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Fréchet approach in second-order optimization

Dušan Bednařík [a], Karel Pastor [b,*]

[a] Department of Mathematics, University of Hradec Králové, Rokitanského 62, 500 03 Hradec Králové, Czech Republic

[b] Department of Mathematical Analysis and Applications of Mathematics, Faculty of Science, Palacký University, Tř. Svobody 26, 771 46 Olomouc, Czech Republic

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ABSTRACT

We state a certain second-order sufficient optimality condition for functions defined in infinite-dimensional spaces by means of generalized Fréchet's approach to second-order differentiability. Moreover, we show that this condition generalizes a certain second-order condition obtained in finite-dimensional spaces.

1. Introduction and preliminary facts

Second-order sufficient optimality conditions play a very important role in optimization theory, see e.g. [1–29] and the references therein.

Various generalized second-order sufficient optimality conditions have been introduced since 80s of the last century. In our previous articles [14,15], we established the generalized second-order sufficient optimality condition by means of a certain second-order derivative of the Peano type for the so-called ℓ-stable functions (see Theorem 1).

Let us give a short survey of notions concerning ℓ-stability. Unless stated otherwise, we assume that X is a normed linear space, S_X is the unit sphere of X and for $x \in X$ and $\delta > 0$, $B(x, \delta)$ denotes the set $\{y \in X;\ \|y - x\| < \delta\}$.

For a function $f : X \to \mathbb{R}$, $x, h \in X$, we denote

$$f^{\ell}(x; h) = \liminf_{t \downarrow 0} \frac{f(x + th) - f(x)}{t}.$$

Lemma 1 ([14, Lemma 4]). *Let $f : X \to \mathbb{R}$ be a continuous function, and let $a, b \in X$. Then there exist $\xi_1, \xi_2 \in (a, b)$ such that*

$$f^{\ell}(\xi_1; b - a) \le f(b) - f(a) \le f^{\ell}(\xi_2; b - a).$$

We say that $f : X \to \mathbb{R}$ is *ℓ-stable at $x \in X$* if there exist a neighbourhood U of x and $K > 0$ such that

$$|f^{\ell}(y; h) - f^{\ell}(x; h)| \le K\|y - x\|, \quad \forall y \in U, \forall h \in S_X.$$

* Corresponding author.
E-mail addresses: dbednarik@seznam.cz (D. Bednařík), pastor@inf.upol.cz (K. Pastor).

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We notice that the property to be an ℓ-stable function at some point requires weaker assumptions than the property to be a $C^{1,1}$ function near this point. Recall that $f : X \to \mathbb{R}$ is a $C^{1,1}$ function near $x \in X$ if it is differentiable on some neighbourhood of x and its derivative $f'(\cdot)$ is Lipschitz here. $C^{1,1}$ functions appear in, e.g., the augmented Lagrange method, the penalty function method and the proximal point method. In [14, Example 6] an example of the function $f : \mathbb{R} \to \mathbb{R}$ which is ℓ-stable at 0 but which is not differentiable on any neighbourhood of 0 was given.

As was shown in [14,15], ℓ-stability at x implies for a continuous function defined on a finite-dimensional space the Lipschitzness on a neighbourhood of x and also the strict differentiability at x. The differentiability properties of the functions which are ℓ-stable at some point and which are defined on an arbitrary normed space were studied in [16].

Theorem 1 *([14,15]). Let a function $f : \mathbb{R}^N \to \mathbb{R}$ be ℓ-stable at $x \in \mathbb{R}^N$. If $f^\ell(x; h) = 0$ for every $h \in S_{\mathbb{R}^N}$, and*

$$\underline{f}_P'^{\ell}(x; h) := \liminf_{t \downarrow 0} \frac{f(x+th) - f(x) - tf^\ell(x; h)}{t^2/2} > 0, \quad \forall h \in S_{\mathbb{R}^N},$$

then x is an isolated minimizer of order 2 for f.

Recall that $x \in X$ is an *isolated minimizer of order k* ($k \in \mathbb{N}$) for a function $f : X \to \mathbb{R}$ if there are neighbourhood U of x and $A > 0$ satisfying $f(y) \geq f(x) + A\|y - x\|^k$ for every $y \in U$. It is easy to verify that each isolated minimizer is a strict local minimizer.

