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Practical Stabilization of Control Systems with Impulse Effects

XINZHI LIU

Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario Canada N2L 3G1

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In this paper, we investigate practical stabilization of control systems with impulse effects. Utilizing piecewise continuous Lyapunov functions and impulsive differential inequalities, we prove some general comparison results through which we establish some sufficient conditions for practical stabilization of control systems with impulse effects. We also use two different measures, which gives a unified approach, and include many interesting special cases. © 1992 Academic Press, Inc.

1. INTRODUCTION

It is known that, in applications, even asymptotic stability by itself is not sufficient, since the domain of attraction may not be large enough to allow the desired derivation to cancel out. As a result, the system may be asymptotically stable in theory but actually unstable in practice [1]. In the stabilization of nonlinear systems many interesting problems deal with bringing states close to certain sets rather than to the particular state $x = 0$. From a practical point of view, a concrete system is considered stable if the deviation of the motions from the equilibrium remain within certain bounds determined by the physical situation. The desired state of a system may be mathematically unstable, and yet the system may oscillate sufficiently near this state and its performance is acceptable. Many aircraft and missiles behave in this manner. To deal with such situations, the notion of practical stability is more useful [4].

To unify various stability concepts and to offer a general framework for study, the stability concepts defined in terms of two measures have been employed fruitfully [5, 6]. This idea is also useful in the study of practical stability, since it can be used to describe various sets and to provide sufficient conditions for practical stability.

Since many evolution processes are characterized by the fact that at

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certain moments of time they experience an abrupt change of state, the study of dynamic systems with impulse effects has been assuming recently a greater importance [2]. We investigate, in this paper, practical stabilization of control systems with impulse effects. Utilizing piecewise continuous Lyapunov functions and impulsive differential inequalities, we prove some general comparison results through which we establish some sufficient conditions for practical stabilization of control systems with impulse effects. We also use two different measures, which gives a unified approach, and include many interesting special cases.

2. Preliminaries

We consider the control system with impulse effects

$$\begin{aligned} x' &= f(t, x, u), \qquad t \neq t_k, \quad k = 1, 2, ..., \\ \Delta x &= I_k(x), \qquad t = t_k, \quad k = 1, 2, ..., \\ x(t_0^+) &= x_0 \end{aligned} \tag{2.1}$$

under the following assumptions:

A_0. (i) $0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $t_k \to \infty$ as $k \to \infty$;

(ii) $f: R_+ \times R^n \times R^m \to R^n$ is continuous in $(t_{k-1}, t_k] \times R^n \times R^m$ and for every $(x, u) \in R^n \times R^m$, $k = 1, 2, ...,$

$$\lim_{(t, y, v) \to (t_{k-1}^+, x, u)} f(t, y, v) = f(t_{k-1}^+, x, u)$$

exists;

(iii) $I_k: R^n \to R^n$ is continuous.

We list the following classes of functions for convenience:

$K = [\sigma \in C[R_+, R_+]: \sigma(s)$ is strictly increasing and $\sigma(0) = 0]$;

$PC = [\sigma: R_+ \to R$ is continuous in $(t_{k-1}, t_k]$ and $\sigma(t_{k-1}^+)$ exists for $k = 1, 2, ...]$;

$\Gamma = [h \in C[R_+ \times R^n, R_+]: \inf h(t, x) = 0]$;

$v_0 = [V: R_+ \times R^n \to R_+$ is continuous in $(t_{k-1}, t_k] \times R^n$, locally Lipschitzian in x and for each $x \in R^n$, $k = 1, 2, ...,$

$$\lim_{(t, y) \to (t_k^+, x)} V(t, y) = V(t_k^+, x)$$

exists].

Let $V \in v_0$. Then for $(t, x) \in (t_{k-1}, t_k) \times R^n$, we define

$$D^+ V(t, x) = \lim_{h \to 0^+} \sup \frac{1}{h} [V(t+h, x+hf(t, x, u)) - V(t, x)]. \quad (2.2)$$

Let E be the admissible control set. Then corresponding to $u^* \in E$, we denote by $x(t) = x(t, t_0, x_0, u^*)$ any solution of (2.1) such that $x(t_0^+) = x_0$. We next define the notions of practical stability in terms of two measures.

