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Beurling–Fourier algebras on compact groups: Spectral theory ☆

Jean Ludwig [a], Nico Spronk [b], Lyudmila Turowska [c,*]

[a] *Laboratoire LMAM, UMR 7122, Département de Mathématiques, Université Paul Verlaine-Metz, Ile du Saulcy, F-57045 Metz, France*

[b] *Department of Pure Mathematics, University of Waterloo, Waterloo, ON N2L 3G1, Canada*

[c] *Department of Mathematical Sciences, Chalmers University of Technology and the University of Gothenburg, SE-41296 Gothenburg, Sweden*

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Abstract

For a compact group G we define the Beurling–Fourier algebra $A_\omega(G)$ on G for weights $\omega : \widehat{G} \to \mathbb{R}^{>0}$. The classical Fourier algebra corresponds to the case ω is the constant weight 1. We study the Gelfand spectrum of the algebra realising it as a subset of the complexification $G_{\mathbb{C}}$ defined by McKennon and Cartwright and McMullen. In many cases, such as for polynomial weights, the spectrum is simply G. We discuss the questions when the algebra $A_\omega(G)$ is symmetric and regular. We also obtain various results concerning spectral synthesis for $A_\omega(G)$.

Keywords: Beurling–Fourier algebra; Spectral synthesis

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* Corresponding author.
E-mail addresses: ludwig@univ-metz.fr (J. Ludwig), nspronk@uwaterloo.ca (N. Spronk), turowska@chalmers.se (L. Turowska).

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1. Introduction

Let G be a compact abelian group with discrete dual group $\widehat{G}$. A weight is a function $\omega : \widehat{G} \to \mathbb{R}^{>0}$ for which $\omega(\sigma\tau) \leqslant \omega(\sigma)\omega(\tau)$ for σ, τ in $\widehat{G}$. Given such an ω, the *Beurling algebra* on $\widehat{G}$ is given by

$$\ell^1_\omega(\widehat{G}) = \left\{ \big(f(\sigma)\big)_{\sigma \in \widehat{G}} \subset \mathbb{C}^{\widehat{G}} \colon \sum_{\sigma \in \widehat{G}} \big|f(\sigma)\big|\omega(\sigma) < \infty \right\}$$

and is easily verified to be a commutative Banach algebra under convolution. We say ω is bounded if $\inf_{\sigma \in \widehat{G}} \omega(\sigma) > 0$. In this case $\ell^1_\omega(\widehat{G})$ is a subalgebra of the group algebra $\ell^1(\widehat{G})$. In particular we can apply the Fourier transform to obtain an algebra $A_\omega(G)$ of continuous functions on G. Beurling algebras have been studied by several people, e.g. Domar [4], Reiter [17]. If $\omega \equiv 1$ we get the classical Fourier algebra $A(G)$, i.e. the space of Fourier transforms of $\ell^1(\widehat{G})$.

For any locally compact group G the Fourier algebra was defined by Eymard [5] as the algebra of matrix coefficients of the left regular representation. In the case that G is compact it is well known that $A(G)$ can be identified with the space of operator fields indexed over the set of irreducible representations:

$$\left\{ \big(f(\pi)\big)_{\pi \in \widehat{G}} \in \prod_{\pi \in \widehat{G}} \mathcal{L}(\mathcal{H}_\pi) \colon \sum_{\pi \in \widehat{G}} d_\pi \big\| f(\pi) \big\|_1 < \infty \right\}$$

where $\mathcal{L}(\mathcal{H}_\pi)$ is the space of linear operators on the Hilbertian representation space $\mathcal{H}_\pi$, and $\|\cdot\|_1$ is the trace norm. In light of the definition of Beurling algebras, above, it is natural to define a weight on $\widehat{G}$ as a function $\omega : \widehat{G} \to \mathbb{R}^{>0}$ which satisfies $\omega(\sigma) \leqslant \omega(\pi)\omega(\pi')$, whenever σ may be realised as a subrepresentation of $\pi \otimes \pi'$. Thus it is natural to define the *Beurling–Fourier algebra* $A_\omega(G)$ so it may be identified with the space of operator fields

$$\left\{ \big(f(\pi)\big)_{\pi \in \widehat{G}} \in \prod_{\pi \in \widehat{G}} \mathcal{L}(\mathcal{H}_\pi) \colon \sum_{\pi \in \widehat{G}} \big\| f(\pi) \big\|_1 d_\pi \omega(\pi) < \infty \right\}.$$

We show that this definition always provides a semisimple commutative Banach algebra; moreover, when ω is bounded — i.e. $\inf_{\pi \in \widehat{G}} \omega(\pi) > 0$ — this is a subalgebra of the Fourier algebra.

To describe the spectrum of $A_\omega(G)$ we require an abstract Lie theory which is built from the Krein–Tannaka duality and was formalised separately by McKennon [15] and Cartwright and McMullen [2] in the 70s. This Lie theory allowed them to develop the complexification $G_{\mathbb{C}}$ even for non-Lie groups G. The Gelfand spectrum of $A_\omega(G)$ is shown to be a subset of $G_{\mathbb{C}}$. In contrast to the Fourier algebra $A(G)$, for which the spectrum is G, $A_\omega(G)$ can have a larger spectrum G_ω. Examples of such weights and groups G are given in Section 4. We explore conditions for which $G_\omega = G$. In Section 4 we prove that for symmetric weights ω the equality holds if and only if the algebra $A_\omega(G)$ is symmetric. In Section 4 we define the notion of exponential growth for a weight ω and show that $G_\omega = G$ if ω is of non-exponential type. Examples of weights of non-exponential growth are polynomial ones defined in Section 5. For such weights we could introduce a smooth functional calculus and use this to show that $A_\omega(G)$ is a regular algebra. This gives us a possibility to study the property of spectral synthesis. Adapting arguments from [14] on the Fourier algebra of compact Lie groups we prove that if E is a compact subset of a Lie group G, then E is a set of weak synthesis if it is of smooth synthesis. Moreover we give an estimate of the corresponding nilpotency degree in terms of the degree of the polynomial weight ω. As a consequence we obtain conditions for a one-point set to be a set of spectral synthesis for $A_\omega(G)$. Finally in the last subsection we study a connection between spectral synthesis and operator synthesis in the spirit of [19].

We note that Lee and Samei in [13] suggest a more general approach to the notion of weight on $\widehat{G}$. Their central weights for compact groups turn out to coincide with our notion of weight. However in their paper they mainly concentrate on the study of properties of operator amenability and Arens regularity.

2. An abstract Lie theory for compact groups

In this section we remark on some consequences of the Krein–Tannaka duality theory for compact groups which allow us to define a "complexification" $G_{\mathbb{C}}$ for any compact group G. This object will be necessary for us to develop a description of the spectrum of general Beurling–Fourier algebras. The theory in this section was thoroughly developed by McKennon [15] and Cartwright and McMullen [2]. We shall be requiring it to an extent that a summary is warranted.

Let G be a compact group with dual object $\widehat{G}$, which, by mild abuse of notation, we treat as a set of unitary irreducible representations: $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$. We let $d_\pi = \dim(\mathcal{H}_\pi)$. For representations σ, π of G, we will use the notation $\sigma \subset \pi$ to denote that σ is unitarily equivalent to a subrepresentation of π.

We let $\mathrm{Trig}(G)$ denote the space of trigonometric polynomials, i.e. the span of matrix coefficients of elements of $\widehat{G}$, which is well known to be an algebra of functions under pointwise operations. We note that $\mathrm{Trig}(G) = \bigoplus_{\pi\in\widehat{G}} \mathrm{Trig}_\pi(G)$, where $\mathrm{Trig}_\pi(G)$ is the span of matrix coefficients of π. For u in $\mathrm{Trig}(G)$, we let

$$\hat{u}(\pi) = \int_G u(s)\pi\left(s^{-1}\right) ds$$

which may be understood to be an element of the space of linear operators $\mathcal{L}(\mathcal{H}_\pi)$. We caution the reader that our notation differs from that in [8, (28.34)]. We shall make an identification between two linear dual spaces $\mathrm{Trig}(G)^\dagger$ and the product $\prod_{\pi\in\widehat{G}} \mathcal{L}(\mathcal{H}_\pi)$ via

$$\left\langle u, (T_\pi)_{\pi\in\widehat{G}}\right\rangle = \sum_{\pi\in\widehat{G}} d_\pi \operatorname{Tr}\left(\hat{u}(\pi)T_\pi\right). \tag{2.1}$$

It follows from the orthogonality relations between matrix coefficients that for a matrix coefficient $\pi_{\xi,\eta}(s) = (\pi(s)\eta, \xi)$

$$\left\langle \pi_{\xi,\eta}, (T_\pi)_{\pi\in\widehat{G}}\right\rangle = (T_\pi\eta, \xi).$$

In the notation of (2.1), we will write for any T in $\mathrm{Trig}(G)^\dagger$ and $\pi \in \widehat{G}$

$$\pi(T) = T_\pi \quad \text{in } \mathcal{L}(\mathcal{H}_\pi). \tag{2.2}$$

In particular, we identify G, qua evaluation functionals on $\mathrm{Trig}(G)$, with $\{(\pi(s))_{\pi\in\widehat{G}}\colon s \in G\}$. Thus we gain the Fourier inversion formula

$$\langle u, s\rangle = \sum_{\pi\in\widehat{G}} d_\pi \operatorname{Tr}\bigl(\hat{u}(\pi)\pi(s)\bigr) = u(s). \tag{2.3}$$

Moreover, by [8, (30.5)], for example

$$G \simeq \bigl\{T \in \operatorname{Trig}(G)^\dagger\colon\ \pi(T) \in \mathcal{U}(\mathcal{H}_\pi) \text{ for each } \pi \text{ in } \widehat{G}\bigr\}. \tag{2.4}$$

We note that $\operatorname{Trig}(G)^\dagger \simeq \prod_{\pi\in\widehat{G}} \mathcal{L}(\mathcal{H}_\pi)$ has an obvious product and involution which respects the formulas $\pi(TT') = \pi(T)\pi(T')$ and $\pi(T^*) = \pi(T)^*$ for π in $\widehat{G}$ and T, T' in $\operatorname{Trig}(G)^\dagger$. Moreover the action of $\operatorname{Trig}(G)^\dagger$ on $\operatorname{Trig}(G)$ given by $\widehat{T\cdot u}(\pi) = \pi(T)\hat{u}(\pi)$ for $\pi \in \widehat{G}$ satisfies $\langle T', T\cdot u\rangle = \langle T'T, u\rangle$. We note that the involution satisfies $\langle T^*, u\rangle = \overline{\langle T, u^*\rangle}$, where $u^*(s) = \overline{u(s^{-1})}$.

With the notation above we define

$$\begin{aligned} G_{\mathbb{C}} &= \bigl\{\theta \in \operatorname{Trig}(G)^\dagger\colon\ \bigl\langle\theta, uu'\bigr\rangle = \langle\theta, u\rangle\bigl\langle\theta, u'\bigr\rangle \text{ for } u, u' \text{ in } \operatorname{Trig}(G)\bigr\} \\ &= \bigl\{\theta \in \operatorname{Trig}(G)^\dagger\colon\ \theta\cdot\bigl(uu'\bigr) = (\theta\cdot u)\bigl(\theta\cdot u'\bigr) \text{ for } u, u' \text{ in } \operatorname{Trig}(G)\bigr\}, \end{aligned} \tag{2.5}$$

$$\begin{aligned} \mathfrak{g}_{\mathbb{C}} &= \left\{X \text{ in } \operatorname{Trig}(G)^\dagger\colon\ \begin{array}{l}\langle X, uu'\rangle = \langle X, u\rangle u'(e) + u(e)\langle X, u'\rangle \\ \text{for all } u, u' \text{ in } \operatorname{Trig}(G)\end{array}\right\} \\ &= \left\{X \text{ in } \operatorname{Trig}(G)^\dagger\colon\ \begin{array}{l}X\cdot(uu') = (X\cdot u)u' + u(X\cdot u') \\ \text{for all } u, u' \text{ in } \operatorname{Trig}(G)\end{array}\right\} \end{aligned} \tag{2.6}$$

and

$$\mathfrak{g} = \bigl\{X \in \mathfrak{g}_{\mathbb{C}}\colon\ X^* = -X\bigr\}$$

where the equivalent descriptions (2.5) and (2.6), of $G_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{C}}$ respectively, can be checked by straightforward calculation. We observe that it is immediate from (2.5) that $G_{\mathbb{C}}$ is closed under the product in $\operatorname{Trig}(G)^\dagger$. It is a standard fact, see [8, (30.26)] for example, that $G_{\mathbb{C}}$ is closed under inversion and hence a group. Moreover, from (2.6) it is immediate that $\mathfrak{g}_{\mathbb{C}}$ is a complex Lie algebra under the usual associative Lie bracket: $[X, X'] = XX' - X'X$. In particular $\mathfrak{g}$ is a real Lie subalgebra. It is obvious that $\operatorname{Trig}(G)^\dagger \simeq \prod_{\pi\in\widehat{G}} \mathcal{L}(\mathcal{H}_\pi)$ is closed under analytic functional calculus. Thus by standard calculation we find that for X in $\operatorname{Trig}(G)^\dagger$, $X \in \mathfrak{g}_{\mathbb{C}}$ if and only if $\exp(tX) \in G_{\mathbb{C}}$ for each t in $\mathbb{R}$; see [2, Prop. 3] (or see comment after (2.10)) for one direction, and differentiate $t \mapsto \exp(tX)\cdot(uu')$ to see the other. Moreover, by further employing (2.4), we see for X in $\operatorname{Trig}(G)^\dagger$ that $X \in \mathfrak{g}$ if and only if $\exp(tX) \in G$ for each t in $\mathbb{R}$.

We record some of the basic properties of the group $G_{\mathbb{C}}$ and the Lie algebras $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}$.

Proposition 2.1. (i) *$G_{\mathbb{C}}$ admits polar decomposition: each θ in $G_{\mathbb{C}}$ can be written uniquely as $\theta = s|\theta|$, i.e. $\pi(\theta) = \pi(s)|\pi(\theta)|$ for each π in $\widehat{G}$. Hence, each such $|\theta|$ is an element of $G_{\mathbb{C}}$.*

(ii) *If $\theta \in G_{\mathbb{C}}$, then $\theta^* \in G_{\mathbb{C}}$ too. If $\theta \in G_{\mathbb{C}}^+ = \{T \in G_{\mathbb{C}}\colon \pi(T) \geqslant 0 \text{ for } \pi \in \widehat{G}\}$, then for each $z \in \mathbb{C}$, $\theta^z \in G_{\mathbb{C}}$ too; moreover if $t \in \mathbb{R}^{\geqslant 0}$, then $\theta^t \in G_{\mathbb{C}}^+$.*

(iii) *If the connected component of the identity G_e is a Lie group, then $\mathfrak{g}$ is isomorphic to the usual Lie algebra of G and $\mathfrak{g}_{\mathbb{C}}$ is its complexification.*

(iv) *We have* $\exp(\mathfrak{g}) \subset G_e$ *and is dense, and* $\exp(i\mathfrak{g}) = G_{\mathbb{C}}^+$. *The map $(s, X) \mapsto s\exp(iX)$: $G \times \mathfrak{g} \to G_{\mathbb{C}}$ is a homeomorphism, where $\mathfrak{g}$ and $G_{\mathbb{C}}$ have relativised topologies as subsets of* $\operatorname{Trig}(G)^\dagger$ *whose topology is the weak topology induced by* (2.1).

Proof. Part (i) and the second part of (ii) can be found in [15, Cor. 1 & Thm. 2]. We note that the proof can be conceptualised a bit differently, the ideas of which we sketch below.