By means of Theorem 1 we generalized the previous results given in [17, Theorem 3.2], [2, Proposition 5.2], [18, Theorem 3.2], and [19, Theorem 2] as was shown in [14]. Theorem 1 also generalizes the unconstrained and scalar case of results presented in [24, Theorem 5] and [25, Theorem 4.2] as it follows from the remarks given in [25].

In the paper [14] we also compared Theorem 1 with the one presented in [20, Theorem 2.9] by L.R. Hung and K.F. Ng. We recall that it was shown in [20, page 388] that Theorem 1 of Chaney in [21] is a weak form of the result of L.R. Hung and K.F. Ng mentioned before.

We would like to recall that Ginchev in [26] (see also [19]) stated the following sufficient and necessary optimality condition for an isolated minimizer of second order using the derivatives of Hadamard type, i.e.

$$f'_-(x; h) = \liminf_{u \to h, t \downarrow 0} \frac{f(x+tu) - f(x)}{t},$$

and

$$f''_-(x; h) = \liminf_{u \to h, t \downarrow 0} \frac{f(x+tu) - f(x) - tf'_-(x; h)}{t^2/2}.$$

Theorem 2. *Let $f : \mathbb{R}^N \to \mathbb{R}$ be an arbitrary function, and let $x \in \mathbb{R}^N$. If for each $h \in S_{\mathbb{R}^N}$ one of the following two conditions hold:*

(i) $f'_-(x; h) > 0$
(ii) $f'_-(x; h) = 0$ *and* $f''_-(x; h) > 0$,

then x is an isolated minimizer of order 2 for f. Conversely, each isolated minimizer of order 2 satisfies these sufficient conditions.

The derivative $f''_-(x; h)$ does not coincide with the classical ones even in the case of C^2 functions in general in contrast to $\underline{f}_P'^{\ell}(x; h)$. For more details about this "complementary principle" in nonsmooth analysis, see [19]. We also recall that in [19] the problem for what class of functions $f : \mathbb{R}^N \to \mathbb{R}$ we can replace the Hadamard derivatives by the respective Dini derivatives in Theorem 2 was presented.

Theorem 1 presented recently answered this question for ℓ-stable at some point functions. Nevertheless, our first goal of this paper is to show that Theorem 1 is a special case of Theorem 2.

Lemma 2. *Let a function $f : \mathbb{R}^N \to \mathbb{R}$ be ℓ-stable at $x \in \mathbb{R}^N$ and $h \in S_{\mathbb{R}^N}$. Then*

$$f^\ell(x; h) = f'_-(x; h).$$

Proof. Since the inequality $f^\ell(x; h) \geq f'_-(x; h)$ is evident, it suffices to show that

$$f^\ell(x; h) \leq f'_-(x; h).$$

Let $\{t_n\}_{n=1}^{+\infty}$, $\{h_n\}_{n=1}^{+\infty}$ be such sequences that

$$\lim_{n \to +\infty} h_n = h, \qquad \lim_{n \to +\infty} t_n = 0,$$

$t_n > 0$ for every $n \in \mathbb{N}$, and

$$\lim_{n \to +\infty} \frac{f(x+t_n h_n) - f(x)}{t_n} = f'_-(x; h).$$

By [14, Proposition 1] f is Lipschitz near x and thus there exists $K > 0$ satisfying

$$\left|\frac{f(x+t_nh)-f(x)}{t_n} - \frac{f(x+t_nh_n)-f(x)}{t_n}\right| = \left|\frac{f(x+t_nh)-f(x+t_nh_n)}{t_n}\right| \le K\|h-h_n\|,$$

for every sufficiently large $n \in \mathbb{N}$. Then

$$\frac{f(x+t_nh)-f(x)}{t_n} \le \frac{f(x+t_nh_n)-f(x)}{t_n} + K\|h-h_n\|,$$

for every sufficiently large $n \in \mathbb{N}$. Hence

$$f^\ell(x;h) \le \lim_{n\to+\infty}\left(\frac{f(x+t_nh_n)-f(x)}{t_n} + K\|h-h_n\|\right) = f'_-(x;h). \quad \square$$

Lemma 3. *Let a function $f : \mathbb{R}^N \to \mathbb{R}$ be ℓ-stable at $x \in \mathbb{R}^N$ and let $f^\ell(x;h) = 0$ for every $h \in S_{\mathbb{R}^N}$. Then*

$$f'^{\ell}_{-P}(x;h) = f''_-(x;h),$$

for every $h \in S_{\mathbb{R}^N}$.