DEFINITION 2.1. Let $h_0, h \in \Gamma$. Then the control system (2.1) is said to be

(i) (h_0, h)-practically stable if, given (η, H) with $0 < \eta < H$, we have $h_0(t_0, x_0) < \eta$ implies $h(t, x(t)) < H$, $t \geq t_0$ for some $t_0 \in R_+$;

(ii) (h_0, h)-uniformly practically stable if (i) holds for every $t_0 \in R_+$;

(iii) (h_0, h)-practically quasi stable if given $(\eta, \beta, T) > 0$ and some $t_0 \in R_+$, we have $h_0(t_0, x_0) < \eta$ implies $h(t, x(t)) < \beta$, $t \geq t_0 + T$;

(iv) (h_0, h)-uniformly practically quasistable if (iii) holds for all $t_0 \in R_+$;

(v) (h_0, h)-strongly practically stable if (i) and (iii) hold simultaneously;

(vi) (h_0, h)-strongly uniformly practically stable if (ii) and (iv) hold together;

(vii) (h_0, h)-practically unstable if (i) does not hold.

A few choices of two measures (h_0, h) help to emphasize the generality of the above definition. It is easy to see that Definition 2.1 reduces to

1. the standard practical stability notion [4] if $h(t, x) = h_0(t, x) = \|x\|$;

2. the practical stability of the prescribed motion $x_0(t)$ of (2.1) if $h(t, x) = h_0(t, x) = \|x - x_0(t)\|$;

3. the partial practical stability if $h(t, x) = \|x\|_s$, $1 \leq s < n$, and $h_0(t, x) = \|x\|$;

4. the orbital practical stability of the orbit C if $h(t, x) = h_0(t, x) = d(x, C)$, d being the distance function;

5. the practical stability of an invariant set A is $h(t, x) = h_0(t, x) = d(x, A)$;

6. the practical stability of conditionally invariant set B with respect to A, where $A \subset B$, if $h(t, x) = d(x, B)$ and $h_0(t, x) = d(x, A)$.

Several other combinations of choices are possible for (h_0, h) in addition to those given above.

3. Comparison Results

Let us begin by stating a Lemma [3] for later use.

Lemma 3.1. *Assume that*

(i) $m, v \in C[[\tau, \tau+\alpha], R]$ $\alpha > 0$, *and*

$$D^+m(t) \leqslant g(t, m(t), v(t)), \qquad t \in [\tau, \tau+\alpha],$$

where $g \in C[[\tau, \tau+\alpha] \times R^2, R]$, g *is nondecreasing in* v *for each* $(t, w) \in [\tau, \tau+\alpha] \times R$;

(ii) $\gamma(t)$ *is the maximal solution of the scalar differential equation*

$$w' = g(t, w, w), \qquad w(\tau) = w^* \geqslant 0, \tag{3.1}$$

existing on $[\tau, \tau+\alpha]$ *and* $v(t) \leqslant \gamma(t)$, $t \in [\tau, \tau+\alpha]$.

Then $m(t) \leqslant \gamma(t)$, $t \in [\tau, \tau+\alpha]$, *provided* $m(\tau) \leqslant w^*$.

We now consider the following comparison system with impulse effects

$$\begin{aligned} w' &= g(t, w, w), && t \neq t_k, \\ w(t_k^+) &= J_k(w(t_k)), && k = 1, 2, ..., \\ w(t_0^+) &= w_0 \geqslant 0, \end{aligned} \tag{3.2}$$

where $g: R_+ \times R \times R \to R$ is continuous in $(t_{k-1}, t_k] \times R \times R$ and for every $(w, z) \in R \times R$, $k = 1, 2, ...$,

$$\lim_{(t, \bar{w}, \bar{z}) \to (t_{k-1}^+, w, z)} g(t, \bar{w}, \bar{z}) = g(t_{k-1}^+, w, z)$$

exists, and $J_k: R \to R$ is nondecreasing for $k = 1, 2, ...$

We denote by $\gamma(t) = \gamma(t, t_0, w_0)$ the maximal solution of (3.2) existing on $[t_0, \infty)$. Then it is easy to see the following result.

Lemma 3.2. *If* $\gamma(t) = \gamma(t, t_0, w_0)$ *is the maximal solution of* (3.2), *then* $\gamma(t)$ *is the maximal solution of* (3.1) *on* $[t_{k-1}, t_k]$ *such that* $w(t_{k-1}^+) = \gamma(t_{k-1}^+)$, $k = 1, 2, ...$

Now we are ready to prove the following comparison result.

