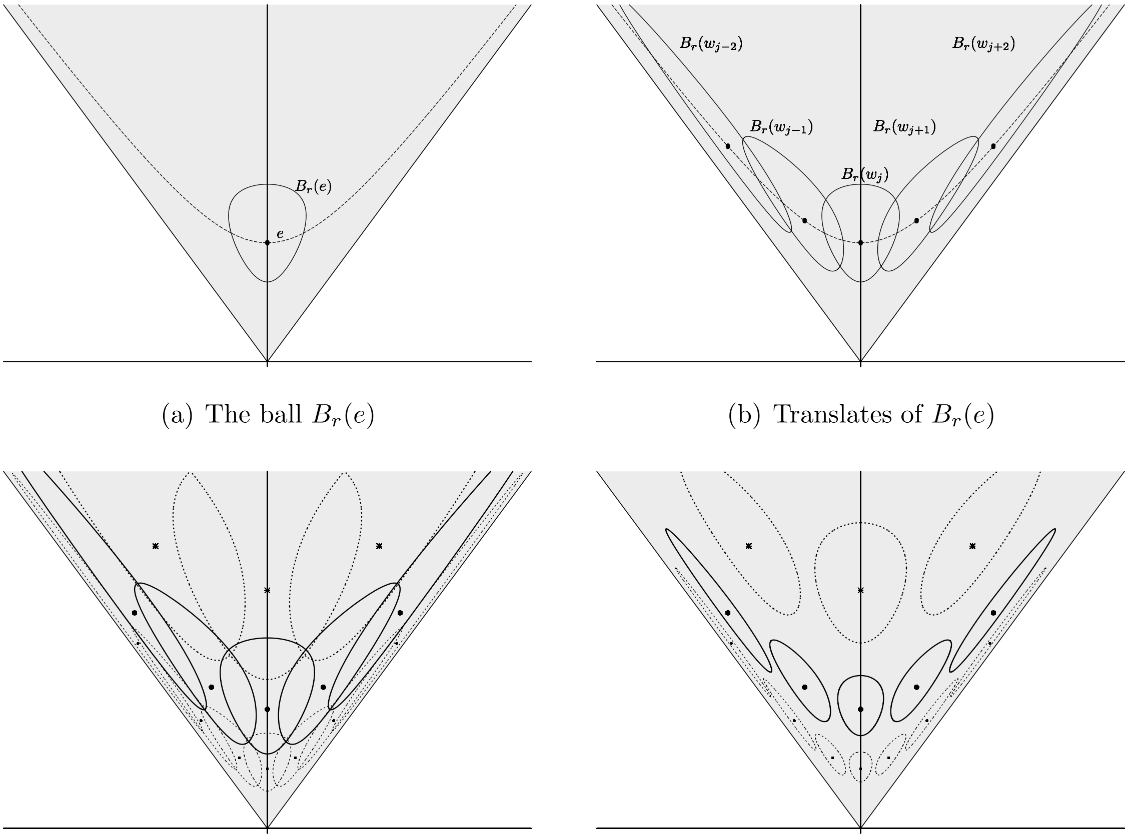
The group $\mathbb{R}_+ A$ is an abelian group with exponential function $\exp : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+ A$ given by $\exp(t, s) = e^t a_s$ (here e^t denotes the usual exponential function on $\mathbb{R}$). Let $V_r = \{(s,t) \in \mathbb{R} \times \mathbb{R} | s^2 + t^2 < r\}$ and define the K-invariant ball $B_r(e) = K \exp(V_r) e \subseteq \Lambda$. For $w = he \in \Lambda$ with $h \in H$ we define the ball of radius r centered at w to be

$$B_r(w) = h B_r(e).$$

The following covering lemma for the cone can be extracted from Lemma 2.6 in [1] and is illustrated in Fig. 1.