It is well known that $\widehat{G \times G}$ is given by Kronecker products $\{\pi \times \pi' \colon \pi, \pi' \in \widehat{G}\}$, and hence $\mathrm{Trig}(G \times G) \simeq \mathrm{Trig}(G) \otimes \mathrm{Trig}(G)$. We let $m : \mathrm{Trig}(G \times G) \to \mathrm{Trig}(G)$ denote pointwise multiplication and

$$m^\dagger : \mathrm{Trig}(G)^\dagger \to \mathrm{Trig}(G \times G)^\dagger \simeq \prod_{\pi, \pi' \in \widehat{G}} \mathcal{L}(\mathcal{H}_\pi \otimes \mathcal{H}_{\pi'})$$

denote its adjoint map. For each π, π' in $\widehat{G}$ and each σ in $\widehat{G}$ there are $m(\sigma, \pi \otimes \pi')$ (this number may be 0) partial isometries $V_{\sigma,i} : \mathcal{H}_\sigma \to \mathcal{H}_\pi \otimes \mathcal{H}_{\pi'}$ with pairwise disjoint ranges, for which

$$\pi \otimes \pi' = \sum_{\sigma \in \widehat{G}} \sum_{i=1}^{m(\sigma, \pi \otimes \pi')} V_{\sigma,i} \sigma(\cdot) V_{\sigma,i}^* \tag{2.7}$$

where we adopt the convention that an empty sum is 0. Then we may calculate for T in $\mathrm{Trig}(G)^\dagger$ that

$$\pi \times \pi'\bigl(m^\dagger T\bigr) = \sum_{\sigma \in \widehat{G}} \sum_{i=1}^{m(\sigma, \pi \otimes \pi')} V_{\sigma,i} \sigma(T) V_{\sigma,i}^*. \tag{2.8}$$

It follows readily that $m^\dagger$ is a $*$-homomorphism; in fact it is the well-known coproduct map, see [20, Ex. 1.2.5]. In particular, it is easy to calculate from the definitions of $G_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{C}}$ that

$$G_{\mathbb{C}} = \bigl\{\theta \in \mathrm{Trig}(G)^\dagger \colon \theta \otimes \theta = m^\dagger(\theta)\bigr\}, \tag{2.9}$$

$$\mathfrak{g}_{\mathbb{C}} = \bigl\{X \in \mathrm{Trig}(G)^\dagger \colon X \otimes I + I \otimes X = m^\dagger(X)\bigr\}, \tag{2.10}$$

where I is the identity element of $\mathrm{Trig}(G)^\dagger$. We note that it is easy to check that exponentiating elements of $\mathfrak{g}_{\mathbb{C}}$ is (2.10) gives elements of the form (2.9).

Now (i) follows from (2.8), (2.9), (2.4) and the uniqueness of polar decomposition. Any multiplicative analytic function $\psi : \mathbb{R}^{>0} \to \mathbb{C}^{\neq 0}$ (resp. multiplicative anti-analytic function $\psi : \mathbb{C} \to \mathbb{C}$) will thus satisfy

$$\psi(\theta) \otimes \psi(\theta) = \psi(\theta \otimes I\, I \otimes \theta) = \psi(\theta \otimes \theta) = \psi\bigl(m^\dagger(\theta)\bigr) = m^\dagger\bigl(\psi(\theta)\bigr)$$

for $\theta \in G_{\mathbb{C}}^+$ (resp. $\theta \in G_{\mathbb{C}}$). Hence, again by (2.9), $\psi(\theta) \in G_{\mathbb{C}}$; moreover $\psi(\theta) \in G_{\mathbb{C}}^+$ if $\psi(\mathbb{R}^{>0}) \subset \mathbb{R}^{>0}$. Thus we obtain (ii).

Part (iii) is [2, Cor. 4], while part (iv) is [2, Prop. 4] (see also [15, Thm. 3]). □

If H is another compact group, and $\sigma : G \to H$ is a continuous homomorphism then σ induces a $*$-homomorphism $\sigma : \mathrm{Trig}(G)^\dagger \to \mathrm{Trig}(H)^\dagger$. Indeed, if we assign $\mathrm{Trig}(G)^\dagger \simeq \prod_{\pi \in \widehat{G}} \mathcal{L}(\mathcal{H}_\pi)$ the linear topology from the dual pairing (2.1) then $\mathrm{span}(G)$ is dense in $\mathrm{Trig}(G)^\dagger$, since $\mathrm{span}(\pi(G)) = \mathcal{L}(\mathcal{H}_\pi)$ for each π by Schur's lemma. The map $\sum_{j=1}^n \alpha_j s_j \mapsto \sum_{j=1}^n \alpha_j \sigma(s_j) : \mathrm{span}(G) \to \mathrm{span}(H)$ is well defined, being the relative adjoint of $u \mapsto u \circ \sigma :$

$\mathrm{Trig}(G) \to \mathrm{Trig}(H)$, and is clearly a $*$-homomorphism. Hence this map extends as claimed. It follows that $\sigma|_{G_{\mathbb{C}}} : G_{\mathbb{C}} \to H_{\mathbb{C}}$ is a homomorphism of groups, while $\sigma|_{\mathfrak{g}_{\mathbb{C}}} : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{h}_{\mathbb{C}}$ is a homomorphism of $\mathbb{C}$-Lie algebras, and $\sigma|_{\mathfrak{g}} : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of $\mathbb{R}$-Lie algebras, where $\mathfrak{h}$ is the Lie algebra of H. Moreover it is immediate that $\exp(\sigma(X)) = \sigma(\exp(X))$ for X in $\mathfrak{g}_{\mathbb{C}}$. Finally, if K is a third compact group and $\tau : H \to K$ is a continuous homomorphism, then $\tau \circ \sigma : G \to K$ extends to a $*$-homomorphism $\tau \circ \sigma : \mathrm{Trig}(G)^\dagger \to \mathrm{Trig}(K)^\dagger$ and restricts to a group, respectively Lie algebra, homomorphism where appropriate.

If $\pi \in \widehat{G}$, we denote the linear dual space of $\mathcal{H}_\pi$ by $\mathcal{H}_{\bar{\pi}}$. For A in $\mathcal{L}(\mathcal{H}_\pi)$, let A^{t} in $\mathcal{L}(\mathcal{H}_{\bar{\pi}})$ denote its linear adjoint. For $s \in G$ we define $\bar{\pi}(s) = \pi(s^{-1})^t$. Then $\bar{\pi}$ is a unitary representation on $\mathcal{H}_{\bar{\pi}}$ called the conjugate representation of π.

Corollary 2.2. *For any θ in $G_{\mathbb{C}}^+$ and π in $\widehat{G}$ we have $\bar{\pi}(\theta) = \pi(\theta^{-1})^{\mathrm{t}}$.*

Proof. Let $\mathcal{U}(\mathcal{H}_\pi)$ denote the unitary group on $\mathcal{H}_\pi$ and $\mathfrak{u}(\mathcal{H}_\pi)$ its Lie algebra, which we may regard in the classical sense by (iii) of the proposition, above. Let $\gamma : \mathcal{U}(\mathcal{H}_\pi) \to \mathcal{U}(\mathcal{H}_{\bar{\pi}})$ be given by $\gamma(u) = (u^{-1})^{\mathrm{t}}$. The map $U \mapsto -U^{\mathrm{t}} : \mathfrak{u}(\mathcal{H}_\pi) \to \mathfrak{u}(\mathcal{H}_{\bar{\pi}})$ is a Lie algebra homomorphism. Moreover, if we write $U = iH$ where H is hermitian we have

$$\exp\bigl(-U^{\mathrm{t}}\bigr) = \exp\bigl(-iH^{\mathrm{t}}\bigr) = \bigl(\exp(iH)^{-1}\bigr)^{\mathrm{t}} = \gamma\bigl(\exp(U)\bigr).$$

We see that $d\gamma(U) = -U^{\mathrm{t}}$, $U \in \mathfrak{u}(\mathcal{H}_\pi)$ and hence its unique $\mathbb{C}$-linear extension to the complexification satisfies $d\gamma(X) = -X^{\mathrm{t}}$ for $X \in \mathfrak{u}(\mathcal{H}_\pi)_{\mathbb{C}}$.

We shall regard $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ as a homomorphism, as above, so $\bar{\pi} = \gamma \circ \pi$. If $\theta \in G_{\mathbb{C}}^+$, we use (iv) in the proposition above to write $\theta = \exp(iX)$ where $X \in \mathfrak{g}$ (so iX is hermitian). Thus we compute

$$\bar{\pi}(\theta) = \exp\bigl(i\bar{\pi}(X)\bigr) = \exp\bigl(i\gamma \circ \pi(X)\bigr) = \exp\bigl(-i\pi(X)^{\mathrm{t}}\bigr) = \pi\bigl(\theta^{-1}\bigr)^{\mathrm{t}}$$

as desired. $\square$

3. Beurling–Fourier algebras. Definition

Let G be a compact group. A *weight* on $\widehat{G}$ is a function $\omega : \widehat{G} \to \mathbb{R}^{>0}$ such that

$$\omega(\sigma) \leqslant \omega(\pi)\omega\bigl(\pi'\bigr) \quad \text{whenever } \sigma \subset \pi \otimes \pi'.$$

We say that ω is *bounded* if $\inf_{\pi \in \widehat{G}} \omega(\pi) > 0$, and *symmetric* if $\omega(\bar{\pi}) = \omega(\pi)$ for each π in $\widehat{G}$. Note that since $1 \subset \pi \otimes \bar{\pi}$, any symmetric weight is automatically bounded with $\omega(\pi)^2 = \omega(\pi)\omega(\bar{\pi}) \geqslant \omega(1)$. Of course, $\omega(1) \geqslant 1$ since $1 \otimes \pi = \pi$ for each π.

We let $\mathrm{Trig}((G)) = \prod_{\pi \in \widehat{G}} \mathrm{Trig}_\pi(G)$ denote the space of formal trigonometric series. For $u \in \mathrm{Trig}((G))$ and π in $\widehat{G}$, the definition $\hat{u}(\pi)$ can be regarded as a formal integral. If ω is a weight on $\widehat{G}$, we define

$$A_\omega(G) = \Biggl\{u \in \mathrm{Trig}((G))\colon\ \|u\|_{A_\omega} = \sum_{\pi \in \widehat{G}} \bigl\|\hat{u}(\pi)\bigr\|_1 d_\pi \omega(\pi) < \infty\Biggr\},$$

where, for each π in $\widehat{G}$, $d_\pi = \dim(\mathcal{H}_\pi)$. It is obvious that $A_\omega(G)$ is a Banach space with norm $\|\cdot\|_{A_\omega}$. We call $A_\omega(G)$ the *Beurling–Fourier algebra* with weight ω. We will see that it is a Banach algebra, below. Note that if ω is the constant weight 1, then $A_1(G) = A(G)$ is the Fourier algebra of G.

Proposition 3.1. *For any weight ω, $A_\omega(G)$ is a Banach algebra under the product extending pointwise multiplication on* $\mathrm{Trig}(G)$. *If ω is bounded, then $A_\omega(G) \subset A(G)$, and hence is an algebra of continuous functions on G.*

Proof. It is clear that $\mathrm{Trig}(G)$ is $\|\cdot\|_{A_\omega}$-dense in $A_\omega(G)$, hence it suffices to verify that $\|\cdot\|_{A_\omega}$ is an algebra norm on $\mathrm{Trig}(G)$. Let us first consider a pair of basic coefficients, $u = (\pi(\cdot)\xi, \eta)$ and $u' = (\pi'(\cdot)\xi', \eta')$, where $\pi, \pi' \in \widehat{G}$, $\xi, \eta \in \mathcal{H}_\pi$, and $\xi', \eta' \in \mathcal{H}_{\pi'}$. Note that by the Schur orthogonality relations

$$\hat{u}(\delta) = \begin{cases} 0 & \text{if } \delta \neq \pi, \\ \frac{1}{d_\pi} T_{\eta,\xi} & \text{if } \delta = \pi, \end{cases}$$

where $T_{\eta,\xi}$ is the rank one operator $h \mapsto (h, \eta)\xi$. Therefore

$$\big\| \big(\pi(\cdot)\xi, \eta\big) \big\|_{A_\omega} = \|T_{\eta,\xi}\|_1 \omega(\pi)$$

and hence

$$\big\| \big(\pi(\cdot)\xi, \eta\big) \big\|_{A_\omega} = \|\xi\| \|\eta\| \omega(\pi).$$

Similar holds for any basic matrix coefficient. There exist, not necessarily distinct, $\sigma_1, \ldots, \sigma_m$ in $\widehat{G}$ such that $V\pi \otimes \pi'(\cdot)V^* = \bigoplus_{j=1}^m \sigma_j(\cdot)$ for some unitary operator V, and, with those, associated pairwise orthogonal projections onto the reducing subspaces, $p_1, \ldots, p_m$ on $V(\mathcal{H}_\pi \otimes \mathcal{H}_{\pi'})$. Let for each i, $V_i = p_i V$ (so $V_i = V^*_{\sigma_i, j}$ in the notation of (2.7)). Hence we have

$$\begin{aligned} uu' &= \big(\pi \otimes \pi'(\cdot)\xi \otimes \xi', \eta \otimes \eta'\big) \\ &= \left(\bigoplus_{j=1}^m \sigma_j(\cdot) \sum_{j'=1}^m V_{j'}\big(\xi \otimes \xi'\big), \ \sum_{j''=1}^m V_{j''}\big(\eta \otimes \eta'\big) \right) \\ &= \sum_{j=1}^m \big(\sigma_j(\cdot) V_j\big(\xi \otimes \xi'\big), V_j\big(\eta \otimes \eta'\big)\big). \end{aligned}$$

Hence, since ω is a weight, we have

$$\begin{aligned} \|uu'\|_{A_\omega} &\leqslant \sum_{j=1}^m \big\| \big(\sigma_j(\cdot) V_j\big(\xi \otimes \xi'\big), V_j\big(\eta \otimes \eta'\big)\big) \big\|_{A_\omega} \\ &= \sum_{j=1}^m \big\| V_j\big(\xi \otimes \xi'\big) \big\| \big\| V_j\big(\eta \otimes \eta'\big) \big\| \omega(\sigma_j) \end{aligned}$$

$$\leqslant \left(\sum_{j=1}^{m} \| V_j(\xi \otimes \xi') \| \| V_j(\eta \otimes \eta') \| \right) \omega(\pi)\omega(\pi')$$

$$\leqslant \left(\sum_{j=1}^{m} \| V_j(\xi \otimes \xi') \|^2 \right)^{1/2} \left(\sum_{j=1}^{m} \| V_j(\eta \otimes \eta') \|^2 \right)^{1/2} \omega(\pi)\omega(\pi')$$

$$= \| \xi \otimes \xi' \| \| \eta \otimes \eta' \| \omega(\pi)\omega(\pi') = \|u\|_{A_\omega} \|u'\|_{A_\omega}.$$

For each π, each T in $\mathcal{L}(\mathcal{H}_\pi)$ with $\|T\|_1 = 1$ is in the convex hull of rank one operators of norm one. Let u in $\mathrm{Trig}(G)$, $\|u\|_{A_\omega} = 1$. Then

$$u(s) = \sum_{\pi \in \widehat{G}} \mathrm{Tr}(\pi(s)\hat{u}(\pi)) d_\pi \quad \text{and} \quad \sum_{\pi \in \widehat{G}} \|\hat{u}(\pi)\|_1 \omega(\pi) d_\pi = 1.$$

Denoting by $T_{\xi,\eta}$ the rank one operator $T_{\xi,\eta} : h \mapsto (h, \xi)\eta$, we have for each $\pi \in \widehat{G}$, $\hat{u}(\pi) = \|\hat{u}(\pi)\|_1 \sum_i \alpha_i^\pi T_{\xi_i^\pi, \eta_i^\pi}$, where $\xi_i^\pi, \eta_i^\pi \in \mathcal{H}_\pi$, $\|\xi_i^\pi\| \|\eta_i^\pi\| = 1$, $\alpha_i^\pi > 0$ for each i and $\sum_i \alpha_i^\pi = 1$. Therefore

$$u(s) = \sum_{\pi \in \widehat{G}} \mathrm{Tr}(\pi(s)\hat{u}(\pi)) d_\pi = \sum_{\pi \in \widehat{G}} \sum_i \alpha_i^\pi \pi_{\xi_i^\pi, \eta_i^\pi}(s) \|\hat{u}(\pi)\|_1 d_\pi.$$

As $\sum_{\pi \in \widehat{G}} \sum_i \alpha_i^\pi \|\hat{u}(\pi)\|_1 \omega(\pi) d_\pi = 1$ we obtain that u is in the convex hull of

$$\{(\pi(\cdot)\xi, \eta): \pi \in \widehat{G},\ \xi, \eta \in \mathcal{H}_\pi,\ \|\xi\| \|\eta\| = 1/\omega(\pi)\}.$$

Now let $u, u' \in A_\omega(G)$, $\|u\|_{A_\omega} = \|u'\|_{A_\omega} = 1$. By the above arguments, u and u' are convex linear combinations $\sum_i \alpha_i u_i$ and $\sum_i \beta_i u_i'$ of matrix coefficients u_i, u_i' of A_ω-norm one. Hence

$$\|uu'\|_{A_\omega} \leqslant \sum_{i,j} \alpha_i \beta_j \|u_i u_j'\|_{A_\omega} \leqslant \sum_{i,j} \alpha_i \beta_j \|u_i\|_{A_\omega} \|u_j'\|_{A_\omega} = 1 = \|u\|_{A_\omega} \|u'\|_{A_\omega},$$

giving sub-multiplicativity of $\|\cdot\|_{A_\omega}$.