Theorem 3.1. *Assume that*

(i) $m, v \in PC$ *and*

$$\begin{aligned} D^+m(t) &\leqslant g(t, m(t), v(t)), && t \neq t_k, \\ M(t_k^+) &\leqslant J_k(m(t_k)), && k = 1, 2, ...; \end{aligned}$$

(ii) *$g(t, w, v)$ is nondecreasing in v for each (t, w) and $\gamma(t)$ is the maximal solution of* (3.2) *existing on* $[t_0, \infty)$ *such that* $v(t) \leqslant \gamma(t)$, $t \geqslant t_0$.

Then $m(t) \leqslant \gamma(t)$, $t \geqslant t_0$, *provided* $m(t_0) \leqslant w_0$.

Proof. It follows from Lemma 3.2 that $\gamma(t)$ is the maximal solution of (3.1) on $[t_{k-1}, t_k]$ such that $w(t_{k-1}^+) = \gamma(t_{k-1}^+)$, for $k = 1, 2, \ldots$. Then, for $t \in (t_0, t_1]$, Lemma 3.1 implies that

$$m(t) \leqslant \gamma_1(t, t_0, w_0), \tag{3.3}$$

where $\gamma_1(t, t_0, w_0)$ is the maximal solution of the differential equation (3.1) existing on $[t_0, t_1]$ such that $\gamma_1(t_0^+, t_0, w_0) = w_0$. Since $J_1(w)$ is nondecreasing, we get from assumption (i) and (3.3) that $m(t_1^+) \leqslant w_1^+$, where $w_1^+ = J_1(\gamma_1(t_1, t_0, w_0))$. Using again Lemma 3.1, we obtain

$$m(t) \leqslant \gamma_2(t, t_1, w_1^+), \quad t \in (t_1, t_2],$$

where $\gamma_2(t, t_1, w_1^+)$ is the maximal solution of (3.1) existing on $[t_1, t_2]$ such that $\gamma_2(t_1^+, t_1, w_1^+) = w_1^+$. We therefore have successively, for $k = 1, 2, \ldots,$

$$m(t) \leqslant \gamma_k(t, t_{k-1}, w_{k-1}^+), \qquad (t \in t_{k-1}, t_k],$$

$\gamma_k(t, t_{k-1}, w_{k-1}^+)$ being the maximal solution of (3.1) existing on $[t_{k-1}, t_k]$ such that $\gamma_k(t_{k-1}^+, t_{k-1}, w_{k-1}^+) = w_k^+$. Thus if we define

$$w(t) = \begin{cases} w_0, & t = t_0 \\ \gamma_1(t, t_0, w_0), & t \in (t_0, t_1], \\ \gamma_2(t, t_1, w_1^+), & t \in (t_1, t_2], \\ \cdots & \\ \gamma_k(t, t_{k-1}, w_{k-1}^+), & t \in (t_{k-1}, t_k], \\ \cdots & \end{cases} \tag{3.4}$$

then it is easy to see that $w(t)$ is a solution of (3.2) and

$$m(t) \leqslant w(t), \qquad t \geqslant t_0.$$

Since $\gamma(t, t_0, w_0)$ is the maximal solution of (3.2), we get immediately

$$m(t) \leqslant \gamma(t, t_0, w_0), \qquad t \geqslant t_0,$$

and the proof is complete.

Let us collect several interesting and useful special cases of Theorem 3.1 in the following corollary.

COROLLARY 3.1. *If in Theorem* 3.1, *we choose that*

(i) $g(t, w, w) \equiv 0$ *and* $J_k(w) = w$ *for all* k, *then* $m(t) \leqslant w_0$, $t \geqslant t_0$;

(ii) $g(t, w, w) \equiv 0$ *and* $J_k(w) = d_k w$, $d_k \geqslant 0$ *for all* k, *then*

$$m(t) \leqslant w_0 \prod_{t_0 < t_k < t} d_k, \qquad t \geqslant t_0;$$

(iii) $g(t, w, w) = -\alpha w$, $\alpha > 0$, $J_k(w) = d_k w$, $d_k \geqslant 0$, *for all* k, *then*

$$m(t) \leqslant w_0 \prod_{t_0 < t_k < t} d_k \exp[-\alpha(t - t_0)], \qquad t \geqslant t_0;$$