Lemma 5.8 *(Whitney cover with lattice points w_j). Given $\delta > 0$, there exists a sequence $\{w_j\} \subseteq \Lambda$ such that $B_{\delta/2}(w_j)$ are disjoint and $B_\delta(w_j)$ cover Λ with the property that there is an N such that any $w \in \Lambda$ belongs to at most N of the balls $B_\delta(w_j)$ (finite intersection property).*

We now construct a smooth partition of unity subordinate to a cover from Lemma 5.8. Let $0 \leqslant \varphi \leqslant 1$ be a smooth function with support in $B_{2\delta}(e)$ such that $\varphi = 1$ on $B_\delta(e)$. Each of the points $w_j \in \Lambda$ can be written $w_j = \gamma_j a_{t_j} n_{c_j} e$ for $g_j = \gamma_j a_{t_j} n_{c_j} \in \mathbb{R}_+ AN$ and now we define $\varphi_j(w) = \varphi(g_j^{-1} w)$. Then the function $\Phi = \sum_j \varphi_j$ is smooth and bounded from above and below (by the finite intersection property), and we can finally define the function ψ_j by letting $\tilde{\psi}_j = \varphi_j / \Phi$. We



(a) The ball $B_r(e)$ (b) Translates of $B_r(e)$

(c) Cover with translates of $B_r(e)$ (d) Translates of $B_{r/2}(e)$ have no overlap

Fig. 1. Covering of the cone.

then see that $\tilde{\psi}_j$ is smooth and with compact support in $B_{2\delta}(w_j)$, $\tilde{\psi}_j = 1$ on $B_{\delta/2}(w_j)$ and $\sum_j \tilde{\psi}_j(w) = 1$ for all $w \in \Lambda$. Such a partition of unity is called a Littlewood–Paley decomposition of the cone subordinate to a Whitney cover.

We note that the convolutions encountered in this section are distributional convolutions in $\mathbb{R}^n$. We are now ready to define the Besov spaces on the light cone as in [1]:

Definition 5.9. Let ψ_j be a Littlewood–Paley decomposition of the cone subordinate to a Whitney cover with lattice points w_j. For $1 \leqslant p, q < \infty$ define the norm

$$\|f\|_{B_s^{p,q}} = \left(\sum_j \mathrm{Det}^{-s}(w_j) \|f * \psi_j\|_p^q \right)^{1/q}$$

then the space $B_s^{p,q}$ consist of the $f \in \mathcal{S}'_\Lambda$ for which $\|f\|_{B_s^{p,q}} < \infty$.

In [1, Lemma 3.8] it is further proven, that $B_s^{p,q}$ does not depend (up to norm equivalence) on the functions ψ_j nor on the Whitney decomposition. We will use this in the sequel.

Theorem 5.10. *The Besov space $B_{n-s-nq/2}^{p,q}$ corresponds to the coorbit $\mathrm{Co}_{\mathcal{S}_\Lambda}^u L_s^{p,q}(G)$ with equivalent norm.*

Proof. First show that $B_{s+nq/2-n}^{p,q} \subseteq \mathrm{Co}_{\mathcal{S}_\Lambda}^u L_s^{p,q}(G)$. Assume that $f \in B_{s+nq/2-n}^{p,q}$ and that $\tilde{\phi}_i$ is a Littlewood–Paley decomposition of the cone with lattice points $w_i = g_i e = \gamma_i a_{t_i} n_{c_i} e$. Further assume that the sets $g_i V$ cover the cone for an open set V with compact closure. Denote by U the subset of H given by $U = \{g \in H | ge \in V\}$ Let $u \in \mathcal{S}_\Lambda$ be a non-zero wavelet for which $\tilde{u}$ has compact support containing the identity. By $u_{\gamma a_t n_c}$ denote the function

$$u_{\gamma a_t n_c}(x) = \gamma^{-n} \overline{u\big((\gamma a_t n_c)^{-1} x\big)},$$

then

$$W_u(f)(\gamma a_t n_c, b) = \gamma^{-n/2} \int f(x)\overline{u\big((\gamma a_t n_c)^{-1}(x-b)\big)}\,dx = \gamma^{n/2} f * u_{\gamma a_t n_c}(b).$$

Let the disjoint sets $V_i \subseteq \Lambda$ cover Λ and satisfy $V_i \subseteq g_i V$. Now choose the subsets U_i to be $U_i = \{g \in H \mid ge \in V_i\}$. We can then write the $L_s^{p,q}$ norm of the wavelet coefficient as