Now if ω is bounded, with $C = \inf_{\pi \in \widehat{G}} \omega(\pi) > 0$, then we have for u in $\mathrm{Trig}_\pi(G)$ that u is contained in $A(G)$ and that

$$\|u\|_A = \sum_{\pi \in \widehat{G}} \|\hat{u}(\pi)\|_1 d_\pi \leqslant \sum_{\pi \in \widehat{G}} \|\hat{u}(\pi)\|_1 \frac{1}{C} \omega(\pi) d_\pi = \frac{1}{C} \|u\|_{A_\omega}. \tag{3.1}$$

Thus $A_\omega(G)$ is contained in $A(G)$ and is therefore a space of continuous functions. □

Example 3.1. (1) If $\omega \equiv 1$ then $A_\omega(G) = A(G)$, the Fourier algebra of G.

(2) If $\omega(\pi) = d_\pi$, $\pi \in \widehat{G}$ (we say that ω is the dimension weight) then $A_\omega(G) = A_\gamma(G)$, an algebra studied by B.E. Johnson [7] and which is the image of the map from $A(G) \otimes^\gamma A(G)$ to $A(G)$ given on elementary tensors by $f \otimes g \mapsto f * \check{g}$, where $\check{g}(t) = g(t^{-1})$; or, as shown in [6], is the image of the map $f \otimes g \mapsto f * g$ from $A(G \times G)$ to $A(G)$.

(3) If $G = \mathbb{T}^n$ and ω is a weight on $\widehat{\mathbb{T}^n} \simeq \mathbb{Z}^n$, then $A_\omega(G) \simeq l^1(\mathbb{Z}^n, \omega)$ is a Beurling algebra (see [17]).

Remark 3.2. For a bounded weight ω on a compact group G we can extend the Fourier inversion formula (2.3) to elements $u \in A_\omega(G)$, since then $A_\omega(G) \subset A(G)$, and series from elements of the latter are summable. That is for $u \in A_\omega(G)$ and $s \in G$ we have that

$$u(s) = \sum_{\pi \in \widehat{G}} \operatorname{Tr}\bigl(\pi(s)\hat{u}(\pi)\bigr) d_\pi.$$

Let ρ denote the right translation on $\operatorname{Trig}(G)$, i.e. for an element u in $\operatorname{Trig}(G)$ we let

$$\rho(t)u(s) := u(st), \quad s, t \in G,$$

and let λ be the left translation on $\operatorname{Trig}(G)$ defined by

$$\lambda(t)u(s) := u\bigl(t^{-1}s\bigr), \quad s, t \in G.$$

Proposition 3.3. *For every weight ω on $\widehat{G}$, right and left translations extend to isometries of $A_\omega(G)$.*

Proof. Indeed, for an element u in $\operatorname{Trig}(G)$ and t in G we have that

$$\widehat{\bigl(\rho(t)u\bigr)}(\pi) = \pi(t)\widehat{u}(\pi), \qquad \widehat{\bigl(\lambda(t)u\bigr)}(\pi) = \widehat{u}(\pi)\pi\bigl(t^{-1}\bigr), \quad \pi \in \widehat{G}$$

and therefore

$$\bigl\|\rho(t)u\bigr\|_{A_\omega} = \sum_{\pi \in \widehat{G}} \bigl\|\pi(t)\widehat{u}(\pi)\bigr\|_1 d_\pi \omega(\pi) = \sum_{\pi \in \widehat{G}} \bigl\|\widehat{u}(\pi)\bigr\|_1 d_\pi \omega(\pi) = \|u\|_{A_\omega}$$

and similarly

$$\bigl\|\lambda(t)u\bigr\|_{A_\omega} = \|u\|_{A_\omega}. \qquad \square$$

Let for a bounded weight ω on $\widehat{G}$

$$L^2_\omega(G) = \Bigl\{\xi \in \operatorname{Trig}((G)):\ \|\xi\|^2_{2,\omega} := \sum_{\pi \in \widehat{G}} \bigl\|\hat{\xi}(\pi)\bigr\|^2_2 d_\pi \omega(\pi) < \infty\Bigr\},$$

where $\|\hat{\xi}(\pi)\|_2$ denotes the Hilbert–Schmidt norm of the operator $\hat{\xi}(\pi)$. The assumption that ω is bounded gives us that $L^2_\omega(G) \subset L^2(G)$. Note also that if for $\xi \in L_2(G)$ we let $\check{\xi}(s) := \xi(s^{-1})$ then $\|\hat{\check{\xi}}(\pi)\|_2 = \|\hat{\xi}(\bar{\pi})\|_2$ and hence for symmetric weight ω, $\xi \in L^2_\omega(G)$ if and only if $\check{\xi} \in L^2_\omega(G)$.

It is well known that the Fourier algebra $A(G)$ coincides with the family of functions

$$A(G) = \Bigl\{s \mapsto (f * g)(s) = \int_G f(t)g\bigl(t^{-1}s\bigr)\,dt:\ f, g \in L^2(G)\Bigr\}.$$

We observe that the product $\omega_1\omega_2$, of two weights ω_1, ω_2, is a weight; and if $\varphi : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ is a non-decreasing sub-multiplicative function, then $\varphi \circ \omega$ is a weight for any weight ω.

Proposition 3.4. *Let ω_1, ω_2 be bounded weights on $\widehat{G}$ and $\omega = (\omega_1\omega_2)^{1/2}$. Then*

$$A_\omega(G) = \left\{ s \mapsto (f * g)(s) = \int_G f(t) g\big(t^{-1}s\big)\,dt\colon\ f \in L^2_{\omega_1}(G) \text{ and } g \in L^2_{\omega_2}(G) \right\}.$$

Proof. Let $u \in A_\omega(G)$. For each $\pi \in \widehat{G}$ consider the polar decomposition of $\hat{u}(\pi)$: $\hat{u}(\pi) = V(\pi)|\hat{u}(\pi)|$. Let $a(\pi) = (\frac{\omega_1(\pi)}{\omega(\pi)})^{1/2} V(\pi)|\hat{u}(\pi)|^{1/2}$ and $b(\pi) = (\frac{\omega_2(\pi)}{\omega(\pi)})^{1/2}|\hat{u}(\pi)|^{1/2}$. We have

$$\sum_{\pi \in \widehat{G}} d_\pi \omega(\pi) \big\| a(\pi) \big\|_2^2 \leqslant \sum_{\pi \in \widehat{G}} d_\pi \omega_1(\pi) \big\| \big|\hat{u}(\pi)\big|^{1/2} \big\|_2^2 = \sum_{\pi \in \widehat{G}} d_\pi \omega_1(\pi) \big\| \hat{u}(\pi) \big\|_1 < \infty$$

and similarly

$$\sum_{\pi \in \widehat{G}} d_\pi \omega(\pi) \big\| b(\pi) \big\|_2^2 \leqslant \sum_{\pi \in \widehat{G}} d_\pi \omega_2(\pi) \big\| \hat{u}(\pi) \big\|_1 < \infty.$$

Thus if $f, g \in L^2(G)$ are such that $\hat{g}(\pi) = a(\pi)$ and $\hat{f}(\pi) = b(\pi)$ then $f \in L^2_{\omega_1}(G)$, $g \in L^2_{\omega_2}(G)$ and $u = f * g$. Hence $A_\omega(G) \subset L^2_{\omega_1}(G) * L^2_{\omega_2}(G)$.

Take now $f \in L^2_{\omega_1}(G)$, $g \in L^2_{\omega_2}(G)$ and let $u = f * g$. Then

$$\begin{aligned}
\sum_{\pi \in \widehat{G}} d_\pi \omega(\pi) \big\| \hat{u}(\pi) \big\|_1 &= \sum_{\pi \in \widehat{G}} d_\pi \omega(\pi) \big\| \hat{g}(\pi) \hat{f}(\pi) \big\|_1 \\
&\leqslant \sum_{\pi \in \widehat{G}} d_\pi \omega_1(\pi)^{1/2} \omega_2(\pi)^{1/2} \big\| \hat{g}(\pi) \big\|_2 \big\| \hat{f}(\pi) \big\|_2 \\
&\leqslant \left(\sum_{\pi \in \widehat{G}} d_\pi \omega_1(\pi) \big\| \hat{g}(\pi) \big\|_2^2 \right)^{1/2} \left(\sum_{\pi \in \widehat{G}} d_\pi \omega_2(\pi) \big\| \hat{f}(\pi) \big\|_2^2 \right)^{1/2} \\
&= \|f\|_{2,\omega_1} \|g\|_{2,\omega_2}.
\end{aligned}$$

Thus $u \in A_\omega(G)$. $\square$

4. The spectrum of a Beurling–Fourier algebra

It follows from the identification $\mathrm{Trig}(G)^\dagger \simeq \prod_{\pi \in \widehat{G}} \mathcal{L}(\mathcal{H}_\pi)$ given in (2.1) that for a weight ω, $A_\omega(G)$ has continuous dual space

$$A_\omega(G)^* = \left\{ T \in \mathrm{Trig}(G)^\dagger\colon\ \|T\|_{A^*_\omega} = \sup_{\pi \in \widehat{G}} \frac{\|\pi(T)\|_{\mathrm{op}}}{\omega(\pi)} < \infty \right\}. \tag{4.1}$$

The definition of $G_{\mathbb{C}}$ then immediately gives the Gelfand spectrum

$$G_\omega = \widehat{A_\omega(G)} = \left\{\theta \in G_{\mathbb{C}}\colon \sup_{\pi\in\widehat{G}} \frac{\|\pi(\theta)\|_{\mathrm{op}}}{\omega(\pi)} < \infty\right\} \tag{4.2}$$

where $< \infty$ may be replaced by $\leqslant 1$, by a well-known theorem of Gelfand that multiplicative functionals on Banach algebras are automatically contractive. Proposition 2.1 provides that $G_{\mathbb{C}}$ is closed under polar decomposition, and that for θ in $G_{\mathbb{C}}$ and $\pi \in \widehat{G}$, $\|\pi(\theta)\|_{\mathrm{op}} = \||\pi(\theta)|\|_{\mathrm{op}} = r(|\pi(\theta)|)$, where the latter is the spectral radius.

The terminology below is motivated by [12].

Proposition 4.1. *If ω is a bounded weight on $\widehat{G}$ then G_ω is a compact subset of $G_{\mathbb{C}}$ which contains G and satisfies the following properties*:

(i) *G_ω is G-*Reinhardt: *for s in G and $\theta \in G_\omega$ we have $s\theta, \theta s \in G_\omega$; and*
(ii) *G_ω is* log-convex: *for $\theta, \theta' \in G_\omega^+ = G_\omega \cap G_{\mathbb{C}}^+$ and $0 \leqslant s \leqslant 1$, we have $\theta^s\theta'^{(1-s)} \in G_\omega$.*

In particular, $(\theta, s) \mapsto \theta s : G_\omega^+ \times G \to G_\omega$ is a homeomorphism.

(iii) *If ω is symmetric then G_ω is inverse-closed.*

Proof. (i) This is immediate from Proposition 3.3 and the fact that operators of translation are multiplicative.

(ii) It follows from Proposition 2.1 that $\theta^s\theta'^{(1-s)} \in G_{\mathbb{C}}$ (though not necessarily in $G_{\mathbb{C}}^+$). It is a standard fact of functional calculus that $\|\pi(\theta)^s\|_{\mathrm{op}} = \|\pi(\theta)\|_{\mathrm{op}}^s$ for each π in $\widehat{G}$, and hence we have

$$\begin{aligned}\sup_{\pi\in\widehat{G}} \frac{\|\pi(\theta)^s\pi(\theta')^{(1-s)}\|_{\mathrm{op}}}{\omega(\pi)} &\leqslant \sup_{\pi\in\widehat{G}} \frac{\|\pi(\theta)\|_{\mathrm{op}}^s\|\pi(\theta')\|_{\mathrm{op}}^{1-s}}{\omega(\pi)} \\ &\leqslant \sup_{\pi\in\widehat{G}} \frac{\max(\|\pi(\theta)\|_{\mathrm{op}}, \|\pi(\theta')\|_{\mathrm{op}})}{\omega(\pi)} < \infty.\end{aligned}$$

It is immediate from (4.2) that $\theta^s\theta'^{(1-s)} \in G_\omega$.

By (i) and (ii) the map $(\theta, s) \mapsto \theta s$ is a bijection between $G_\omega^+ \times G$ and G_ω, and it follows from Proposition 2.1(iv) that this map is a homeomorphism.

(iii) If ω is symmetric, then for θ in $G_{\mathbb{C}}$, Corollary 2.2 shows that

$$\sup_{\pi\in\widehat{G}} \frac{\|\pi(\theta^{-1})\|_{\mathrm{op}}}{\omega(\pi)} = \sup_{\pi\in\widehat{G}} \frac{\|\bar{\pi}(\theta)\|_{\mathrm{op}}}{\omega(\pi)},$$

and hence $\theta^{-1} \in G_\omega$ if and only if $\theta \in G_\omega$. $\square$

If ω is bounded, then it is clear from Proposition 3.1 that $A_\omega(G)$ is semisimple. If ω is not bounded this is not as clear, but still true. The following is motivated by [10, 2.8.2].

Theorem 4.2. *The algebra $A_\omega(G)$ is semisimple.*

Proof. We first note that the constant function 1 is in $A_\omega(G)$. Hence the spectrum G_ω is non-empty. Let $\theta \in G_\omega$.

We note that if $u \in A_\omega(G)$ and $T \in A_\omega(G)^*$ then $T \cdot u \in A(G)$. Indeed

$$\|T \cdot u\|_A = \sum_{\pi \in \widehat{G}} \big\|\pi(T)\hat{u}(\pi)\big\|_1 d_\pi \leqslant \sum_{\pi \in \widehat{G}} \big\|\pi(T)\big\|_{\mathrm{op}} \big\|\hat{u}(\pi)\big\|_1 d_\pi$$
$$\leqslant \sum_{\pi \in \widehat{G}} \frac{\|\pi(T)\|_{\mathrm{op}}}{\omega(\pi)} \big\|\hat{u}(\pi)\big\|_1 d_\pi \omega(\pi) \leqslant \|T\|_{A_\omega^*} \|u\|_{A_\omega} < \infty.$$

Thus for θ in G_ω, as above, $\theta \cdot u \in A(G)$ for u in $A_\omega(G)$. Suppose $u \neq 0$. Since $u = \theta^{-1} \cdot (\theta \cdot u)$ (formally, in $\mathrm{Trig}((G))$), we have that $\theta \cdot u \neq 0$ in $A(G)$, and hence there is some s in G for which

$$\langle u, s\theta\rangle = \theta \cdot u(s) \neq 0.$$

Since $s\theta \in G_\omega$ for all s in G by Proposition 4.1, it follows that $A_\omega(G)$ admits no radical elements, and thus is semisimple. □

The following fact, which will be useful for the following examples, is well known. If $\tilde{\omega} : \mathbb{N} \to \mathbb{R}^{>0}$ is a *weight*, i.e. $\tilde{\omega}(n+m) \leqslant \tilde{\omega}(n)\tilde{\omega}(m)$, then

$$\rho_{\tilde{\omega}} = \lim_{n \to \infty} \tilde{\omega}(n)^{1/n} = \inf_{n \in \mathbb{N}} \tilde{\omega}(n)^{1/n}. \tag{4.3}$$

See, for example, [3, A.1.26]. Furthermore, if $\tilde{\omega}$ is bounded below, then $\rho_{\tilde{\omega}} \geqslant 1$.