(iv) $g(t, w, w) = \lambda'(t) w$, $J_k(w) = d_k w$, $d_k \geqslant 0$ *for all* k, $\lambda \in C^1[R_+, R_+]$ *and* $\lambda'(t) \geqslant 0$, *then*

$$m(t) \leqslant w_0 \prod_{t_0 < t_k < t} d_k \exp[\lambda(t) - \lambda(t_0)], \qquad t \geqslant t_0;$$

(v) $g(t, w, w) = -\alpha w + b$, $\alpha, b > 0$, $J_k(w) = d_k(w) = d_k w$, $d_k \geqslant 0$ *for all* k, *then*

$$m(t) \leqslant w_0 \prod_{j=1}^{k} d_j e^{-\alpha(t - t_0)} + \frac{b}{\alpha} \sum_{j=1}^{k} \prod_{i=j}^{k} d_i (e^{-\alpha(t - t_i)} - e^{-\alpha(t - t_{j-1})})$$
$$+ \frac{b}{\alpha} (w - e^{-\alpha(t - t_k)}), \qquad t \in (t_k, t_{k+1}].$$

If we drop the requirement, in Theorem 3.1, that $g(t, w, v)$ is nondecreasing in v, then we have the following result.

THEOREM 3.2. *Assume that*

(i) $m, v \in PC$ *and*

$$D^+ m(t) \leqslant g(t, m(t), v(t)), \qquad t \neq t_k,$$
$$m(t_k^+) \leqslant J_k(m(t_k)), \qquad k = 1, 2, \ldots;$$

(ii) $\gamma(t)$ *is the maximal solution of*

$$\begin{aligned} &w' = g(t, w, v(t)), && t \neq t_k, \\ &w(t_k^+) = J_k(w(t_k)), && k = 1, 2, \ldots \\ &w(t_0^+) = w_0 \leqslant 0, && \end{aligned} \tag{3.5}$$

existing on $[t_0, \infty)$.

Then $m(t) \leqslant \gamma(t)$, $t \geqslant t_0$, *provided* $m(t_0) \leqslant w_0$.

The proof of Theorem 3.2 is similar to that of Theorem 3.1; we omit the details here.

4. Stabilization Criteria

Having the comparison results developed in Section 3 at our disposal, we are ready to prove, in this section, some results which offer sufficient conditions in a unified way for various practical stabilization criteria of the control system (2.1). We first consider the control set

$$E = \{u \in R^m,\ U(t, u) \leqslant \gamma(t),\ t \geqslant t_0\}, \tag{4.1}$$

where $U\colon R_+ \times R^m \to R_+$ is continuous on $(t_{k-1}, t_k] \times R^m$ and for every $u \in R^m$, $k = 1, 2, \ldots,$

$$\lim_{(t, v) \to (t_{k-1}^+, u)} U(t, v) = U(t_k^+, u)$$

exists, and $\gamma(t)$ is the maximal solution of (3.2).

THEOREM 4.1. *Assume that*

(i) $0 < \eta < H$ *are given*;

(ii) $h_0, h \in \Gamma$ *and* h_0 *is finer than* h, *i.e.* $h(t, x) \leqslant \phi(h_0(t, x))$, $\phi \in k$ *whenever* $h_0(t, x) < \eta$;

(iii) $V \in v_0$ *and there exist* $a, b \in K$ *such that*

$$b(h(t, x)) \leqslant V(t, x), \qquad \text{if} \quad h(t, x) < \rho,\ \rho > H,$$

$$V(t, x) \leqslant a(h_0(t, x)), \qquad \text{if} \quad h_0(t, x) < \eta;$$

(iv) *for* $(t, x) \in (t_{k-1}, t_k) \times R^n$ *and* $u(t) \in E$,

$$D^+ V(t, x) \leqslant g(t, V(t, x), U(t, u(t))), \qquad \text{if} \quad h(t, x) < \rho,$$

where $g(t, w, v)$ *is nondecreasing in* v *for each* $(t, w) \in R_+ \times R$ *and*

$$V(t_k^+, x_k^+) \leqslant J_k(V(t_k, x_k)), \qquad \text{if} \quad h(t_k, x_k) < \rho;$$

(v) $\phi(\eta) < H$ *and* $a(\eta) < b(H)$;

(vi) $h(t, x) < H$ *implies* $h(t, x + I_k(x)) < \rho$ *for all* k.