$$\begin{aligned}
\big\|W_u(f)\big\|_{L_s^{p,q}} &= \bigg(\int_H \gamma^{s+nq/2-n} \|f * u_{\gamma a_t n_c}\|_p^q \frac{d\gamma\,dt\,dc}{\gamma^{n+1}}\bigg)^{1/q} \\
&= \bigg(\int_H \gamma^{n-s-nq/2} \|f * u_{\gamma^{-1} a_t n_c}\|_p^q \frac{d\gamma\,dt\,dc}{\gamma^1}\bigg)^{1/q} \\
&\leqslant \bigg(\sum_i \int_{U_i} \gamma^{n-s-nq/2} \|f * u_{\gamma^{-1} a_t n_c}\|_p^q \frac{d\gamma\,dt\,dc}{\gamma}\bigg)^{1/q} \\
&\leqslant C\bigg(\sum_i \gamma_i^{n-s-nq/2} \int_{U_i} \|f * u_{\gamma^{-1} a_t n_c}\|_p^q \frac{d\gamma\,dt\,dc}{\gamma}\bigg)^{1/q},
\end{aligned}$$

where we have used that γ is comparable to $\gamma_i = \mathrm{Det}(w_i)$ when $\gamma a_t n_c \in U_i$. For any j define $\tilde{\phi}_{i,j} = \ell_{g_j}\tilde{\phi}_i$. Since $\{\tilde{\phi}_i\}_i$ is a Littlewood–Paley decomposition of the cone the systems $\{\tilde{\phi}_{i,j}\}_j$ (with index j) and $\{\tilde{\phi}_{i,j}\}_i$ (with index i) are also Littlewood–Paley decompositions of the cone. For fixed i we thus can write $\|f * u_{\gamma^{-1} a_t n_c}\|_p$ as

$$\|f * u_{\gamma^{-1} a_t n_c}\|_p = \bigg\|\sum_{j\in J} f * u_{\gamma^{-1} a_t n_c} * \phi_{i,j}\bigg\|_p \leqslant \sum_{j\in J} \|f * u_{\gamma^{-1} a_t n_c} * \phi_{i,j}\|_p.$$

The index set J in this sum is finite, since both $\tilde{u}$ and $\tilde{\phi}$ are compactly supported and w_i are well-spread. Further the index set J can be chosen large enough that it neither depends on i nor on $\gamma a_t n_c \in U_i$. The $L^1(\mathbb{R}^n)$-norm of $u_{\gamma^{-1} a_t n_c}$ is uniformly bounded from above, in fact $\|u_{\gamma^{-1} a_t n_c}\|_{L^1(\mathbb{R}^n)} = \|u\|_{L^1(\mathbb{R}^n)}$, so we obtain that

$$\|f * u_{\gamma^{-1} a_t n_c}\|_p \leqslant \sum_{j\in J} \|f * \phi_{i,j}\|_p.$$

Therefore

$$\begin{aligned}
\big\|W_u(f)\big\|_{L_s^{p,q}} &\leqslant C\bigg(\sum_i \gamma_i^{n-s-nq/2} \int_{U_i} \bigg(\sum_{j\in J} \|f * \phi_{i,j}\|_p\bigg)^q \frac{d\gamma\,dt\,dc}{\gamma}\bigg)^{1/q} \\
&\leqslant C\bigg(\sum_i \gamma_i^{n-s-nq/2} \bigg(\sum_{j\in J} \|f * \phi_{i,j}\|_p\bigg)^q\bigg)^{1/q},
\end{aligned}$$

where we have used that the $U_i \subseteq g_i U$ have uniformly bounded measure. The triangle inequality for the ℓ^q-norm then yields

$$\big\|W_u(f)\big\|_{L_s^{p,q}} \leqslant C \sum_{j\in J} \bigg(\sum_i \gamma_i^{n-s-nq/2} \|f * \phi_{i,j}\|_p^q\bigg)^{1/q}.$$

Now set $\gamma_{i,j} = \mathrm{Det}(g_j w_i) = \gamma_i \gamma_j$, then, since the sum over J is finite, each γ_i is comparable to $\gamma_{i,j}$ for all j, and finally we obtain

$$\begin{aligned}
\|f\|_{L_s^{p,q}} &\leqslant C \sum_{j\in J} \bigg(\sum_i \gamma_i^{n-s-nq/2} \|f * \phi_{i,j}\|_p^q\bigg)^{1/q} \\
&\leqslant C \sum_{j\in J} \bigg(\sum_i \gamma_{i,j}^{n-s-nq/2} \|f * \phi_{i,j}\|_p^q\bigg)^{1/q} \\
&= C \sum_{j\in J} \bigg(\sum_i \mathrm{Det}(g_i w_i)^{-(s+nq/2-n)} \|f * \phi_{i,j}\|_p^q\bigg)^{1/q}.
\end{aligned}$$