Example 4.3. Let $G = \mathbb{T}^n$ and ω be a bounded weight on $\mathbb{Z}^n \simeq \widehat{G}$. Let

$$\ell^1\big(\mathbb{Z}^n, \omega\big) = \left\{ f : \mathbb{Z}^n \to \mathbb{C}\colon \|f\|_{1,\omega} = \sum_{\mu \in \mathbb{Z}^n} \big|f(\mu)\big|\omega(\mu) < \infty \right\}$$

with convolution as multiplication. The Fourier transform identifies $\ell^1(\mathbb{Z}^n, \omega)$ with $A_\omega(\mathbb{T}^n)$ (formally, in the case that ω is not bounded). Let $\varepsilon_1, \ldots, \varepsilon_n$ denote the standard integer basis of $\mathbb{Z}^n$. Since $\delta_{\varepsilon_1}, \ldots, \delta_{\varepsilon_n}$ and their inverses generate $l_1(\mathbb{Z}^n, \omega)$, any character $\chi \in \widehat{\ell^1(\mathbb{Z}^n, \omega)} \simeq \mathbb{T}^n_\omega$ is determined by the values $z_j = \chi(\delta_{\varepsilon_j})$, $j = 1, \ldots, n$. Hence the Gelfand transform converts f in $\ell^1(\mathbb{Z}^n, \omega)$ into a Laurent series

$$\sum_{\mu \in \mathbb{Z}^n} f(\mu) z^\mu \quad \big(z^\mu = z_1^{\mu_1} \ldots z_n^{\mu_n}\big).$$

These series converge, simultaneously for all f in $\ell^1(\mathbb{Z}^n, \omega)$, if and only if $|z^\mu|^k = |z^{k\mu}| \leqslant \omega(k\mu)$ for each μ in $\mathbb{Z}^n$. Thus it follows from an application of (4.3) that

$$\mathbb{T}^n_\omega \simeq \big\{ z \in \mathbb{C}^n\colon 1/\rho_\omega(-\mu) \leqslant \big|z^\mu\big| \leqslant \rho_\omega(\mu) \text{ for all } \mu \in \mathbb{Z}^n \big\}$$

where $\rho_\omega(\mu) = \lim_{k\to\infty} \omega(k\mu)^{1/k}$. If $n \geqslant 2$ the family of defining inequalities can be simplified to choices of μ for which $\gcd(\mu_1, \ldots, \mu_n) = 1$. However if $n = 1$ we obtain the usual annulus of convergence with inner radius $1/\rho_\omega(-1)$ and outer radius $\rho_\omega(1)$.

We observe that in the case that $n \geqslant 2$ and $\omega(\mu) = \lambda^{\mu_1}$ for some $\lambda > 1$, then $\mathbb{T}^n_\omega \supsetneq \mathbb{T}^n$, but is not an open subset of $\mathbb{C}^n$. For any λ in $(\mathbb{R}^{\geqslant 1})^n$ with $\lambda_1 \ldots \lambda_n > 1$, the weight $\omega(\mu) = \lambda^\mu$ defines an exponential weight.

If $\alpha > 0$, the weight $\omega_\alpha(\mu) = (1 + \|\mu\|_1)^\alpha$ is the classical polynomial weight. These weights will be generalised in Section 5.

Example 4.4. Let $G = \mathcal{T} \rtimes \mathbb{Z}_2 = \{(s,a)\colon\ s \in \mathcal{T},\ a \in \mathbb{Z}_2\}$ with multiplication $(s,a)(t,b) = (st^a, ab)$, where $\mathbb{Z}_2 = \{1, -1\}$. It is a straightforward application of the "Mackey machine" that $\widehat{G} = \{1, \sigma, \pi_n,\ n \geqslant 1\}$, where σ is a one-dimensional representation given by $\sigma((s,a)) = a$ and π_n, $n \geqslant 1$ is a two-dimensional representation defined by

$$\pi_n((s,a)) = \begin{pmatrix} s^n & 0 \\ 0 & s^{-n} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{(1-a)/2}.$$

We have that the corresponding characters are

$$\chi_\sigma((s,a)) = a, \qquad \chi_{\pi_n}((s,a)) = \begin{cases} s^n + s^{-n}, & a = 1, \\ 0, & a = -1. \end{cases}$$

Taking into account that $\chi_{\pi\otimes\rho} = \chi_\pi \chi_\rho$, we obtain that $1 \otimes \sigma = \sigma$, $1 \otimes \pi_n = \pi_n$, $\sigma \otimes \sigma = 1$, $\sigma \otimes \pi_n \approx \pi_n$, $\pi_n \otimes \pi_m \approx \pi_{m+n} \oplus \pi_{|m-n|}$ if $m \neq n$ and $\pi_n \otimes \pi_n \approx \pi_{2n} \oplus 1 \oplus 1$. Hence any weight ω on $\widehat{G}$ is governed only by the relations that $\omega(1), \omega(\sigma) \geqslant 1$, and $\tilde{\omega}(n) = \omega(\pi_n)$ defines a weight on $\{0\} \cup \mathbb{N}$ which satisfies $\tilde{\omega}(0) = \omega(1)$ and $\tilde{\omega}(|n-m|) \leqslant \tilde{\omega}(n)\tilde{\omega}(m)$. Since $\bar{\pi} \approx \pi$ for each π in $\widehat{G}$, any weight ω is automatically symmetric.

We observe that $G_\mathbb{C} \simeq \mathbb{C}^* \rtimes \mathbb{Z}_2$. Indeed, we appeal to [2, Prop. 9] to see that $(G_\mathbb{C})_e \simeq (G_e)_\mathbb{C} \simeq \mathbb{C}^*$, as $G_e \simeq \mathbb{T}$, and that $G_\mathbb{C}/(G_\mathbb{C})_e \simeq (G/G_e)_\mathbb{C}$, where $(G/G_e)_\mathbb{C} \simeq (\mathbb{Z}_2)_\mathbb{C} \simeq \mathbb{Z}_2$ by definition of the complexification. In particular we can identify $G^+_\mathbb{C} = \{(\lambda, 1)\colon\ \lambda > 0\}$ and we have $G_\mathbb{C} = GG^+_\mathbb{C}$. Thus for $\lambda > 0$ and s in $\mathcal{T}$, the corresponding element θ in $G_\mathbb{C}$ is given by $1(\theta) = 1$, $\sigma(\theta) = \sigma(s)$ and $\pi_n(\theta) = \pi_n(s)\Lambda(n)$, where $\Lambda(n) = \left(\begin{smallmatrix} \lambda^n & 0 \\ 0 & \lambda^{-n} \end{smallmatrix}\right)$, so $\Lambda(n) = \pi_n(|\theta|)$.

Let $\omega : \widehat{G} \to \mathbb{R}^{>0}$ be any weight and $\rho_\omega = \lim_{n\to\infty} \omega(\pi_n)^{1/n}$. For example if $\alpha > 1$, let $\omega(1) = \omega(\sigma) = 1$ and $\omega(\pi_n) = \alpha^n$, and then $\rho_\omega = \alpha$. For θ as above, $\|\pi_n(\theta)\|_{\mathrm{op}} = \|\Lambda(n)\|_{\mathrm{op}} = \max(\lambda^n, \lambda^{-n})$. For each n in $\mathbb{N}$ (4.3) implies

$$\sup_{n\in\mathbb{N}} \frac{\|\pi_n(\theta)\|_{\mathrm{op}}}{\tilde{\omega}(n)} = \sup_{n\in\mathbb{N}} \frac{\max(\lambda^n, \lambda^{-n})}{\tilde{\omega}(n)} \leqslant 1 \quad \Leftrightarrow \quad \max\bigl(\lambda, \lambda^{-1}\bigr) \leqslant \rho_\omega.$$

Hence $G_\omega \simeq \{(z,a)\colon\ a \in \mathbb{Z}_2,\ z \in \mathbb{C},\ 1/\rho_\omega \leqslant |z| \leqslant \rho_\omega\}$. In particular, if $\rho_\omega > 1$, then $G_\omega \supsetneq G$.

Example 4.5. Let $G = \mathrm{SU}(2)$. We have that $\widehat{G} = \{\pi_n\colon\ n \in \{0\} \cup \mathbb{N}\}$ where $\pi_0 = 1$ and π_1 is the standard representation. The representations satisfy the well-known tensor relations

$$\pi_n \otimes \pi_{n'} \approx \bigoplus_{j=0}^{(n+n'-|n-n'|)/2} \pi_{|n-n'|+2j}.$$

Thus any weight $\omega : \widehat{G} \to \mathbb{R}^{>0}$ is given by a weight $\tilde{\omega} : \{0\} \cup \mathbb{N} \to \mathbb{R}^{>0}$ which satisfies $\tilde{\omega}(|n-n'|+2j) \leqslant \tilde{\omega}(n)\tilde{\omega}(n')$ for $j = 0, \ldots, (n+n'-|n-n'|)/2$. For example, any exponential weight $\tilde{\omega}(n) = \lambda^n$ for some $\lambda \geqslant 1$, suffices. We have for every n that $\bar{\pi}_n \approx \pi_n$, so every weight is automatically symmetric.

It is well known that $\mathfrak{su}(2)_{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$. It follows from Proposition 2.1(iii) that $\mathrm{SL}_2(\mathbb{C}) \simeq \pi_1(i\mathfrak{su}(2))\pi_1(G) = \pi_1(G_{\mathbb{C}})$. Then, since π_1 generates $\widehat{G}$ in the sense that every irreducible representation of G is a subrepresentation of the n-fold tensor product of π_1 for some $n \in \mathbb{N}$, we find from (2.7) that $\pi_1(G_{\mathbb{C}})$ determines $G_{\mathbb{C}}$. Hence $\mathrm{SL}_2(\mathbb{C}) \simeq G_{\mathbb{C}}$.

Now given a weight ω, let $\rho_\omega = \lim_{n\to\infty} \tilde{\omega}(n)^{1/n}$. Any element of $\mathrm{SL}_2(\mathbb{C})^+$ is, up to unitary equivalence, $\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ for some $\lambda > 0$. Taking successive tensor products $\Lambda^{\otimes n} = \pi_1{}^{\otimes n}(\Lambda)$ we see by induction that $\|\pi_n(\Lambda)\|_{\mathrm{op}} = \max(\lambda^n, \lambda^{-n})$. Thus, using reasoning as in the example above, and then (4.2) and comments thereafter, we see that

$$G_\omega \simeq \bigl\{x \in \mathrm{SL}_2(\mathbb{C}) \colon \sigma\bigl(|x|\bigr) = \bigl\{\lambda, \lambda^{-1}\bigr\},\ 1/\rho_\omega \leqslant \lambda \leqslant \rho_\omega\bigr\}.$$

Example 4.6. Suppose G_e is non-trivial so $G_{\mathbb{C}} \supsetneq G$. If $\theta \in G_{\mathbb{C}} \setminus G$ let $\omega_\theta(\pi) = \|\pi(\theta)\|_{\mathrm{op}} = \|\pi(|\theta|)\|_{\mathrm{op}}$. It is immediate from (2.7) that ω_θ is a weight, and from Proposition 2.1(i) we may choose θ to be positive. It follows from Corollary 2.2 that ω_θ is symmetric if and only if $\|\pi(\theta)^{-1}\|_{\mathrm{op}} = \|\pi(\theta)\|_{\mathrm{op}}$ for each $\pi \in \widehat{G}$. For the cases in Examples 4.3, 4.4 and 4.5 above, these weights generalise the exponential weights.

We can take advantage of Proposition 4.1 to see that G_ω contains some analytic structure when it is bigger than G. We note that by Proposition 4.1 G_ω^+ is *logarithmically star-like* about e: for θ in G_ω^+ and $0 \leqslant s \leqslant 1$, $\theta^s = \theta^s e^{1-s} \in G_\omega^+$. We will call θ in $G_\omega^+ \setminus \{e\}$ a *relative interior point* of G_ω^+ if $\theta^{1+\varepsilon} \in G_\omega^+$ for some $\varepsilon > 0$. If $G_\omega \supsetneq G$, then G_ω^+ always admits relative interior points.

Theorem 4.7. *Suppose that ω is bounded and $G_\omega \supsetneq G$. Then for any relative interior point θ in $G_\omega^+ \setminus \{e\}$ there are real numbers $\alpha < \beta$ such that for every u in $A_\omega(G)$, $u_\theta(z) = \langle \theta^z, u\rangle$ defines a holomorphic function on $S_{\alpha,\beta} = \{z \in \mathbb{C} \colon \alpha < \operatorname{Re} z < \beta\}$.*

Proof. We let $\alpha = \inf\{s \in \mathbb{R} \colon \theta^s \in G_\omega\}$ and $\beta = \sup\{s \in \mathbb{R} \colon \theta^s \in G_\omega\}$. Proposition 2.1(iv) provides $X \in i\mathfrak{g}$, for which $\exp(X) = \theta$. We note for $z = s + it$ in $S_{\alpha,\beta}$, that $\theta^z = \theta^s \exp(itX) \in G_\omega$. Now, if $u \in \mathrm{Trig}_\pi(G)$ then $u_\theta(z) = \mathrm{Tr}(\hat{u}(\pi)\exp(z\pi(X)))d_\pi$ defines a holomorphic function on $S_{\alpha,\beta}$ for which

$$\sup_{z \in S_{\alpha,\beta}} \bigl|u_\theta(z)\bigr| \leqslant \sup_{\theta' \in G_\omega} \bigl|\bigl\langle \theta', u\bigr\rangle\bigr| \leqslant \|u\|_{A_\omega} = \|\hat{u}\|_1 d_\pi \omega(\pi).$$

Hence if we consider u in $A_\omega(G)$, we see that

$$u_\theta(z) = \sum_{\pi \in \widehat{G}} \mathrm{Tr}\bigl(\hat{u}(\pi)\pi(\theta)^z\bigr) d_\pi$$

converges uniformly on $S_{\alpha,\beta}$ and hence defines a holomorphic function. $\square$

Remark 4.8. We will say that the unit e of G is a relative interior point of G_ω^+ if there is θ in $G_\omega^+ \setminus \{e\}$ such that $\theta^{-\varepsilon} \in G_\omega^+$ for some $\varepsilon > 0$. We note that if ω is symmetric and $G_\omega \supsetneq G$, then it follows from Corollary 2.2 if $\theta \in G_\omega^+$ then $\theta^{-1} \in G_\omega^+$ showing that e is a relative interior point of G_ω^+. Even in the case that e is a relative interior point, the above procedure, applied to e produces only constant holomorphic functions. If θ in $G_\omega^+ \setminus \{e\}$ is a relative interior point, for which $\theta^{-\varepsilon} \in G_\omega$, the holomorphic function u_θ satisfies $u_\theta(0) = u(e)$.

Definition 4.9. An involutive Banach algebra is called *symmetric* if for every self-adjoint element u, the spectrum $\sigma(u) \subset \mathbb{R}$.

If ω is a symmetric weight then $u \mapsto \bar{u}$ defines an isometric involution on $A_\omega(G)$, where $\bar{u}(s) = \overline{u(s)}$ for s in G. Indeed, it is easy to check that for every π in $\widehat{G}$, and u in $A_\omega(G)$ that $\|\widehat{\bar{u}}(\pi)\|_1 = \|\hat{u}(\bar{\pi})\|_1$. It is then immediate from the definition of the norm that $\|\bar{u}\|_{A_\omega} = \|u\|_{A_\omega}$.