Then the practical stability properties of (3.2) with respect to $(a(\eta), b(H))$ imply the corresponding (h_0, h)-practical stability properties of (2.1) with respect to (η, H).

Proof. We only show (h_0, h)-practical stability and (h_0, h)-strongly practical stability of (2.1). The remaining cases can be verified similarly. Let $t_0 \geqslant 0$ and $t_0 \in (t_j, t_{j+1}]$ for some $j \geqslant 1$. For convenience, we designate, $t_i = t_{j+i}$ if $t_0 \neq t_{j+1}$, $t_i = t_{j+1+i}$, if $t_0 = t_{j+1}$, $i = 1, 2, \ldots$. Suppose that the comparison system (3.2) is practically stable with respect to $(a(\eta), b(H))$. Then we have that $w_0 < a(\eta)$ implies

$$w(t, t_0, w_0) < b(H), \qquad t \geqslant t_0, \tag{4.2}$$

where $w(t, t_0, w_0)$ is any solution of (3.2) existing on $[t_0, \infty)$. Choose $(t_0, x_0) \in R_+ \times R^n$ such that $h_0(t_0, x_0) < \eta$. Then by assumptions (ii) and (v), we have

$$h(t_0, x_0) \leqslant \phi(h_0(t_0, x_0)) \leqslant \phi(\eta) < H.$$

We claim that

$$h(t, x(t)) < H, \qquad \text{for all} \quad t \geqslant t_0, \tag{4.3}$$

where $x(t) = x(t, t_0, x_0, u^*)$ is any solution of (2.1) with $h_0(t_0, x_0) < \eta$. If this is not true, then there would exist a $u^0 = u^0(t)$ and a corresponding solution $x(t) = x(t, t_0, x_0, u^0)$ of (2.1) with $h_0(t_0, x_0) < \eta$ and a $t^* > t_0$ such that $t_k < t^* \leqslant t_{k+1}$ for some k, satisfying

$$H \leqslant h(t^*, x(t^*)) \qquad \text{and} \qquad h(t, x(t)) < H,\ t_0 \leqslant t \leqslant t_k. \tag{4.4}$$

It then follows from assumption (vi) that we can find a t^0 such that $t_k < t^0 \leqslant t^*$ and

$$H \leqslant h(t^0, x(t^0)) < \rho. \tag{4.5}$$

Setting $m(t) = V(t, x(t))$, $t_0 \leqslant t \leqslant t^0$, and $w_0 = V(t_0, x_0)$, then assumption (iii) yields, by standard computation, the differential inequality

$$\begin{aligned} &D^+ m(t) \leqslant g(t, m(t), U(t, u^0(t))), \qquad t_0 \leqslant t \leqslant t^0,\ t \neq t_i,\ i = 1, 2, \ldots, k, \\ &m(t_i^+) \leqslant J_i(m(t_i)), \qquad i = 1, 2, \ldots, k. \end{aligned} \tag{4.6}$$

Since $u^0 \in E$ and g is nondecreasing in v, we have, from (4.6), that

$$\begin{aligned} &D^+ m(t) \leqslant g(t, m(t), \gamma(t)), \qquad t_0 \leqslant t \leqslant t^0,\ t \neq t_i \\ &m(t_i^+) \leqslant J_i(m(t_i)), \qquad i = 1, 2, \ldots, k. \end{aligned} \tag{4.7}$$

It then follows from Theorem 3.1 that

$$m(t) \leqslant \gamma(t), \qquad t_0 \leqslant t \leqslant t^0, \tag{4.8}$$

where $\gamma(t) = \gamma(t, t_0, w_0)$ is the maximal solution of (3.2). We are then led, from (4.2), (4.5), and (4.8) to a contradiction

$$b(H) \leqslant V(t^0, x(t^0)) \leqslant \gamma(t^0) < b(H), \tag{4.9}$$

proving the control system (2.1) is (h_0, h)-practically stable.