Proof. We take an arbitrary $h \in S_{\mathbb{R}^N}$ and notice that $f'_-(x;h) = 0$ by Lemma 2. Since the inequality

$$f'^{\ell}_{-P}(x;h) \ge f''_-(x;h)$$

is evident, it suffices to show that $f'^{\ell}_{-P}(x;h) \le f''_-(x;h)$.

Let $\{t_n\}_{n=1}^{+\infty}$, $\{h_n\}_{n=1}^{+\infty}$ be such sequences that

$$\lim_{n\to+\infty} h_n = h, \qquad \lim_{n\to+\infty} t_n = 0,$$

$t_n > 0$ for every $n \in \mathbb{N}$, and

$$\lim_{n\to+\infty} \frac{2}{t_n^2}(f(x+t_nh_n)-f(x)) = f''_-(x;h).$$

Due to Lemma 1 and the property of ℓ-stability, for every sufficiently large $n \in \mathbb{N}$ we can find $\xi_n \in (x+t_nh, x+t_nh_n)$ such that

$$\begin{aligned}\left|\frac{2}{t_n^2}(f(x+t_nh)-f(x)) - \frac{2}{t_n^2}(f(x+t_nh_n)-f(x))\right| &= \left|\frac{2}{t_n^2}(f(x+t_nh)-f(x+t_nh_n))\right| \\ &\le \frac{2}{t_n^2} f^\ell(\xi_n; t_n(h-h_n)) \\ &= \frac{2}{t_n}(f^\ell(\xi_n; h-h_n) - f^\ell(x; h-h_n)) \\ &\le \frac{2}{t_n}K\|\xi_n - x\|\|h-h_n\|. \end{aligned} \quad (1)$$

Since $\xi_n \in (x+t_nh, x+t_nh_n)$, for every $n \in \mathbb{N}$ there exists $\alpha_n \in (0,1)$ satisfying

$$\begin{aligned}\frac{2}{t_n}K\,\|\xi_n - x\|\|h-h_n\| &= \frac{2}{t_n}Kt_n\|\alpha_nh + (1-\alpha_n)h_n\|\|h-h_n\| \\ &= 2K\|\alpha_nh + (1-\alpha_n)h_n\|\|h-h_n\|.\end{aligned} \quad (2)$$

Using formulas (1) and (2), we have that for every sufficiently large $n \in \mathbb{N}$ it holds

$$\frac{2}{t_n^2}(f(x+t_nh)-f(x)) \le \frac{2}{t_n^2}(f(x+t_nh_n)-f(x)) + 2K\|\alpha_nh + (1-\alpha_n)h_n\|\|h-h_n\|.$$

Hence

$$\begin{aligned} f'^{\ell}_{-P}(x;h) &\le \liminf_{n\to+\infty} \frac{2}{t_n^2}(f(x+t_nh)-f(x)) \\ &\le \lim_{n\to+\infty}\left(\frac{2}{t_n^2}(f(x+t_nh_n)-f(x)) + 2K\|\alpha_nh + (1-\alpha_n)h_n\|\|h-h_n\|\right) \\ &= f''_-(x;h), \end{aligned}$$

because $\lim_{n\to+\infty}\|h-h_n\| = 0$. $\square$

Now, using Lemmas 2 and 3, Theorem 2 implies Theorem 1.

All the previously mentioned results were stated for the functions which are defined on finite-dimensional spaces. In fact, there are not many unconstrained second-order sufficient optimality conditions established in terms of generalized second-order derivatives for infinite dimension. So, maybe now it is time to generalize Theorem 1 with respect to infinite dimension. In this way we will show that $\underline{f}_P'^{\ell}(x; h)$ has a certain Fréchet property which is the main aim of our paper.

2. Fréchet approach

We will start this section with a definition concerning the Fréchet differentiability and we establish second-order sufficient optimality condition in infinite dimension.