Theorem 4.10. *Let ω be a symmetric weight on G. The Beurling–Fourier algebra $A_\omega(G)$ is symmetric if and only if $G_\omega = G$.*

Proof. If $G = G_\omega$, then it is obvious that $A_\omega(G)$ is symmetric.

If $G_\omega \supsetneq G$, then by Theorem 4.7, for any relative interior point θ in $G_\omega^+ \setminus \{e\}$, the function $u \mapsto u_\theta$ is a homomorphism from $A_\omega(G)$ into $\mathrm{Hol}(S_{\alpha,\beta})$, the space of holomorphic functions on an open strip $S_{\alpha,\beta}$. Since for $z \neq 1$ but sufficiently close to 1, $\theta^z \neq \theta$, there is u in $A_\omega(G)$ for which $\langle u, \theta^z\rangle \neq \langle u, \theta\rangle$. Moreover, since $A_\omega(G)$ is generated by its self-adjoint elements, i.e. $2u = (u + \bar{u}) + (u - \bar{u})$ for each u, there must be a self-adjoint element u for which $\langle u, \theta^z\rangle \neq \langle u, \theta\rangle$ for some z. Hence u_θ is a non-constant holomorphic function, whence $u_\theta(S_{\alpha,\beta})$ is open in $\mathbb{C}$. Since $u_\theta(S_{\alpha,\beta}) \subset \sigma(u)$, the latter cannot be contained in $\mathbb{R}$. □

In the end of this section we give general conditions on weights ω for which the spectrum G_ω of $A_\omega(G)$ coincides with (is different from) G.

4.1. Some functorial properties

The Beurling–Fourier algebras admit natural Beurling–Fourier algebras when restricted to subgroups. If H is a closed subgroup of G and $\omega : \widehat{G} \to \mathbb{R}^{>0}$ then we define $\omega_H : \widehat{H} \to \mathbb{R}^{>0}$ by

$$\omega_H(\sigma) = \inf_{\substack{\pi \in \widehat{G} \\ \sigma \subset \pi|_H}} \omega(\pi).$$

Note that by [8, 27.46], ω_H is well-defined.

Proposition 4.11. *ω_H is a weight on $\widehat{H}$.*

Proof. Let $\sigma, \sigma', \tau \in \widehat{H}$ with $\tau \subset \sigma \otimes \sigma'$ and $\varepsilon > 0$ be given. Find π, π' in $\widehat{G}$ such that $\sigma \subset \pi|_H$ with $\omega(\pi) < \omega_H(\sigma) + \varepsilon$ and $\sigma' \subset \pi'|_H$ with $\omega(\pi') < \omega_H(\sigma') + \varepsilon$. Then $\tau \subset \pi \otimes \pi'|_H$ and hence

$$\omega_H(\tau) \leqslant \inf_{\substack{\rho \subset \pi \otimes \pi' \\ \tau \subset \rho|_H}} \omega(\rho) \leqslant \omega(\pi)\omega(\pi') \leqslant \big(\omega_H(\sigma) + \varepsilon\big)\big(\omega_H(\sigma') + \varepsilon\big).$$

Since $\varepsilon > 0$ can be chosen arbitrarily and independently of σ, σ', it follows that $\omega_H(\tau) \leqslant \omega_H(\sigma)\omega_H(\sigma')$. □

Let $\iota : H \to G$ denote the injection map which, as in Section 2, extends to a homomorphism $\iota : \mathrm{Trig}(H)^\dagger \to \mathrm{Trig}(G)^\dagger$.

Proposition 4.12. (i) *The "restriction" map $u \mapsto u \circ \iota$ is a Banach algebra quotient map from $A_\omega(G)$ onto $A_{\omega_H}(H)$.*

(ii) *The spectrum H_{ω_H} of $A_{\omega_H}(H)$ is homeomorphic to $G_\omega \cap \iota(H_{\mathbb{C}})$.*

Proof. (i) The extension $\iota : \mathrm{Trig}(H)^\dagger \to \mathrm{Trig}(G)^\dagger$, satisfies

$$\pi \circ \iota(T) \simeq \bigoplus_{\substack{\sigma \in \widehat{H} \\ \sigma \subset \pi|_H}} \sigma(T)^{\oplus m(\sigma,\pi)}$$

where $m(\sigma,\pi)$ is the multiplicity of σ in $\pi|_H$. We have that $\iota(A_{\omega_H}(H)^*) \subset A_\omega(G)^*$. Indeed $\sigma \subset \pi|_H$ implies $\omega_H(\sigma) \leqslant \omega(\pi)$ and hence $\frac{\|\sigma(T)\|_{\mathrm{op}}}{\omega_H(\sigma)} \geqslant \frac{\|\sigma(T)\|_{\mathrm{op}}}{\omega(\pi)}$ so we have

$$\|\iota(T)\|_{A_\omega^*} = \sup_{\pi \in \widehat{G}} \frac{\|\pi \circ \iota(T)\|_{\mathrm{op}}}{\omega(\pi)} = \sup_{\pi \in \widehat{G}} \max_{\substack{\sigma \in \widehat{H} \\ \sigma \subset \pi|_H}} \frac{\|\sigma(T)\|_{\mathrm{op}}}{\omega(\pi)} \leqslant \sup_{\sigma \in \widehat{H}} \frac{\|\sigma(T)\|_{\mathrm{op}}}{\omega_H(\sigma)}.$$

Hence $\iota|_{A_{\omega_H}(H)^*} : A_{\omega_H}(H)^* \to A_\omega(G)^*$ is contractive. Moreover, $\iota|_{A_{\omega_H}(H)^*}$ is an isometry. Indeed, given T in $A_{\omega_H}(H)^*$ and $\varepsilon > 0$, there is σ in $\widehat{H}$ such that $\frac{\|\sigma(T)\|_{\mathrm{op}}}{\omega_H(\sigma)} > \|T\|_{A_{\omega_H}^*} - \varepsilon$. Moreover there is $\pi \in \widehat{G}$ such that $\sigma \subset \pi|_H$ and $\omega(\pi) < \omega_H(\sigma) + \varepsilon$. Then

$$\|\iota(T)\|_{A_\omega} \geqslant \frac{\|\sigma(T)\|_{\mathrm{op}}}{\omega(\pi)} > \frac{\|\sigma(T)\|_{\mathrm{op}}}{\omega_H(\sigma) + \varepsilon} \geqslant \left(\|T\|_{A_{\omega_H}^*} - \varepsilon\right) \frac{1}{1 + \frac{\varepsilon}{\omega_H(\sigma)}}$$

from which it follows that $\iota|_{A_{\omega_H}(H)^*}$ is an isometry.

Since $\mathrm{Trig}(G)$ is dense in $A_\omega(G)$, it follows that the preadjoint $u \mapsto u \circ \iota = u|_H$ of $\iota|_{A_{\omega_H}(H)^*}$ extends to a quotient map from $A_\omega(G)$ onto $A_{\omega_H}(H)$.

(ii) It is noted in [2, Cor. 3] that the map $\iota|_{H_{\mathbb{C}}} : H_{\mathbb{C}} \to G_{\mathbb{C}}$ is a topological embedding. Also, $H_{\omega_H} = H_{\mathbb{C}} \cap A_{\omega_H}(H)^*$. Hence $\iota(H_{\omega_H}) = G_{\mathbb{C}} \cap A_\omega(G)^* = G_\omega$. $\square$

The connected component of the identity warrants particular consideration.

Corollary 4.13. *Let ω be a bounded weight. The connected component of G_ω containing e, $(G_\omega)_e$ is naturally isomorphic with $(G_e)_{\omega_{G_e}}$. In particular, $G_\omega \supsetneq G$ if and only if $(G_\omega)_e \supsetneq G_e$.*

Proof. It follows from Proposition 4.1(ii) that $G_\omega^+ \subset (G_\omega)_e$. The same proposition shows that $(\theta, s) \mapsto \theta s : G_\omega^+ \times G \to G_\omega$ is a homeomorphism, and hence $G_\omega^+ G_e = (G_\omega)_e$. However, since $G_{\mathbb{C}}^+ G_e = \iota(G_e)_{\mathbb{C}}$, we get, from Proposition 4.12 above, that $G_\omega^+ G_e = G_\omega \cap \iota(G_e)_{\mathbb{C}} = (G_e)_{\omega_{G_e}}$.

In particular, we have $G_\omega \supsetneq G$ if and only if $G_\omega^+ \supsetneq \{e\}$. Hence this condition is equivalent to $(G_\omega)_e \supsetneq G_e$. $\square$

Let N be a closed normal subgroup of G and $q : G \to G/N$ be the quotient map. The map $\pi \mapsto \pi \circ q : \widehat{G/N} \to \widehat{G}$ clearly preserves decomposition into irreducible components. Thus if $\omega : \widehat{G} \to \mathbb{R}^{>0}$ is a weight we may define a weight $\omega^N : \widehat{G/N} \to \mathbb{R}^{>0}$ by

$$\omega^N(\pi) = \omega(\pi \circ q).$$

As above, we let $\iota : N \to G$ denote the injection which extends naturally to a map $\iota : \mathrm{Trig}(N)^\dagger \to \mathrm{Trig}(G)^\dagger$. We note that since N is normal in G, $\mathfrak{n} = \{X \in \mathfrak{g}\colon \exp(tX) \in N \text{ for all } t \in \mathbb{R}\}$ is a Lie ideal in $\mathfrak{g}$, whence $\mathfrak{n}_{\mathbb{C}}$ is a Lie ideal in $\mathfrak{g}_{\mathbb{C}}$, from which it follows that $N_{\mathbb{C}} \cong \iota(N_{\mathbb{C}}) = N\exp(i\mathfrak{n})$ is normal in $G_{\mathbb{C}}$.

Proposition 4.14. (i) *The map $u \mapsto u \circ q : A_{\omega^N}(G/N) \to A_\omega(G)$ is an isometric homomorphism.*

(ii) *On G_ω, let $\theta \sim_{N_{\mathbb{C}}} \theta'$ if $\theta^{-1}\theta' \in \iota(N_{\mathbb{C}})$. Then the quotient space $G_\omega/N_{\mathbb{C}}$ may be identified with a closed subset of $(G/N)_{\omega^N}$.*

Proof. (i) Define $P_N u(s) = \int_N u(sn)\,dn$. Then, since by Proposition 3.3 translations are isometries on $A_\omega(G)$, we have that P_N defines a bounded linear operator on $A_\omega(G)$. Moreover, $P_N^2 = P_N$ and $P_N(A_\omega(G)) = A_\omega(G{:}N)$, the subalgebra of elements constant on cosets of N. It remains to prove the latter is isometrically isomorphic to $A_{\omega_{G/N}}(G/N)$.

Let us note that if $\pi \in \widehat{G} \setminus (\widehat{G/N} \circ q)$ then $\pi|_N$ never contains the trivial representation of N. Indeed if for $\xi \in \mathcal{H}_\pi$ we have $\pi(n)\xi = \xi$ for all n in N, then for any s in G and any n in N, we have $\pi(n)\pi(s)\xi = \pi(s)\pi(s^{-1}ns)\xi = \pi(s)\xi$. Then either $\xi = 0$, or ξ is a cyclic vector for $\pi(G)$, in which case $\pi(n) = I$ for all n in N, but this contradicts our assumption about π.

Thus if $u \in A_\omega(G)$ we have for π in $\widehat{G}$, s in G

$$\begin{aligned} P_N \operatorname{Tr}\bigl(\hat{u}(\pi)\pi(s)\bigr) &= \operatorname{Tr}\Bigl(\hat{u}(\pi)\pi(s)\int_N \pi(n)\,dn\Bigr) \\ &= \begin{cases} 0 & \text{if } \pi \notin \widehat{G/N} \circ q, \\ \operatorname{Tr}(\hat{u}(\pi)\pi(s)) & \text{if } \pi \in \widehat{G/N} \circ q \end{cases} \end{aligned}$$

by the Schur orthogonality relations. Thus

$$A_\omega(G{:}N) = \left\{ u \in \mathrm{Trig}((G))\colon \begin{array}{l} \hat{u}(\pi) = 0 \text{ for } \pi \in \widehat{G} \setminus \widehat{G/N} \circ q \text{ and} \\ \sum_{\pi \in \widehat{G/N}\circ q} \|\hat{u}(\pi)\|_1 d_\pi \omega(\pi) < \infty \end{array} \right\}$$

which is clearly isometrically isomorphic to $A_{\omega_{G/N}}(G/N)$.

(ii) We consider the extended map $q : \mathrm{Trig}(G)^\dagger \to \mathrm{Trig}(G/N)^\dagger$. We have that $\ker(q)|_{G_{\mathbb{C}}} = \iota(N_{\mathbb{C}})$. Indeed, we note that since $q(N) = \{e\}$, $q(\mathfrak{n}) = \{0\}$, and hence $q \circ \iota(N_{\mathbb{C}}) = q(N\exp(i\mathfrak{n})) = \{e\}$. Conversely, if $\theta = s|\theta| \in \ker(q)|_{G_{\mathbb{C}}}$, then $q(s)q(|\theta|)$ is the polar decomposition of $q(\theta) = e$, hence $q(s) = e = q(|\theta|)$. Thus $s \in N$. Moreover we write $|\theta| = \exp(iX)$ for some X in $\mathfrak{g}$. We see for $t, t_0 \in \mathbb{R}$ that $e = q(|\theta|)^{tt_o} = q(\exp(itt_0X)) = \exp(itq(t_0X))$, and, taking derivative at $t = 0$ we obtain that $iq(t_0X) = 0$. Thus we see that $X \in \mathfrak{n}$, hence $\theta = s\exp(iX) \in \iota(N_{\mathbb{C}})$.

Now, from (i) above, $q(G_\omega)$ will be a closed subset of $(G/N)_{\omega^N}$. We see that for θ, θ' in $G_\omega \subset G_{\mathbb{C}}$, $q(\theta) = q(\theta')$ if and only if $q(\theta^{-1}\theta') \in \iota(N_{\mathbb{C}})$, i.e. $\theta \sim_{N_{\mathbb{C}}} \theta'$. $\square$

That $q : G \to G/N$ extends to an open quotient map $q|_{G_{\mathbb{C}}} : G_{\mathbb{C}} \to (G/N)_{\mathbb{C}}$ is noted in [2, Cor. 3] and requires the somewhat delicate lifting one-parameter subgroup result [15, Thm. 4]. It is unclear that this result preserves the rate of growth of positive elements.

Conjecture 4.15. *In* (ii)*, above, we have that* $G_\omega / \sim_{N_{\mathbb{C}}} = (G/N)_{\omega^N}$*. In particular, if for some Lie quotient* G/N *of* G*,* $(G/N)_{\omega^N} \supsetneq G/N$*, then* $G_\omega \supsetneq G$*.*

4.2. Growth of weights

We wish to find conditions on the weight ω which characterise when $G_\omega \supsetneq G$ and $G_\omega = G$. We begin with some notation. If $S \subset \widehat{G}$ we let

$$S^{\otimes n} = \{\pi \in \widehat{G}\colon \pi \subset \sigma_1 \otimes \cdots \otimes \sigma_n \text{ where } \sigma_1, \ldots, \sigma_n \in S\}, \qquad \langle S \rangle = \bigcup_{n \in \mathbb{N}} S^{\otimes n}.$$

We say that $\widehat{G}$ is finitely generated if $\widehat{G} = \langle S \rangle$ for some finite $S \subset \widehat{G}$.