Let us suppose next that (3.2) is strongly practically stable with respect to $(a(\eta), b(H))$. This implies that (2.1) is (h_0, h)-practically stable. Consequently, we have that $h_0(t_0, x_0) < \eta$ implies

$$h(t, x(t)) < H, \qquad t \geqslant t_0, \tag{4.10}$$

$x(t) = x(t, t_0, x_0, u^*)$ being any solution of (2.1) with $h_0(t_0, x_0) < \eta$. Let $0 < \beta < H$ and $T > 0$ be given. Since (3.2) is practically quasistable, given $b(\beta) > 0$ and $T > 0$, we have

$$w_0 < a(\eta) \text{ implies } w(t, t_0, w_0) < b(\beta), \qquad t \geqslant t_0 + T. \tag{4.11}$$

Let (t_0, x_0) be chosen such that $h_0(t_0, x_0) < \eta$. In view of (4.10), arguments leading to (4.9) yield

$$V(t, x(t)) \leqslant \gamma(t, t_0, a(h_0(t_0, x_0))), \qquad t \geqslant t_0,$$

from which and (4.11) it follows that

$$b(h(t, x(t)) \leqslant V(t, x(t)) < b(\beta), \qquad t \geqslant t_0 + T,$$

which proves

$$h(t, x(t)) < \beta, \qquad t \geqslant t_0 + T.$$

Hence the control system (2.1) is (h_0, h)-strongly practically stable and the proof is complete.

As an example, we consider the linear control system

$$\begin{aligned} x' &= Ax + Bu + \sigma(t), \qquad t \neq t_k, \\ \Delta x &= C_k x, \; t = t_k, \qquad k = 1, 2, ..., \\ x(t_0^+) &= x_0, \end{aligned} \tag{4.12}$$

where A, B are $n \times n$ and $n \times m$ matrices, C_k is $n \times n$ matrix for each k, and $\sigma: R_+ \to R^n$ is piecewise continuous.

THEOREM 4.2. *Assume that*

(i) $0<\eta<H$ *are given*;

(ii) $\mu(A) = \lim_{h\to 0}(1/h)[\|I+hA\|-1] \leqslant -\alpha$, $\alpha>0$, $\|B\|=b$, $\|\sigma(t)\|\leqslant l$ *and* $\|C_k\|=c_k$ *for* $k=1, 2, \ldots$;

(iii) $\alpha-b=\delta>0$ *and* $\eta\prod_{k=1}^{\infty}(1+c_k)+(l/\delta)\sum_{k=1}^{\infty}\prod_{i=k}^{\infty}(1+c_i)+(l/\delta)<H$.

Then the linear control system (4.12) *is practically stable.*

Proof. Take $V(t,x)=\|x\|$ and $h_0(t,x)=h(t,x)=\|x\|$. Then it is easy to compute that

$$g(t,w,w)=(-\alpha+b)\,w+l \qquad \text{and} \qquad J_k(w)=(1+c_k)\,w.$$

Thus to prove the theorem, it is enough to prove the comparison system

$$\begin{aligned} w' &= -\delta w+l, && t\neq t_k \\ w(t_k^+) &= (1+c_k)\,w(t_k), && k=1,2,\ldots, \\ w(t_0^+) &= w_0, \end{aligned} \tag{4.13}$$

is practically stable with respect to (η, H). It is easy to compute that the solutions of (4.13) are of the form

$$\begin{aligned} w(t,t_0,w_0) &= w_0\prod_{j=1}^{k}(1+c_j)\,e^{-\delta(t-t_0)} \\ &\quad +\frac{l}{\delta}\sum_{j=1}^{k}\prod_{i=j}^{k}(1+c_i)\,e^{-\delta(t-t_j)} \\ &\quad -\frac{l}{\delta}\sum_{j=1}^{k}\prod_{i=j}^{k}(1+c_i)\,e^{-\delta(t-t_{j-1})} \\ &\quad +\frac{l}{\delta}(1-e^{-\delta(t-t_k)}),\ t\in(t_k,t_{k+1}]. \end{aligned}$$

Thus it follows from assumption (iii) that $w_0<\eta$ implies $w(t,t_0,w_0)<H$, $t\geqslant t_0$.

Hence, we get from Theorem 4.1 the corresponding practical stability of (4.12) and the proof is complete.

There are many interesting special cases of Theorem 4.1, which we state below as a corollary.