Definition 1. Let $f : X \to \mathbb{R}$ be a function, and let $x \in X$. We say that $\underline{f}_P'^{\ell}(x; \cdot)$ is Fréchet if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $h \in S_X$ and for every $0 < t < \delta$ there holds

1. $\underline{f}_P'^{\ell}(x; h) - \frac{2}{t^2}(f(x+th) - f(x) - tf^{\ell}(x; h)) < \varepsilon$, if $\underline{f}_P'^{\ell}(x; h) < +\infty$.
2. $\frac{2}{t^2}(f(x+th) - f(x) - tf^{\ell}(x; h)) > \frac{1}{\varepsilon}$, if $\underline{f}_P'^{\ell}(x; h) = +\infty$.

It was shown e.g. in [22, page 484] that if f is C^2 with (classical) second-order Fréchet derivative $f''(x)$, then

$$\underline{f}_P'^{\ell}(x; h) = f''(x; h, h), \quad \forall h \in X.$$

Therefore $\underline{f}_P'^{\ell}(x; h) < +\infty$ for every $h \in S_X$ and condition 1 from Definition 1 is satisfied in this case.

Theorem 3. *Let $f : X \to \mathbb{R}$ be a function, $x \in X$, and let $c > 0$. If $f^{\ell}(x; h) = 0$ for every $h \in S_X$, $\underline{f}_P'^{\ell}(x; \cdot)$ is Fréchet, and*

$$\underline{f}_P'^{\ell}(x; h) \geq c, \quad \forall h \in S_X,$$

then x is an isolated minimum of second order for f.

Proof. We put $\varepsilon = \min\{\frac{c}{2}, 1\}$. There exists $\delta > 0$ such that for every $h \in S_X$ and for every $0 < t < \delta$, we have

$$\frac{2}{t^2}(f(x+th) - f(x) - tf^{\ell}(x; h)) > \varepsilon.$$

Since $f^{\ell}(x; h) = 0$ for every $h \in S_X$, the previous inequality implies

$$f(x+th) - f(x) > \frac{\varepsilon}{2}t^2, \quad \forall h \in S_X, \forall 0 < t < \delta.$$

Thus, x is an isolated minimizer of second order for f. □

A series of the following lemmas yields to the properties of $\underline{f}_P'^{\ell}(x; \cdot)$ for an ℓ-stable function which are presented in Proposition 1.

Lemma 4 *([23, Lemma 2.1]). Let $f : X \to \mathbb{R}$ be Lipschitz near $x \in X$. Then the function $h \mapsto f^{\ell}(x; h)$ is continuous on X.*

Lemma 5 *([23, Lemma 2.2]). Let $f : \mathbb{R}^N \to \mathbb{R}$ be Lipschitz near $x \in \mathbb{R}^N$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $h \in S_{\mathbb{R}^N}$*

$$0 < t < \delta \Longrightarrow f^{\ell}(x; h) - \varepsilon < \frac{f(x+th) - f(x)}{t}.$$

Lemma 6. *Let $f : X \to \mathbb{R}$ be a function which is continuous near $x \in X$ and ℓ-stable at x. If $f^{\ell}(x; h) = 0$ for every $h \in S_X$, then the function $g : X \to \mathbb{R}$ defined by*

$$g(y) = \begin{cases} \dfrac{f(y) - f(x)}{\|y - x\|}, & \text{if } y \neq x, \\ 0, & \text{if } y = x, \end{cases}$$

is Lipschitz near x.

Proof. Step 1. We can suppose without any loss of generality that $x = 0$ and $f(0) = 0$. We have to prove that the following function $g : X \to \mathbb{R}$,

$$g(y) = \begin{cases} \dfrac{f(y)}{\|y\|}, & \text{if } y \neq 0, \\ 0, & \text{if } y = 0, \end{cases}$$

is Lipschitz near 0.

If f is ℓ-stable at 0, there are $K > 0$ and $\delta > 0$ such that

$$|f^\ell(y; h) - f^\ell(0; h)| \le K\|y\|, \quad \forall y \in B(0, \delta), \forall h \in S_X.$$

Now, let us choose arbitrary $y, z \in B(0, \delta)$ such that $y \ne z$.