Proposition 4.16. *G is a Lie group if and only if $\widehat{G}$ is finitely generated.*

Proof. This is [8, (30.48)]. □

We let for any continuous unitary representation ρ of G, and any finite subset S of $\widehat{G}$

$$\omega(\rho) = \sup_{\sigma \in \widehat{G},\, \sigma \subset \rho} \omega(\sigma) \quad \text{and} \quad \omega(S) = \sup_{\sigma \in S} \omega(\sigma)$$

so that $\omega(\rho) = \omega(\{\pi \in \widehat{G}\colon \pi \subset \rho\})$. We note that if S is a finite subset of $\widehat{G}$, then $S^{\otimes(n+m)} = S^{\otimes n} \otimes S^{\otimes m}$, i.e. any π in $S^{\otimes(n+m)}$ may be realised as a subrepresentation of $\pi' \otimes \pi''$ for some π' in $S^{\otimes n}$ and π'' in $S^{\otimes m}$. Hence the function $\tilde{\omega} : \mathbb{N} \to \mathbb{R}^{>0}$ given by $\tilde{\omega}(n) = \omega(S^{\otimes n})$ is a weight. Thus we can appeal to (4.3) to define

$$\rho_\omega(S) = \lim_{n \to \infty} \omega\big(S^{\otimes n}\big)^{1/n}$$

and for a single π in $\widehat{G}$, we define $\rho_\omega(\pi) = \lim_{n\to\infty} \omega(\pi^{\otimes n})^{1/n}$.

We say that ω is *non-exponential* if for every $\pi \in \widehat{G}$

$$\rho_\omega(\pi) = 1 \tag{4.4}$$

and we say that ω has *exponential growth* otherwise.

Proposition 4.17. *Let ω be a bounded weight on $\widehat{G}$. Then the following are equivalent*:

(i) *ω has exponential growth*; *and*
(ii) *there is some finite subset S of $\widehat{G}$ for which $\rho_\omega(S) > 1$.*

Further, if G is a Lie group then (i) *and* (ii) *are equivalent to*:

(iii) *$\rho_\omega(S) > 1$ for every generating set $S \subset \widehat{G}$.*

Proof. Recall that a bounded weight always has $\rho_\omega(\pi) \geqslant 1$; see remark after (4.3). That (i) implies (ii) is obvious. In the case that G is a Lie group it is obvious that (iii) implies (ii). For a Lie group G, if S' is a generating set, then for some m, $S'^{\otimes m} \supset S$ and hence

$$\rho_\omega(S') = \lim_{n\to\infty} \omega(S'^{\otimes n})^{1/n} \geqslant \lim_{n\to\infty} \omega(S'^{\otimes mn})^{1/mn} = \rho_\omega(S'^{\otimes m})^{1/m} \geqslant \rho_\omega(S)^{1/m}.$$

Hence (ii) implies (iii). It remains to show that (ii) implies (i), in general.

Let $S = \{\sigma_1, \dots, \sigma_m\}$. Suppose $\rho_\omega(S) > 1$. By (4.3) we have that $\omega(S^{\otimes n}) \geqslant \rho_\omega(S)^n$, and hence there is a sequence $(\pi_n)_{n\in\mathbb{N}}$ such that

$$\pi_n \in S^{\otimes n} \quad \text{for each } n, \quad \text{and} \quad \omega(\pi_n) \geqslant \rho_\omega(S)^n.$$

Then for each n there are $l_{1,n}, \dots, l_{m,n}$ in $\{0\} \cup \mathbb{N}$ such that $\pi_n \in \sigma_1^{\otimes l_{1,n}} \otimes \dots \otimes \sigma_m^{\otimes l_{m,n}}$ and $l_{1,n} + \dots + l_{m,n} = n$. We have

$$\omega(\pi_n) \leqslant \omega(\sigma_1^{\otimes l_{1,n}}) \dots \omega(\sigma_m^{\otimes l_{m,n}}).$$

It follows from the "pigeon-hole principle" that for some $j = 1, \dots, m$, there is a sequence $n_1 < n_2 < \cdots$ for which $\omega(\pi_{n_k})^{1/m} \leqslant \omega(\sigma_j^{\otimes l_{j,n_k}})$. Since $\rho_\omega(S)^{n_k/m} \leqslant \omega(\sigma_j^{\otimes l_{j,n_k}})$, we may assume $n_1, n_2, \dots$ are chosen so $l_{j,n_1} < l_{j,n_2} < \cdots$. Thus, since $l_{j,n_k} \leqslant n_k$ and $\omega(\sigma_j^{\otimes l_{j,n_k}}) > 1$, we have

$$\omega(\pi_{n_k})^{1/mn_k} \leqslant \omega(\sigma_j^{\otimes l_{j,n_k}})^{1/n_k} \leqslant \omega(\sigma_j^{\otimes l_{j,n_k}})^{1/l_{j,n_k}}$$

and we find

$$1 < \rho_\omega(S)^{1/m} = \lim_{k\to\infty} \omega(\pi_{n_k})^{1/mn_k} \leqslant \lim_{k\to\infty} \omega(\sigma_j^{\otimes l_{j,n_k}})^{1/l_{j,n_k}}$$

where the latter is $\rho_\omega(\sigma_j)$, again by (4.3). □

Example 4.18. (1) Consider exponential weights ω_θ of Example 4.6. As $\theta \in G_{\mathbb{C}}$ we may appeal to (2.8) and (2.9) to see that

$$\omega_\theta(\pi^{\otimes n}) = \max_{\sigma\subset\pi^{\otimes n}} \|\sigma(\theta)\|_{\mathrm{op}} = \|\pi^{\otimes n}(\theta)\|_{\mathrm{op}} = \|\pi(\theta)^{\otimes n}\|_{\mathrm{op}} = \|\pi(\theta)\|_{\mathrm{op}}^n$$

and hence $\rho_{\omega_\theta}(\pi) = \|\pi(\theta)\|_{\mathrm{op}}$. In particular such a weight is bounded only if $\inf_{\pi\in\widehat{G}} \|\pi(\theta)\|_{\mathrm{op}} \geqslant 1$, and of non-exponential growth only if $\|\pi(\theta)\|_{\mathrm{op}} = 1$ for all π in $\widehat{G}$.

Corollary 2.2 shows that ω_θ is symmetric exactly when, for each π, the smallest and largest eigenvalues μ_π, λ_π of $|\pi(\theta)|$ satisfy $\lambda_\pi = \mu_\pi^{-1}$.

(2) Let G be a compact group and let $\omega(\pi) = d_\pi$, $\pi \in \widehat{G}$, be the dimension weight. The weight ω is non-exponential by Example 5.2 and Proposition 5.5 below.

Other examples of weights of non-exponential growth are given in the next section.

Proposition 4.19. *Let ω be a non-exponential symmetric weight on $\widehat{G}$. Then $G_\omega = G$.*

Proof. Assume $G_\omega \neq G$. Then by Proposition 4.1 there exists $\theta \in G_\omega^+$ such that $\sup_{\pi\in\widehat{G}} \frac{\|\pi(\theta)\|_{\mathrm{op}}}{\omega(\pi)} \leqslant 1$ and $\|\pi(\theta)\|_{\mathrm{op}} > 1$ for some $\pi \in \widehat{G}$. Then

$$\omega\bigl(\pi^{\otimes n}\bigr)^{1/n} = \sup_{\sigma\subset\pi^{\otimes n}} \omega(\sigma)^{1/n} \geqslant \sup_{\sigma\subset\pi^{\otimes n}} \bigl\|\sigma(\theta)\bigr\|_{\mathrm{op}}^{1/n} = \bigl\|\pi(\theta)^{\otimes n}\bigr\|_{\mathrm{op}}^{1/n} = \bigl\|\pi(\theta)\bigr\|_{\mathrm{op}}$$

giving $\lim_{n\to\infty} \omega(\pi^{\otimes n})^{1/n} > 1$, a contradiction. □

Question. Is it true in general that if a weight ω is exponential then $G_\omega \neq G$?

5. Polynomial weights

In this section we introduce the polynomial weights which are of fundamental importance. For ease *we will always assume that a weight ω on $\widehat{G}$ is symmetric*. In particular these weights are bounded and thus $G \subset G_\omega$.

5.1. Definition and basic theory

The following description of the dual space of a connected compact Lie group G has been taken from [21]. Let $\mathfrak{g}$ be the Lie algebra of G. Then $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}_1$ with $\mathfrak{z}$ the center of $\mathfrak{g}$ and $\mathfrak{g}_1 = [\mathfrak{g},\mathfrak{g}]$ a compact Lie algebra. Let $\langle\cdot,\cdot\rangle$ be an inner product on $\mathfrak{g}$ satisfying (1) $\langle\mathfrak{g}_1,\mathfrak{z}\rangle = (0)$ and (2) $\langle\cdot,\cdot\rangle_{|\mathfrak{g}_1\times\mathfrak{g}_1} = -B_{\mathfrak{g}_1}$ (here $B_{\mathfrak{k}}$ denotes the Killing form of a Lie algebra $\mathfrak{k}$).

Let $X_1,\ldots,X_n$ be an orthonormal basis of $\mathfrak{g}$, such that $\{X_1,\ldots,X_r\}$ is a basis of $\mathfrak{z}$. Set

$$\Omega = \sum_i X_i^2 \in \mathrm{Trig}(G)^\dagger. \tag{5.1}$$

Then Ω is independent of the choice of the orthonormal basis of $\mathfrak{g}$ and Ω is central in $\mathrm{Trig}(G)^\dagger$. (Normally, the element Ω is defined in the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$, but for our purposes it is sufficient to regard its image in the associative algebra $\mathrm{Trig}(G)^\dagger$.)

Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{g}_1$ and let $T = \exp\mathfrak{t}$. Let also $\lambda_1,\ldots,\lambda_r$ be complex-valued linear forms on $\mathfrak{z}$ defined by $\lambda_j(X_i) = 2\pi(-1)^{1/2}\delta_{i,j}$, $1 \leqslant i,j \leqslant r$. Let P be a Weyl chamber of T. Let $\Lambda_1,\ldots,\Lambda_l$ be defined by $\frac{2\Lambda_i(H_{\alpha_j})}{\alpha_j(H_{\alpha_j})} = \delta_{i,j}$, where $\alpha_1,\ldots,\alpha_l$ are the simple roots relative to P and the H_{α_j} the corresponding vectors in $\mathfrak{t}$. To every γ in the dual space $\widehat{G}$ of G corresponds a unique element (highest weight) $\Lambda_\gamma = \sum_i n_i\lambda_i + \sum_j m_j\Lambda_j$ with the n_i integers and the m_j non-negative integers. Set

$$\|\gamma\|_1 = \sum_i |n_i| + \sum_j m_j.$$

We let now $\pi_i = \chi_i$ be the character of the group G associated to the highest weight λ_i, $i = 1,\ldots,r$; and let γ_j be the irreducible representation associated to the weight Λ_j, $j = 1,\ldots,l$; the existence and uniqueness of such γ_j follows from [21, Thm. 4.5.3] and the fact that Λ_j is a dominant integral weight.

Let $S = \{\pm\chi_i, \gamma_j,\ i = 1,\ldots,r,\ j = 1,\ldots,l\}$. For each highest weight Λ let π_Λ be the corresponding irreducible representation. It is well known that for two irreducible representations

π_Λ, π_M of G the tensor product representation $\pi_\Lambda \otimes \pi_M$ contains the representation $\pi_{\Lambda+M}$ exactly once and all its irreducible components π_N satisfy the relation $N = \Lambda + M'$, where M' is a weight of π_M. Moreover, $M' = M - \sum k_i \alpha_i$ with k_i non-negative integer (see [11, p. 111]).

Therefore S generates $\widehat{G}$. This allows us to define the function τ_S on $\widehat{G}$

$$\tau_S(\pi) = k, \quad \text{if } \pi \in S^{\otimes k} \setminus S^{\otimes(k-1)}. \tag{5.2}$$

The function τ_S is subadditive on $\widehat{G}$. Indeed, for two irreducible representations γ_1, γ_2 of G, we have that

$$\gamma_1 \otimes \gamma_2 \subset S^{\otimes \tau_S(\gamma_1)} \otimes S^{\otimes \tau_S(\gamma_2)} = S^{\otimes(\tau_S(\gamma_1)+\tau_S(\gamma_2))}.$$

Hence

$$\tau_S(\gamma_1 \otimes \gamma_2) \leqslant \tau_S(\gamma_1) + \tau_S(\gamma_2).$$

Let

$$\omega_S = 1 + \tau_S.$$

One can easily see that the function $\omega_S = 1 + \tau_S$ is a bounded weight on $\widehat{G}$ which we shall call the fundamental polynomial weight.

By the arguments above, for any highest weight $\Lambda_\gamma = \sum_i n_i \lambda_i + \sum_j m_j \Lambda_j$ corresponding to the irreducible representation γ of G we see that

$$\gamma \subset \prod_i \chi_i^{n_i} \otimes \prod_j \gamma_j^{\otimes m_j} \subset S^{\otimes \|\gamma\|_1}$$

and hence we have the inequalities:

$$\tau_S(\gamma) \leqslant \|\gamma\|_1 \quad \text{and} \quad \omega_S(\gamma) \leqslant 1 + \|\gamma\|_1, \quad \gamma \in \widehat{G}. \tag{5.3}$$

Clearly, for every power α ($\alpha \in \mathbb{R}^{>0}$) the function $\omega_\alpha = \omega_S^\alpha$ is also a weight on $\widehat{G}$. We observe that if S' is another generating set for $\widehat{G}$, then there are constants k_1, k_2 such that $k_1\omega_S \leqslant \omega_{S'} \leqslant k_2\omega_S$. For example, if k is such that $S' \subset S^{\otimes k}$, then $\tau_{S'} \leqslant k\tau_S$ and hence $\omega_{S'} \leqslant k\omega_S$. Hence $A_{\omega_S}(G) = A_{\omega_{S'}}(G)$.

Definition 5.1. Let G be a compact Lie group. A weight ω on $\widehat{G}$ is said to have *polynomial growth*, if for some $\alpha, C > 0$, $\omega(\gamma) \leqslant C(1 + \|\gamma\|_1)^\alpha$, $\gamma \in \widehat{G}$.

If G is any compact group with a weight ω on $\widehat{G}$ then we say that ω is of *polynomial growth* if for every closed normal subgroup N such that G/N is a Lie group the restriction weight ω^N has polynomial growth.

We know from [21, Lemma 5.6.4] (with the notations of that lemma) that for every $\gamma \in \widehat{G}$

$$-\gamma(\Omega) = \big(\langle \Lambda_\gamma + \rho, \Lambda_\gamma + \rho\rangle - \langle \rho, \rho\rangle\big)\mathbb{I}_{\mathcal{H}_\gamma} =: c(\gamma) I_\gamma \tag{5.4}$$

where ρ is half the sum of the positive roots of G related to the Weyl chamber P of T. Then by [21, Lemma 5.6.6], there are positive constants c_1, c_2 such that

$$c_1\|\gamma\|_1^2 \leqslant c(\gamma) \leqslant c_2\|\gamma\|_1^2 \tag{5.5}$$

and by [21, Lemma 5.6.7], the series

$$\sum_{\gamma\in\widehat{G}} d_\gamma^2\big(1+\|\gamma\|_1^2\big)^{-s} \tag{5.6}$$

converges if $s > \frac{d(G)}{2}$, here $d(G)$ denotes the dimension of the group G.

Example 5.2. Let G be a connected compact group and let ω be the dimension weight, i.e. $\omega(\pi) = d_\pi$. Then ω is of polynomial growth. In fact, if N is a closed normal subgroup such that G/N is a Lie group, then for $\gamma \in \widehat{G/N}$ we have $\omega^N(\gamma) = \omega(\gamma \circ q) = d_\gamma$. Since by (5.6), $d_\gamma \leqslant C(1+\|\gamma\|_1)^{(d(G/N)/2)+\varepsilon}$ for some $\varepsilon > 0$ and $C > 0$, the weight has polynomial growth.

Next result will give us another weight of polynomial growth.