Each of the $\{\phi_{i,j}\}_i$ form a Littlewood–Paley decomposition of the cone, so the terms

$$\left(\sum_i \operatorname{Det}(g_j w_i)^{-(s+nq/2-n)} \|f * \phi_{i,j}\|_p^q\right)^{1/q}$$

are Besov space norms. Each norm is comparable to $\|f\|_{B^{p,q}_{s+nq/2-n}}$ by [1, Lemma 3.8 and expression (3.20)]. This shows, that there is a $C > 0$ such that

$$\left\|W_u(f)\right\|_{L^{p,q}_s} \leqslant C\|f\|_{B^{p,q}_{s+nq/2-n}}.$$

It remains to show that $\operatorname{Co}^u_{\mathcal{S}_\Lambda} L^{p,q}_s(G) \subseteq B^{p,q}_{s+nq/2-n}$. Let $\tilde{\phi}$ be the smooth function with support in $B_{2\delta}(e)$ used to generate a Littlewood–Paley decomposition. The coorbit spaces are independent of the wavelet u, so we choose u and a compact neighborhood $U \subseteq H$ such that $U \operatorname{supp}(\tilde{\phi})$ is contained in $\tilde{u}^{-1}(\{1\})$. It holds that the $g_i U$'s have finite overlap (the g_i's come from the lattice points $w_i = g_i e$ which are well-spread). This means that $\operatorname{supp}(\tilde{\phi}_i)$ is contained in $(\tilde{u}_{\gamma^{-1}a_t n_c})^{-1}(\{1\})$ for $\gamma a_t n_c \in g_i U$. Therefore $\tilde{\phi}_i \tilde{u}_{\gamma^{-1}a_t n_c} = \tilde{\phi}_i$ for all $\gamma a_t n_c \in g_i U$. We exploit this to see that

$$\begin{aligned}
\|f * \phi_i\|_p^q &= \frac{1}{|U|}\int_{g_i U} \|f * \phi_i\|_p^q \frac{d\gamma\, dt\, dc}{\gamma} \\
&= \frac{1}{|U|}\int_{g_i U} \|f * \phi_i * u_{\gamma^{-1}a_t n_c}\|_p^q \frac{d\gamma\, dt\, dc}{\gamma} \\
&\leqslant C \int_{g_i U} \|f * u_{\gamma^{-1}a_t n_c}\|_p^q \frac{d\gamma\, dt\, dc}{\gamma},
\end{aligned}$$

where $\frac{d\gamma\, dt\, dc}{\gamma}$ is the invariant measure on the group H. In the last step we used that $\|\phi_i\|_{L^1(\mathbb{R}^n)}$ is uniformly bounded (see [1, Proposition 3.2(3)]). For $\gamma a_t n_c \in g_i U$ we see that γ is comparable to γ_i. Further the sets $g_i U$ overlap a finite amount of times, so we obtain the estimate

$$\begin{aligned}
\sum_i \gamma_i^{-(s+nq/2-n)} \|f * \phi_i\|_p^q &\leqslant C \int_H \gamma^{-(s+nq/2-n)} \|f * u_{\gamma^{-1}a_t n_c}\|_p^q \frac{d\gamma\, dt\, dc}{\gamma} \\
&= C \int_H \gamma^{s+nq/2-n} \|f * u_{\gamma a_t n_c}\|_p^q \frac{d\gamma\, dt\, dc}{\gamma}.
\end{aligned}$$

We use this to find the estimate of the Besov space norm

$$\begin{aligned}
\|f\|_{B^{p,q}_{s+nq/2-n}} &= \left(\sum_i \operatorname{Det}(w_i)^{-(s+nq/2-n)} \|f * \phi_i\|_p^q\right)^{1/q} \\
&= \left(\sum_i \gamma_i^{-(s+nq/2-n)} \|f * \phi_i\|_p^q\right)^{1/q} \\
&\leqslant C\left(\int_H \gamma^{s+nq/2} \|f * u_{\gamma a_t n_c}\|_p^q \frac{d\gamma\, dt\, dc}{\gamma^{n+1}}\right) \\
&= C\left\|W_u(f)\right\|_{L^{p,q}_s}.
\end{aligned}$$

This proves the equivalence of the norms of the two spaces. □

Remark 5.11. The wavelet characterization of Besov spaces on forward light cones seems to generalize to all symmetric cones. We will deal with this in future work.

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