Step 2. If $\|y\| = \|z\|$, due to Lemma 1 there is $\xi \in (z, y)$ such that

$$\begin{aligned}\|g(z) - g(y)\| &= \frac{|f(z) - f(y)|}{\|z\|} \le \frac{|f^\ell(\xi; z - y)|}{\|z\|} \\ &= \frac{|f^\ell(\xi; z - y) - f^\ell(0; z - y)|}{\|z\|} \le \frac{K\|\xi\|\|z - y\|}{\|z\|}.\end{aligned}$$

Since $\|\xi\| < \|z\|$, the previous calculation implies

$$\|g(z) - g(y)\| \le K\|z - y\|.$$

Step 3. In the next special case — $y = 0, z \ne 0$ — due to Lemma 1 we can find $\xi \in (0, z)$ such that

$$\begin{aligned}\|g(z) - g(y)\| &= \frac{|f(z) - f(y)|}{\|z\|} \le \frac{|f^\ell(\xi; z) - f^\ell(0; z)|}{\|z\|} \\ &\le \frac{K\|\xi\|\|z\|}{\|z\|} = K\|\xi\| \le K\|z - y\|.\end{aligned}$$

Step 4. Now, we suppose that there exists $t > 0$ with the property $z = y + ty$. Using Lemma 1, we can find $\xi_1 \in (y, y + ty)$ and $\xi_2 \in (0, y)$ satisfying

$$\begin{aligned}\|g(z) - g(y)\| &= \left|\frac{f(y + ty)}{\|y + ty\|} - \frac{f(y)}{\|y\|}\right| = \left|\frac{f(y + ty) - f(y)(1 + t)}{\|y\|(1 + t)}\right| \\ &= \frac{1}{1 + t}\left|\frac{f(y + ty) - f(y) - t(f(y) - f(0))}{\|y\|}\right| \\ &\le \frac{1}{1 + t}\left|\frac{f(y + ty) - f(y)}{\|y\|}\right| + \frac{t}{1 + t}\left|\frac{f(y) - f(0)}{\|y\|}\right| \\ &\le \frac{t}{1 + t}\left(\left|f^\ell\left(\xi_1; \frac{y}{\|y\|}\right) - f^\ell\left(0; \frac{y}{\|y\|}\right)\right| + \left|f^\ell\left(\xi_2; \frac{y}{\|y\|}\right) - f^\ell\left(0; \frac{y}{\|y\|}\right)\right|\right) \\ &\le \frac{t}{1 + t}K(\|\xi_1\| + \|\xi_2\|) \le \frac{Kt}{1 + t}(\|y + ty\| + \|y\|) \\ &= \frac{2 + t}{1 + t}Kt\|y\| \le 2Kt\|y\| = 2K\|z - y\|.\end{aligned}$$

Since the case $y = z$ is clear and because of Step 3, the Lipschitzness of function g will be proved when we show that for arbitrary $y, z \in B(0, \delta), y \ne 0, z \ne 0, y \ne z$, we have

$$\|g(z) - g(y)\| \le 5K\|z - y\|. \tag{3}$$

We can suppose that $\|z\| \ge \|y\|$. We put

$$s = \|z\| - \|y\|, \qquad z_1 = z - \frac{s}{2}\frac{z}{\|z\|}, \qquad y_1 = y + \frac{s}{2}\frac{y}{\|y\|}.$$

Then $\|z_1\| = \|y_1\|$ and Steps 2 and 4 imply

$$\begin{aligned}\|g(z) - g(y)\| &\le \|g(z) - g(z_1)\| + \|g(z_1) - g(y_1)\| + \|g(y_1) - g(y)\| \\ &\le 2K\|z - z_1\| + K\|z_1 - y_1\| + 2K\|y_1 - y\|.\end{aligned} \tag{4}$$

Using elementary geometry, we can obtain that

$$\|z - z_1\| \le \|z - y\|, \qquad \|z_1 - y_1\| \le \|z - y\| \quad \text{and} \quad \|y_1 - y\| \le \|z - y\|.$$

Thus formula (4) implies the considered inequality (3). $\square$

Proposition 1. *Let $f : X \to \mathbb{R}$ be a function which is continuous near $x \in X$ and ℓ-stable at x. If $f^\ell(x; h) = 0$ for every $h \in S_X$, the function $h \mapsto f'^\ell_{\_P}(x; h)$ is continuous on X.*

Moreover, if $X = \mathbb{R}^N$, then $f'^\ell_{\_P}(x; \cdot)$ is Fréchet.