Lemma 5.3. *If $\sigma \subset \gamma_1 \otimes \gamma_2$ for irreducible representations σ, γ_1 and γ_2 of G then $c(\sigma)^{1/2} \leqslant c(\gamma_1)^{1/2} + c(\gamma_2)^{1/2}$.*

Proof. Let Λ, M and N be the highest weights of γ_1, γ_2 and σ respectively. We set $\gamma_1 = \pi_\Lambda$, $\gamma_2 = \pi_M$ and $\sigma = \pi_N$. It is known that $\Lambda + M$ is the highest weight of the tensor product $\pi_\Lambda \otimes \pi_M$ and $N = \Lambda + M - \sum k_i\alpha_i$ with non-negative integers k_i. Let $\beta = \sum k_i\alpha_i$. As N is the highest weight, N is a dominant integral weight of $\pi_\Lambda \otimes \pi_M$ and hence $\langle N, \beta\rangle \geqslant 0$, where $\langle\cdot,\cdot\rangle$ is the inner product on $\mathfrak{t}_{\mathbb{R}}^*$. This implies

$$\langle \Lambda + M, \Lambda + M\rangle = \langle N + \beta, N + \beta\rangle = \langle N, N\rangle + 2\langle N, \beta\rangle + \langle \beta, \beta\rangle \geqslant \langle N, N\rangle.$$

Moreover, if ρ is half the sum of positive roots of G relative to the Weyl chamber P, we have $\langle \beta, \rho\rangle \geqslant 0$ (see e.g. [11, Prop. 4.33]) and $\langle N, \rho\rangle = \langle \Lambda + M, \rho\rangle - \langle \beta, \rho\rangle \leqslant \langle \Lambda + M, \rho\rangle$. Therefore by (5.4)

$$\begin{aligned}
c(\pi_N) &= \langle N, N\rangle + 2\langle N, \rho\rangle \leqslant \langle \Lambda + M, \Lambda + M\rangle + 2\langle \Lambda + M, \rho\rangle \\
&= \langle \Lambda, \Lambda\rangle + 2\langle \Lambda, \rho\rangle + \langle M, M\rangle + 2\langle M, \rho\rangle + 2\langle \Lambda, M\rangle \\
&\leqslant c(\pi_\Lambda) + c(\pi_M) + 2\langle \Lambda, \Lambda\rangle^{1/2}\langle M, M\rangle^{1/2} \\
&\leqslant c(\pi_\Lambda) + c(\pi_M) + 2c(\pi_\Lambda)^{1/2}c(\pi_M)^{1/2} = \big(c(\pi_\Lambda)^{1/2} + c(\pi_M)^{1/2}\big)^2
\end{aligned}$$

and hence $c(\pi_N)^{1/2} \leqslant c(\pi_\Lambda)^{1/2} + c(\pi_M)^{1/2}$. $\square$

Theorem 5.4. *Let $\omega(\gamma) := 1 + c(\gamma)^{1/2}$, $\gamma \in \widehat{G}$. Then ω is a bounded weight equivalent to the fundamental weight ω_S and hence $A_\omega(G) = A_{\omega_S}(G)$.*

Proof. Clearly, by Lemma 5.3, ω is a weight. It follows by induction that $c(\sigma)^{1/2} \leqslant c(\gamma_1)^{1/2} + \cdots + c(\gamma_n)^{1/2}$ whenever $\sigma \subset \gamma_1 \otimes \cdots \otimes \gamma_n$, $\sigma, \gamma_i \in \widehat{G}$, $i = 1, \ldots, n$. By (5.5), $\omega(\gamma) \leqslant C(1 + \|\gamma\|_1)$, $\gamma \in \widehat{G}$, for some constant $C > 0$, i.e. ω is a polynomial weight, and

$$\|\sigma\|_1 \leqslant C\big(\|\gamma_1\|_1 + \cdots + \|\gamma_n\|_1\big).$$

Assume now that $\tau_S(\sigma) = k$ and take the γ_i's in S such that $\sigma \subset \gamma_1 \otimes \cdots \otimes \gamma_k$. Then $\|\gamma_i\|_1 = 1$ for every i and so

$$\|\sigma\|_1 \leqslant C\big(\|\gamma_1\|_1 + \cdots + \|\gamma_k\|_1\big) = Ck = C\tau_S(\sigma) \leqslant C\|\sigma\|_1,$$

the last inequality is due to (5.3). Applying again (5.5) we obtain the statement. □

Proposition 5.5. *A polynomial weight ω on $\widehat{G}$ is non-exponential and hence $G_\omega = G$.*

Proof. We assume first that G is a Lie group. Take $\pi \in \widehat{G}$. Then since by Theorem 5.4 $\|\sigma\|_1 \leqslant cn\|\pi\|_1$ for each $\sigma \subset \pi^{\otimes n}$ and some constant c independent of n, we have

$$\begin{aligned}\omega\big(\pi^{\otimes n}\big)^{1/n} &= \sup_{\sigma \subset \pi^{\otimes n}} \omega(\sigma) \leqslant \sup_{\sigma \subset \pi^{\otimes n}} C\big(1 + \|\sigma\|_1\big)^{\alpha/n} \\ &\leqslant C\big(1 + cn\|\pi\|_1\big)^{\alpha/n} \xrightarrow{n\to\infty} 1.\end{aligned}$$

Let G be an arbitrary compact group with polynomial weight on its dual space $\widehat{G}$. Take $\pi \in \widehat{G}$. Then $G/\ker(\pi)$ is a Lie group. Let $N = \ker(\pi)$ and let π_N be the representation of G/N on $\mathcal{H}_\pi$ corresponding to π. We have

$$\omega\big(\pi^{\otimes n}\big)^{1/n} = \omega^N\big(\pi_N^{\otimes n}\big)^{1/n}.$$

By the previous argument, $\lim_{n\to\infty} \omega(\pi^{\otimes n})^{1/n} = 1$. That $G_\omega = G$ follows from Proposition 4.19. □

We observe the following, which was also proved in [16].

Corollary 5.6. *Let G be a compact group and let ω be the dimension weight. Then $G_\omega = G$.*

Proof. Follows from Example 5.2 and Proposition 5.5. □

Proposition 5.7. *Let G be a compact Lie group, let ω be a symmetric weight such that $a = \inf_{\gamma\in\widehat{G}} \omega(\gamma)^{1/\|\gamma\|_1} > 1$. Then $G_\omega \neq G$.*

Proof. Let $X_1, \ldots, X_n$ and Ω be as in (5.1). Since each X_i is skew-hermitian we have from (5.4) that $0 \leqslant -\gamma(X_i)^2 \leqslant c(\gamma)I_\gamma$ for any $\gamma \in \widehat{G}$. Moreover, there exists $\gamma \in \widehat{G}$ such that $\gamma(X_n) \neq 0$. Set $\theta = \exp i\lambda X_n$. Then, as in Proposition 2.1, we have $\theta \in G_{\mathbb{C}}^+$. Since $i\pi(X_n) \leqslant c(\pi)^{1/2} I_{\pi_n} \leqslant c\|\pi\|_1 I_{\pi_n}$ (the last inequality is due to (5.5)), we have for $\pi \in \widehat{G}$ that

$$\frac{\|\pi(\theta)\|_{\mathrm{op}}}{\omega(\pi)} \leqslant \frac{\|\exp i\lambda\pi(X_n)\|_{\mathrm{op}}}{a^{\|\pi\|_1}} \leqslant \frac{e^{\lambda c\|\pi\|_1}}{a^{\|\pi\|_1}}$$

for each $\pi \in \widehat{G}$. Taking now λ such that $e^{\lambda c} \leqslant a$ we obtain that $\frac{\|\pi(\theta)\|_{\mathrm{op}}}{\omega(\pi)} \leqslant 1$, and hence $\theta \in G_\omega \setminus G$. $\square$

In what follows we shall frequently use an L_2-function E_m, $m > d(G)/4$, defined on G by

$$\gamma(E_m) = \frac{1}{(1+c(\gamma))^m}\mathbb{I}_\gamma, \quad \forall\gamma \in \widehat{G}. \tag{5.7}$$

The existence of E_m follows from the Plancherel theorem and the convergence result of the series (5.6). We say that function $u : G \to \mathbb{C}$ is *$2n$-times Ω-differentiable* if u belongs to the space

$$L^2_{\Omega,n} := \{g \in L^2(G)\colon\ (1-\Omega)^n g \in L^2(G)\}.$$

We note that by (5.3) $L^2_\omega(G) \subset L^2_{\Omega,n}(G)$ for any polynomial weight ω such that $\omega(\gamma) \leqslant C(1 + \|\gamma\|_1)^{2n}$.

Proposition 5.8. *Let $\alpha > 0$ and let ω be a polynomial weight such that $\omega(\gamma) \leqslant C(1 + \|\gamma\|_1)^\alpha$, $\gamma \in \widehat{G}$, for some $C > 0$. Then $L^2_{\Omega,n}(G) \subset A_\omega(G)$ if $n > \frac{d(G)}{4} + \frac{\alpha}{2}$.*

Proof. Let $E_n \in L^2(G)$ be the function defined by (5.7). Then for $g \in L^2_{\Omega,n}(G)$, $n > \frac{d(G)}{4} + \frac{\alpha}{2}$ we have

$$E_n * (1-\Omega)^n g = g. \tag{5.8}$$

Since $(1-\Omega)^n g \in L^2(G)$ to see that $g \in A_\omega(G)$, by Proposition 3.4 it is enough to prove that $E_n \in L^2_{\omega^2}(G)$. By (5.5) we have

$$\sum_{\gamma\in\widehat{G}} d_\gamma^2\omega(\gamma)^2\|\gamma(E_n)\|_2^2 = \sum_{\gamma\in\widehat{G}} \frac{d_\gamma^2\omega(\gamma)^2}{(1+c(\gamma))^{2n}} \leqslant C\sum_{\gamma\in\widehat{G}} \frac{d_\gamma^2}{(1+c(\gamma))^{2n-\alpha}}$$

and by (5.6) the series is convergent if $2n - \alpha > \frac{d(G)}{2}$. $\square$

5.2. A smooth functional calculus and regularity of $A_\omega(G)$

Definition 5.9. For $\pi \in \widehat{G}$ denote by χ_π the normalised character of π i.e.

$$\chi_\pi(s) = d_\pi \operatorname{Tr}(\pi(s)), \quad s \in G.$$

Then we have for any $\sigma \in \widehat{G}$

$$\sigma(\chi_\pi) = \begin{cases} 0 & \text{if } \sigma \neq \pi, \\ \mathbb{I}_\pi & \text{if } \sigma = \pi. \end{cases}$$

Let Q_ω denote the linear operator on $\mathrm{Trig}(G)$ defined by

$$Q_\omega(u) = \sum_{\pi \in \widehat{G}} \sqrt{\omega(\pi)}\chi_\pi * u, \quad u \in \mathrm{Trig}(G),$$

and let R_ω be its inverse:

$$R_\omega(u) = \sum_{\pi \in \widehat{G}} \frac{1}{\sqrt{\omega(\pi)}}\chi_\pi * u, \quad u \in \mathrm{Trig}(G).$$

Then we can extend the linear operator Q_ω (resp. R_ω) to an isometry $Q_\omega : L^2_\omega(G) \to L^2(G)$ (resp. $R_\omega : L^2(G) \to L^2_\omega(G)$) and for $\xi \in L^2(G)$ we have that

$$\xi \in L^2_\omega(G) \quad \Leftrightarrow \quad Q_\omega(\xi) \in L^2(G)$$

and

$$\|\xi\|_{2,\omega} = \bigl\| Q_\omega(\xi)\bigr\|_2.$$

We can consider Q_ω (resp. R_ω) as a convolution operator with the central distribution $q_\omega = \sum_{\pi\in\widehat{G}} \sqrt{\omega(\pi)}\chi_\pi$ (resp. with the central distribution $r_\omega = \sum_{\pi\in\widehat{G}} \frac{1}{\sqrt{\omega(\pi)}}\chi_\pi$) and we shall write Q_ω and R_ω as convolution operators and then

$$Q_\omega * u = u * Q_\omega, \qquad R_\omega * u = u * R_\omega, \quad u \in \mathrm{Trig}(G).$$

For a real number a, let $[a]$ be the integer part of a and let $d(G)$ denote the dimension of the group G. Let for $\alpha > 0$

$$r(G,\alpha) = \left[\frac{d(G)}{2} + \alpha\right] + 1. \tag{5.9}$$

The following theorem is an adaptation of Theorem 3.1 in [14].

Theorem 5.10. *Let G be a connected compact Lie group, let $\omega \leqslant c\omega_S^\alpha$ be a symmetric polynomial weight on $\widehat{G}$ and let $u = \bar{u}$ be a self-adjoint element of $A(G)_{\omega_S^{r(G,\alpha)}}$. Then there exists a positive constant $C = C(u)$ such that*

$$\bigl\|e^{itu}\bigr\|_{A_\omega(G)} \leqslant C\bigl(1+|t|\bigr)^{d(G)/2+\alpha}, \quad t \in \mathbb{R}. \tag{5.10}$$

Proof. For $N \in \mathbb{N}^*$ we let

$$\widehat{G}_N = \bigl\{\gamma \in \widehat{G};\ \|\gamma\| \leqslant N\bigr\}$$

and consider $F_N \in L^2(G)$ given by

$$\gamma(F_N) = I_\gamma, \quad \forall \gamma \in \widehat{G}_N, \qquad \gamma(F_N) = 0 \quad \text{otherwise}.$$

Take an L^2-function E_m on G, $m > d(G)/4$, given by (5.7) and let $E_{m,N}$ denote an $L^2(G)$-function for which

$$\gamma(E_{m,N}) = \gamma(E_m), \quad \forall \gamma \notin \widehat{G}_N, \qquad \gamma(E_{m,N}) = 0 \quad \text{otherwise.}$$

We decompose e^{itu}, $t \in \mathbb{R}$, into

$$e^{itu} = a_{t,N} + b_{t,N},$$

where $a_{t,N}$ is defined by

$$\gamma(a_{t,N}) = \gamma(g_t), \quad \forall \gamma \in \widehat{G}_N, \qquad \gamma(a_{t,N}) = 0 \quad \text{otherwise,}$$

and $b_{t,N} = e^{itu} - a_{t,N}$. Clearly, $a_{t,N}$ is a C^∞-function and hence $b_{t,N}$ is of class $r(G,\alpha)$.

In order to estimate the norm $\|a_{t,N}\|_{A_\omega}$ we note first that

$$a_{t,N} = F_N * a_{t,N} = F_N * Q_\omega * R_\omega * a_{t,N}.$$

Hence, by Proposition 3.4 and the assumption that u is a real-valued function,

$$\begin{aligned}\|a_{t,N}\|_{A_\omega(G)} &\leqslant \|F_N * Q_\omega\|_{2,\omega} \|R_\omega * a_{t,N}\|_{2,\omega} \leqslant \|F_N\|_{2,\omega^2} \big\|e^{itu}\big\|_2 \\ &\leqslant \|F_N\|_{2,\omega^2} \big\|e^{itu}\big\|_{L^\infty(G)} = \|F_N\|_{2,\omega^2}.\end{aligned}$$

Using estimation given in the proof of [14, Thm. 3.1] we obtain

$$\|F_N\|^2_{2,\omega^2} = \sum_{\|\gamma\| \leqslant N} \omega^2(\gamma) d_\gamma^2 \leqslant c_1 N^{2\alpha} \sum_{j=0}^{N} d_\gamma^2 \leqslant c_2 N^{2\alpha + d(G)}$$

and hence $\|a_{t,N}\|_{A_\omega(G)} \leqslant C N^{\frac{d(G)}{2} + \alpha}$ for a certain constant $C > 0$.