COROLLARY 4.1 (in Theorem 4.1). 1. *The functions* $g(t,w,w)\equiv 0$, $J_k(w)=d_k w$, $d_k\geqslant 0$ *for all* k *are admissible to yield* (h_0, h)*-uniform practical stability* *of* (2.1) *provided the infinite product* $\prod_{k=1}^{\infty} d_k$ *converges. In particular,* $d_k = 1$ *for all k is admissible*;

2. $g(t, w, w) = \lambda'(t)\, w$, $\lambda \in c^1[R_+, R_+]$, $J_k(w) = d_k w$, $d_k \geqslant 0$ *for all k are admissible to imply* (h_0, h)*-practical stability of* (2.1) *provided*

$$\lambda(t_k) + \ln d_k \leqslant \lambda(t_{k-1}) \qquad \text{for all } k.$$

Let $v \in PC$ be given. We shall next consider the control set

$$\Omega = \{u \in R^m;\ U(t, u) \leqslant v(t),\ t \geqslant t_0\}.$$

THEOREM 4.3. *Assume that*

(i) $0 < \eta < H$ *are given*;

(ii) $h_0, h \in \Gamma$ *and for some* $\phi \in K$, $h(t, x) \leqslant \phi(h_0(t, x))$ *if* $h_0(t, x) < \eta$;

(iii) $V \in v_0$ *and* $a, b \in K$ *such that*

$$b(h(t, x)) \leqslant V(t, x), \qquad \text{if } \ h(t, x) < \rho,\ \rho > H,$$

$$V(t, x) \leqslant a(h_0(t, x)), \qquad \text{if } \ h_0(t, x) < \eta;$$

(iv) *for* $(t, x) \in (t_{k-1}, t_k) \times R^n$ *and* $u \in \Omega$,

$$D^+ V(t, x) \leqslant g(t, V(t, x), U(t, u(t))), \qquad \text{if } \ h(t, x) < \rho,$$

and

$$V(t, x) \leqslant a(h_0(t, x)), \qquad \text{if } \ h_0(t, x) < \eta;$$

(v) $\phi(\eta) < H$, $a(\eta) < b(H)$, *and* $h(t, x) < H$ *implies* $h(t, x + I_k(x)) < \rho$ *for all* k;

(vi) *there exists a control function* $v \in PC$ *such that any solution* $w(t) = w(t, t_0, w_0, v)$ *of* (3.5) *satisfies*

$$w_0 \leqslant a(\eta) \text{ implies } w(t) < b(H), \qquad t \geqslant t_0, \tag{4.14}$$

and

$$w(t_0 + T) \leqslant b(\beta),\ 0 < \beta < H, \qquad \text{for some } T = T(t_0, w_0) > 0. \tag{4.15}$$

Then there exist admissible controls $u = u(t) \in \Omega$ *such that the control system* (2.1) *is* (h_0, h)*-practically stable and all solutions* $x(t) = x(t, t_0, x_0, u)$ *starting in* $\Omega_1 = \{x \in R^n, h(t, x) < \eta, t \geqslant t_0\}$ *are transferred to the region* $\Omega_2 = \{x \in R^n\colon h(t, x) \leqslant \beta\}$ *in a finite time* $T^* = T^*(t_0, x_0) = T(t_0, V(t_0, x_0))$, *that is, the system* (2.1) *is controllable.*

Proof. Let $h_0(t_0, x_0) \leqslant \phi(h_0(t_0, x_0) < \eta$ and $U(t, u(t)) \leqslant v(t)$, $t \geqslant t_0$. Then we have $h(t_0, x_0) \leqslant \phi(h_0(t_0, x_0)) < H$. We claim that for any solution $x(t) = x(t, t_0, x_0, u)$ of (2.1), we have

$$h_0(t_0, x_0) < \eta \text{ implies } h(t, x(t)) < H \qquad \text{for} \quad t \geqslant t_0. \tag{4.16}$$

If this is false, there would exist a $u^0 \in \Omega$ and a corresponding solution $x(t) = x(t, t_0, x_0, u^0)$ of (2.1) and a $t^* > t_0$ such that $t_k < t^* \leqslant t_{k+1}$ for some k, satisfying

$$H \leqslant h(t^*, x(t^*)) \qquad \text{and} \qquad h(t, x(t)) < H,\ t_0 \leqslant t \leqslant t_k. \tag{4.17}$$