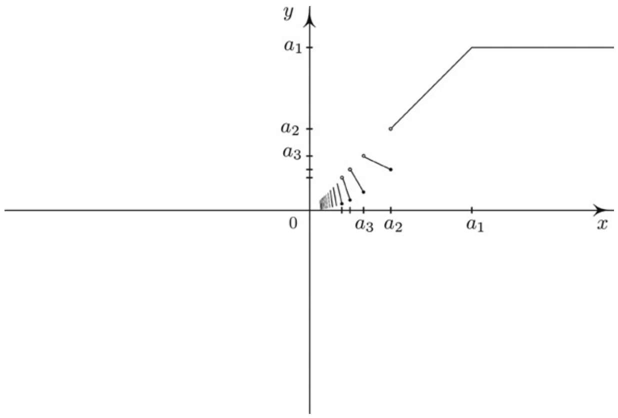

Fig. 1. Function φ.

Proof. Let us consider the function $g : X \to \mathbb{R}$ such that

$$g(y) = \begin{cases} \dfrac{2(f(y) - f(x))}{\|y - x\|}, & \text{for } y \neq x, \\ 0, & \text{for } y = x. \end{cases}$$

Then

$$\begin{aligned} g^{\ell}(x; h) &= \liminf_{t \downarrow 0} \frac{g(x + th) - g(x)}{t} = \liminf_{t \downarrow 0} \frac{2\frac{f(x+th)-f(x)}{\|th\|} - 0}{t} \\ &= \liminf_{t \downarrow 0} \frac{f(x + th) - f(x) - tf^{\ell}(x; h)}{t^2/2} = f''^{\ell}_{-P}(x; h), \end{aligned} \tag{5}$$

for every $h \in S_X$. Due to Lemmas 4 and 6, the function $h \mapsto f''^{\ell}_{-P}(x; h)$ is continuous on X.

The second part of theorem follows from formula (5) and Lemmas 5 and 6. □

Using Proposition 1 together with the fact that an ℓ-stable function at some point defined on a finite-dimensional space is continuous near this point [15], the compactness of $S_{\mathbb{R}^N}$ implies the following consequence.

Corollary 1. *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a function which is ℓ-stable at $x \in \mathbb{R}^N$. If $f^{\ell}(x; h) = 0$ for every $h \in S_{\mathbb{R}^N}$ and*

$$f''^{\ell}_{-P}(x; h) > 0, \quad \forall h \in S_{\mathbb{R}^N},$$

then there exists $c > 0$ satisfying

$$f''^{\ell}_{-P}(x; h) \geq c, \quad \forall h \in S_{\mathbb{R}^N}.$$

Now, by Proposition 1 and Corollary 1, Theorem 1 is a special case of Theorem 3.

Finishing the paper, we show an example of nonconvex and noncontinuous function for which we can use Theorem 3. We use some ideas from [14, Example 2]

Example 1. Consider a sequence $a_n = 1/n$, $n = 1, 2, \ldots$. Then

$$\lim_{n \to \infty} \frac{a_{n+1} + a_n^2}{a_{n+1} + a_n} = \frac{1}{2} > 0.$$

Let us define a function $\varphi : [0, \infty) \to \mathbb{R}$ as follows (see Fig. 1 for the construction of φ).

$$\varphi(u) = \begin{cases} a_1, & \text{if } u > a_1, \\ \dfrac{a_n^2 - a_{n+1}}{a_n - a_{n+1}}(u - a_{n+1}) + a_{n+1}, & \text{if } u \in (a_{n+1}, a_n], \\ 0, & \text{if } u = 0. \end{cases}$$

Next, we will define a function $f : \mathbb{R} \to \mathbb{R}$ via the Riemann integral :

$$f(x) := \int_0^{|x|} \varphi(u) \mathrm{d}u, \quad x \in \mathbb{R}.$$

Since φ is a piecewise affine function, the integral exists.

It is easy to show that f is differentiable at 0 with derivative equal to 0. Now we claim that $\underline{f}'^{\ell}_{P}(0; \pm 1) > 0$. So it suffices to show that

$$\liminf_{t \downarrow 0} \frac{f(t)}{t^2/2} > 0.$$

Note that there is $\epsilon > 0$ such that for each $n \in \mathbb{N}$ it holds:

$$\frac{a_{n+1} + a_n^2}{a_{n+1} + a_n} \geq \epsilon > 0.$$

Now consider $t \in [a_{j+1}, a_j)$ for some $j \in \mathbb{N}$ and fix $k \in \mathbb{N}$, $k \geq j + 2$. Let S_n denote an area of a trapezoid over the interval (a_{n+1}, a_n), $n = j + 1, \ldots, k$, bounded by a graph of φ. Let R denote an area of a trapezoid over the interval (a_{j+1}, t) bounded by the graph of φ. Now we can write down the formula for the integral:

$$\int_{a_k}^{t} \varphi(u)\mathrm{d}u = \left(\sum_{n=j+1}^{k} S_n \right) + R.$$

Further $\tilde{S}_n$ stands for an area of a trapezoid over the interval (a_{n+1}, a_n), $n = j + 1, \ldots, k$ bounded by the linear function $y = x$, and $\tilde{R}$ stands for an area of a trapezoid over the interval (a_{j+1}, t) bounded also by the function $y = x$. Now it can be shown that

$$\begin{aligned}\int_{a_k}^{t} \varphi(u)\mathrm{d}u &= \left(\sum_{n=j+1}^{k} S_n \right) + R \geq \epsilon \sum_{n=j+1}^{k} \tilde{S}_n + \epsilon \tilde{R} \\ &= \epsilon \left(\sum_{n=j+1}^{k} \tilde{S}_n + \tilde{R} \right).\end{aligned}$$

Letting $k \to +\infty$, we will get:

$$\begin{aligned}f(t) &= \int_{0}^{t} \varphi(u)dt \geq \epsilon \left(\sum_{n=j+1}^{\infty} \tilde{S}_n + \tilde{R} \right) \\ &= \epsilon \frac{t^2}{2}.\end{aligned}$$

Hence $2f(t)/t^2 \geq \epsilon > 0$, where $t \in [a_{j+1}, a_j)$. Since this holds for almost any $j \in \mathbb{N}$ and for all $t \in [a_{j+1}, a_j)$, we have for any $\delta > 0$ sufficiently small,

$$\inf \left\{ 2\frac{f(t)}{t^2} : t \in (0, \delta) \right\} \geq \epsilon > 0.$$

Hence $\liminf_{t \downarrow 0} 2f(t)/t^2 \geq \epsilon > 0$.

Now, let us suppose that X is an arbitrary infinite-dimensional space, A_n is a nonempty subset of S_X for every $n \in \mathbb{N}$, and

$$\bigcup_{n=1}^{+\infty} A_n = S_X.$$

Let us define the function $g : X \to \mathbb{R}$ by the following way.

$$g(th) = nf(t), \quad \forall n \in \mathbb{N}, \forall h \in A_n, \forall t \in \mathbb{R}.$$

It follows immediately from the construction of f that

$$g^{\ell}(0; h) = f^{\ell}(0; 1) = 0, \quad \forall h \in S_X,$$

and for every $h \in A_n$, $n \in \mathbb{N}$, we have

$$\underline{g}'^{\ell}_{P}(0; h) \geq n\varepsilon \geq \varepsilon.$$

Moreover, since g is defined by means of f in every direction, we have that $\underline{g}'^{\ell}_{P}(0; h)$ is Fréchet. Therefore, using Theorem 3, we obtain that 0 is an isolated minimizer for g.

We notice that it follows from the construction of φ and definition of f and g that the function g is not convex. Further, considering arbitrary $h_1 \in A_1$, we have

$$g(th_1) > 0, \quad \forall t > 0,$$

and

$$g(th_n) = ng(th_1),$$

for every $h_n \in A_n$, $n \in \mathbb{N}$. Thus, the function g is not continuous. ♣

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