It then follows from assumption (v) that we can find a t^0 such that $t_k < t^0 \leqslant t^*$ and

$$H \leqslant h(t^0, x(t^0)) < \rho. \tag{4.18}$$

Setting $m(t) = V(t, x(t))$, for $t_0 \leqslant t \leqslant t^0$, then assumption (iv) yields

$$D^+ m(t) \leqslant g(t, m(t), v(t)), \qquad t \neq t_i,\ t_0 \leqslant t \leqslant t^0,$$

$$m(t_i^+) \leqslant J_i(m(t_i)), \qquad i = 1, 2, ..., k,$$

which implies by Theorem 3.2 the estimate

$$m(t) \leqslant \gamma(t, t_0, w_0, v), \qquad t_0 \leqslant t \leqslant t^0, \tag{4.19}$$

provided $m(t_0) \leqslant w_0$, where $\gamma(t, t_0, w_0, v)$ is the maximal solution of (3.5). Choosing $w_0 = V(t_0, x_0)$, we then get from (iii), (4.17)–(4.19), the relation

$$b(h(t, x(t)) \leqslant V(t, x(t)) \leqslant \gamma(t, t_0, w_0, v), \qquad t_0 \leqslant t \leqslant t^0. \tag{4.20}$$

Now we are led to the following contradiction, in view of (4.14) and (4.18),

$$b(H) \leqslant b(h(t^0, x(t^0))) \leqslant \gamma(t_0, t_0, w_0, v) < b(H),$$

which proves the practical stability of (2.1). As a result, (4.20) holds for all $t \geqslant t_0$, and therefore the assumption (4.15) yields

$$h(t_0 + T^*, x(t_0 + T^*)) \leqslant \beta,$$

where $T^* = T(t_0, V(t_0, x_0))$. The proof is hence complete.

As an example, we let the comparison system (3.5) be of the following form

$$\begin{aligned} &w' = \bar{a}(t)\, w + \bar{b}(t),\ t \neq t_k, && \bar{a}, \bar{b} \in PC, \\ &w(t_k^+) = d_k w(t_k),\ d_k \geqslant 0, && k = 1, 2, ..., \\ &w(t_0^+) = w_0. \end{aligned} \tag{4.21}$$

The solution of (4.21) is

$$w(t)=\begin{cases} w_0 \exp\left(\int_{t_0}^{t} \bar{a}(s)\,ds\right)+\exp\left(\int_{t_0}^{t} \bar{a}(s)\,ds\right) \\ \quad \cdot\int_{t_0}^{t} \exp\left(-\int_{t_0}^{s} \bar{a}(\sigma)\,d\sigma\right) \bar{b}(s)\,v(s)\,ds, & t_0 \leqslant t \leqslant t_1, \\ w_{k-1}^{+} \exp\left(\int_{t_{k-1}}^{t} \bar{a}(s)\,ds\right)+\exp\left(\int_{t_{k-1}}^{t} \bar{a}(s)\,ds\right) \\ \quad \cdot\int_{t_{k-1}}^{t} \exp\left(-\int_{t_{k-1}}^{s} \bar{a}(\sigma)\,d\sigma\right) \bar{b}(s)\,v(s)\,ds, & t_{k-1} < t \leqslant t_k, \\ w_{k-1}^{+}=d_k w(t_{k-1}), & k=2, 3, \ldots. \end{cases} \tag{4.22}$$

Let $0<b(\eta)<b(H)$ and $0<b(\beta)<b(H)$ be given. We choose $v \in PC$ such that

$$\int_{t_{k-1}}^{t_k} \exp\left(-\int_{t_{k-1}}^{s} \bar{a}(\sigma)\,d\sigma\right) \bar{b}(s)\,v(s)\,ds \leqslant \gamma_k .$$

If $\bar{a}(t) \leqslant 0$ for $t \geqslant t_0$ and

$$b(\eta) \prod_{k=1}^{\infty} d_k + \sum_{k=1}^{\infty} \gamma_k \prod_{j=k}^{\infty} d_j \leqslant b(H),$$

then we have from (4.22)

$$w(t, t_0, w_0) < b(H),\ t \leqslant t_0, \qquad \text{provided } w_0 < b(\eta),$$

i.e., (4.21) is practically stable.

If in addition, there exists $T>0$ such that $t_0+T=t_{k+1}$

$$\exp\left(\int_{t_{i-1}}^{t_i} \bar{a}(s)\,ds\right) \leqslant \alpha_{i-1}, \qquad i=1, 2, \ldots, k+1,$$

and

$$b(\eta) \prod_{i=1}^{k} d_i \alpha_{i-1} + \sum_{i=1}^{k} \prod_{j=1}^{k} d_j \alpha_{j-1} \gamma_i + \alpha_{k+1} \gamma_{k+1} < b(\beta),$$

then

$$w(t_0+T, t_0, w_0) < b(\beta),$$

which shows that the comparison system (4.21) is controllable.

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