Now for the norm of the element $b_{t,N}$ we use the equalities

$$b_{,N} = E_m * (1 - \Omega)^m b_{t,N} = E_{m,N} * (1 - \Omega)^m e^{itu}$$

which hold for $m = \frac{1}{2} r(G,\alpha)$, as $b_{t,N}$ is of class $r(G,\alpha)$. Thus

$$\begin{aligned}\|b_{t,N}\|_{A_\omega(G)} &= \big\|E_{m,N} * (1 - \Omega)^m e^{itu}\big\|_{A_\omega(G)} \\ &= \big\|Q_\omega * E_{m,N} * R_\omega * (1 - \Omega)^m e^{itu}\big\|_{A_\omega(G)} \\ &\leqslant \|E_{m,N}\|_{2,\omega^2} \big\|(1 - \Omega)^m e^{itu}\big\|_2.\end{aligned}$$

The arguments in [14, Proof of Theorem 3.1] give the following estimate

$$\big\|(1 - \Omega)^m e^{itu}\big\|_2 \leqslant C\big(1 + |t|^{2m}\big), \quad t \in \mathbb{R},$$

for some constant $C > 0$ and for all $m \in \frac{1}{2}\mathbb{N}$, and the estimate for the $A_\omega(G)$-norm of $b_{t,N}$, $t \in \mathbb{R}$,

$$\begin{aligned}
\|b_{t,N}\|_{A_\omega(G)} &\leqslant C_1\biggl(\sum_{\|\gamma\|>N}\frac{d_\gamma^2\omega(\gamma)^2}{(1+c(\gamma))^{2m}}\biggr)^{1/2}\bigl(1+|t|\bigr)^{2m}\\
&\leqslant C_2\biggl(\sum_{j>N} j^{(d(G)-1)}\frac{j^{2\alpha}}{(1+j)^{4m}}\biggr)^{1/2}\bigl(1+|t|\bigr)^{2m}\\
&\leqslant C_3\frac{1}{N^{2m-\frac{d(G)+2\alpha}{2}}}\bigl(1+|t|\bigr)^{2m}.
\end{aligned}$$

Letting now N to be the smallest integer $\geqslant |t|$ we obtain

$$\begin{aligned}
\bigl\|e^{itu}\bigr\|_{A_\omega(G)} &\leqslant \|a_{t,N}\|_{A_\omega(G)} + \|b_{t,N}\|_{A_\omega(G)}\\
&\leqslant \frac{C}{2}\bigl(1+|t|\bigr)^{(d(G)/2)+\alpha} + \frac{C}{2}\bigl(1+|t|\bigr)^{(d(G)/2)+\alpha}
\end{aligned}$$

for a new constant $C > 0$. $\square$

Theorem 5.11. *Let G be a connected compact group and let ω be a symmetric polynomial weight on $\widehat{G}$. Then $A_\omega(G)$ is a regular Banach algebra.*

Proof. Let E, F be two closed subsets of G such that $E \cap F = \emptyset$. We must find an element $v \in A_\omega(G)$, such that $v = 0$ on E and $v = 1$ on F. Since G is connected, we can find a closed normal subgroup N of G, such that G/N is a Lie group and such that $EN \cap FN = \emptyset$. Hence by Proposition 4.14 we can assume that G is a connected compact Lie group. The algebra $\mathrm{Trig}(G)$ is uniformly dense in $C(G)$. Hence there exists a trigonometric polynomial u on G, such that $u = \bar{u}$ and such that $|u(x)| < \frac{1}{10}$, $x \in E$, and $u(y) > \frac{9}{10}$, $y \in F$. We apply now the functional calculus of C^k functions to u. Choose a function $\varphi : \mathbb{R} \to \mathbb{R}$ with compact support of class $C^{(d(G)/2)+\alpha+2}$, vanishing on the interval $[-\frac{2}{10}, \frac{2}{10}]$ and taking the value 1 on the interval $[\frac{8}{10}, \frac{12}{10}]$. Then the integral

$$v = \int_{\mathbb{R}} \hat{\varphi}(t) e^{2\pi i t u}\, dt$$

converges in $A_\omega(G)$ by Theorem 5.10. Moreover, by the Fourier inversion formula,

$$\begin{aligned}
v(x) &= \int_{\mathbb{R}} \hat{\varphi}(t) e^{2\pi i t u(x)}\, dt = \varphi\bigl(u(x)\bigr) = 0, \quad x \in E,\\
v(y) &= \int_{\mathbb{R}} \hat{\varphi}(t) e^{2\pi i t u(y)}\, dt = \varphi\bigl(u(y)\bigr) = 1, \quad y \in F. \qquad \square
\end{aligned}$$

6. Spectral synthesis

Let A be a unital semisimple, regular, commutative Banach algebra with X_A as spectrum. We will identify A with a subalgebra of $C(X_A)$ in our notation. If $E \subset X_A$ is a closed subset, let

$$I_A(E) = \{a \in A \mid a^{-1}(0) \text{ contains } E\},$$
$$J_A^0(E) = \{a \in A \mid a^{-1}(0) \text{ contains a nbhd of } E\} \quad \text{and} \quad J_A(E) = \overline{J_A^0(E)}.$$

It is known that $I_A(E)$ and $J_A(E)$ are the largest and the smallest closed ideals with E as hull, i.e., if I is a closed ideal such that $\{x \in X_A\colon f(x) = 0 \text{ for all } f \in I\} = E$ then

$$J_A(E) \subset I \subset I_A(E).$$

We say that E is a *set of spectral synthesis* for A if $J_A(E) = I_A(E)$ and of *weak synthesis* if the quotient algebra $I_A(E)/J_A(E)$ is nilpotent (see [22]).

Let A^* be the dual of A. For $a \in A$ we set $\mathrm{supp}(a) = \overline{\{x \in X_A\colon a(x) \neq 0\}}$ and $\mathrm{null}(a) = \{x \in X_A\colon a(x) = 0\}$. For $\tau \in A^*$ and $a \in A$ define $a\tau$ in A^* by $\langle a\tau, b\rangle := \langle \tau, ab\rangle$ and define the support of τ by

$$\mathrm{supp}(\tau) = \{x \in X_A\colon a\tau \neq 0 \text{ whenever } a(x) \neq 0\}.$$

It is known that $\mathrm{supp}(\tau)$ consists of all $x \in X_A$ such that for any neighbourhood U of x there exists $a \in A$ for which $\mathrm{supp}(a) \subset U$ and $\langle \tau, a\rangle \neq 0$. Then, for a closed set $E \subset X_A$

$$J_A(E)^\perp = \{\tau \in A^*\colon \mathrm{supp}(\tau) \subset E\}$$

and E is spectral for A if and only if $\langle \tau, a\rangle = 0$ for any $a \in A$ and $\tau \in A^*$ such that $\mathrm{supp}(\tau) \subset E \subset \mathrm{null}(a)$.

We say that $a \in A$ *admits spectral synthesis* if $a \in J_A(\mathrm{null}(a))$, which is equivalent to $\langle \tau, a\rangle = 0$ for any $\tau \in A^*$, $\mathrm{supp}(\tau) \subset \mathrm{null}(a)$.

Let G be a connected compact Lie group, ω be a symmetric weight on $\widehat{G}$ of polynomial growth and let $A_\omega(G)$ be the corresponding Beurling–Fourier algebra. Then $X_{A_\omega(G)} = G$ by Proposition 5.5, and $A_\omega(G)$ is a semisimple regular commutative Banach algebra of functions on G by Theorem 5.11 and Theorem 4.2. In what follows we write $I_\omega(E)$ for $I_{A_\omega(G)}(E)$ and $J_\omega(E)$ for $J_{A_\omega(G)}(E)$. Let $\mathcal{D}(G)$ be the space of smooth functions on G. For a closed subset E of G, we denote by $J_{\mathcal{D}}(E)$ the space of all elements of $\mathcal{D}(G)$ vanishing on E. Note that $\mathcal{D}(G) \subset A_\omega(G)$ by virtue of Proposition 5.8.

6.1. Smooth synthesis

In the rest of the section we always assume that G is a connected compact Lie group and ω is a symmetric weight of polynomial growth on $\widehat{G}$.

Definition 6.1. The closed subset E of G is said to be of smooth synthesis for $A_\omega(G)$ if $\overline{J_{\mathcal{D}}(E)} = I_\omega(E)$.

The proof of the next theorem is similar to the one of [14, Thm. 4.3].

Theorem 6.2. *Let G be a connected compact Lie group of dimension n. Let M be a smooth submanifold of dimension $m < n$ and let E be a compact subset of M. If ω is a symmetric weight on $\widehat{G}$ such that $\omega \leqslant C\omega_S^\alpha$ for some C, $\alpha > 0$, then $\overline{J_{\mathcal{D}}(E)}^{[\frac{m}{2}+\alpha]+1} = J_\omega(E)$.*

Proof. As $J_\omega(E)$ is the smallest closed ideal with E as hull, in order to prove the statement it is enough to see the inclusion $J_{\mathcal{D}}(E)^{[\frac{m}{2}+\alpha]+1} \subset J_\omega(E)$. We note first that for the right translation $\rho(t)$ the mapping $t \mapsto \rho(t)f \in A_\omega(G)$ is C^∞. In fact, for $m > \frac{d(G)}{4} + \frac{\alpha}{2}$ we have $E_m \in L^2_{\omega^2}(G)$ (see (5.7) for the definition of E_m) and $f = E_m * g$, where $g = (1-\Omega)^m f \in \mathcal{D}(G)$. Hence, for $t \in G$, we have

$$\begin{aligned}\rho(t)f(x) &= f(xt)\\ &= \int_G E_m(s)g\big(s^{-1}xt\big)\,ds\\ &= \int_G E_m(s)\rho(t)g\big(s^{-1}x\big)\,ds\\ &= E_m * \big(\rho(t)g\big)(x)\end{aligned}$$

which gives the statement.

For $0 < \varepsilon < \|f\|_\infty$ let

$$W_\varepsilon = \big\{x \in G\colon\ \big\|\rho(x)f - f\big\|_{A_\omega(G)} < \varepsilon\big\}$$

and

$$\Omega_\varepsilon = \big\{x \in G\colon\ \big\|\rho(x)f - f\big\|_\infty < \varepsilon\big\}.$$

If $A = \inf_{\pi \in \widehat{G}} \omega(\pi)$ then by (3.1)

$$W_\varepsilon \subset \Omega_{\varepsilon/A}.$$

Since the mapping $g \mapsto \rho(g)f \in A_\omega(G)$ is C^∞, there exist a constant $K > 0$, an open neighbourhood W of 0 in the Lie algebra $\mathfrak{g}$ of G, such that

$$\big\|\rho(\exp X)f - f\big\|_{A_\omega(G)} \leqslant K\|X\|$$

for every $X \in \mathfrak{g}$ and some fixed norm $\|\cdot\|$ on $\mathfrak{g}$. Let for $\varepsilon > 0$, $V_\varepsilon = \exp B_\varepsilon$, where B_ε denotes the ball of radius $\frac{\varepsilon}{2K}$ of center 0 in $\mathfrak{g}$. There exist constants $C_1 > C_2 > 0$ such that for every $\varepsilon > 0$

$$C_1\varepsilon^n > |V_\varepsilon| > C_2\varepsilon^n$$

and $V_\varepsilon \subset W_\varepsilon \subset \Omega_{\varepsilon/A}$. Let $f \in J_{\mathcal{D}}(E)$. Then for every $x = x_0 v \in EV_\varepsilon$, $x_0 \in E$, $v \in V_\varepsilon$, we have $f(x_0) = 0$ and

$$\begin{aligned}\big|f(x)\big| = \big|f(x_0v)\big| &\leqslant \big|f(x_0v) - f(x_0)\big| + \big|f(x_0)\big|\\ &= \big|\big(\rho(v)f - f\big)(x_0)\big| \leqslant \big\|\big(\rho(v)f - f\big)\big\|_\infty < \varepsilon A^{-1}.\end{aligned}$$

Hence

$$\left| f^{[\frac{m}{2}+\alpha]+1}(x) \right| \leqslant \left(\varepsilon A^{-1}\right)^{[\frac{m}{2}+\alpha]+1}. \tag{6.1}$$

Let $\nu = f^{[\frac{m}{2}+\alpha]+1}$ on EV_ε and $\nu = 0$ elsewhere. Take a non-negative function $b(X) \in C_c^\infty(\mathfrak{g})$ supported in B_1 and let $b_\varepsilon(X) = b(X/\varepsilon)$, $X \in \mathfrak{g}$. Set $u(x) = Cb_\varepsilon(\log x)$, where C is a constant such that $\int_G u(x)\,dx = 1$. Then $\operatorname{supp}(u) \subset V_\varepsilon$. Since

$$\begin{aligned}
\|u\|_{2,\omega^2} &= \sum_{\pi\in\widehat{G}} d_\pi \omega(\pi)^2 \left\|\hat{u}(\pi)\right\|_2^2 \leqslant C \sum_{\pi\in\widehat{G}} d_\pi \left(1+c(\pi)\right)^\alpha \left\|\hat{u}(\pi)\right\|_2^2 \\
&= C \sum_{\pi\in\widehat{G}} d_\pi \left\| \left(\widehat{(1-\Omega)^{\alpha/2}u}\right)(\pi) \right\|_2^2 = C \left\|(1-\Omega)^{\alpha/2}u\right\|_2^2
\end{aligned}$$

we have $u \in L^2_{\omega^2}(G)$ and $\|u\|_{2,\omega^2}$ behaves like $\varepsilon^{-n/2-\alpha}$ if $\alpha/2$ is an integer, i.e.,

$$\left\|(1-\Omega)^m u\right\|_2^2 = \sum_{\pi\in\widehat{G}} \left(1+c(\pi)\right)^{2m} \left\|\hat{u}(\pi)\right\|_2^2 d_\pi \leqslant C_m \varepsilon^{-n-4m}$$

for some constant C_m depending on the non-negative integer m.

This gives

$$\sum_{\pi\in\widehat{G}} \left(\varepsilon^2\left(1+c(\pi)\right)\right)^{2m} \left\|\hat{u}(\pi)\right\|_2^2 d_\pi \leqslant C_m \varepsilon^{-n}.$$

Let $\alpha > 0$ be arbitrary. Then for non-negative integers l, m such that $2l \leqslant \alpha < 2m$ we have

$$\begin{aligned}
\varepsilon^{2\alpha} \left\|(1-\Omega)^{\alpha/2}u\right\|_2^2 &= \sum_{\pi\in\widehat{G}} \left(\varepsilon^2\left(1+c(\pi)\right)\right)^\alpha \left\|\hat{u}(\pi)\right\|_2^2 d_\pi \\
&= \sum_{\pi:\varepsilon^2(1+c(\pi))<1} \left(\varepsilon^2\left(1+c(\pi)\right)\right)^\alpha \left\|\hat{u}(\pi)\right\|_2^2 d_\pi \\
&\quad + \sum_{\pi:\varepsilon^2(1+c(\pi))\geqslant 1} \left(\varepsilon^2\left(1+c(\pi)\right)\right)^\alpha \left\|\hat{u}(\pi)\right\|_2^2 d_\pi \\
&\leqslant \sum_{\pi\in\widehat{G}} \left(\varepsilon^2\left(1+c(\pi)\right)\right)^{2l} \left\|\hat{u}(\pi)\right\|_2^2 d_\pi \\
&\quad + \sum_{\pi\in\widehat{G}} \left(\varepsilon^2\left(1+c(\pi)\right)\right)^{2m} \left\|\hat{u}(\pi)\right\|_2^2 d_\pi \\
&\leqslant C_l \varepsilon^{-n} + C_m \varepsilon^{-n} \leqslant C\varepsilon^{-n}.
\end{aligned}$$

Therefore

$$\left\|(1-\Omega)^{\alpha/2}u\right\|_2^2 \leqslant C\varepsilon^{-n-2\alpha} \quad \text{and} \quad \|u\|_{2,\omega^2}^2 \leqslant D\varepsilon^{-n-2\alpha}$$

for some $C, D > 0$.

Consider now the function

$$\varphi(s) = \left(f^{[\frac{m}{2}+\alpha]+1} - \nu\right) * \check{u}(s) = \int_G \left(f^{[\frac{m}{2}+\alpha]+1} - \nu\right)(st)u(t)\,dt.$$

As $f^{[\frac{m}{2}+\alpha]+1} - \nu \in L^2(G)$ and $u \in L^2_{\omega^2}(G)$, it follows from Proposition 3.4 and the remark before that $\varphi \in A_\omega(G)$, and $\varphi(s) = 0$ if $s \cdot \operatorname{supp}(u) \subset EV_\varepsilon$. As $E \subset \{s\colon s \cdot \operatorname{supp}(u) \subset EV_\varepsilon\}$ and the set $\{s\colon s \cdot \operatorname{supp}(u) \subset EV_\varepsilon\}$ is open, $\operatorname{supp}(\varphi)$ is disjoint from E and therefore $\varphi \in J_\omega(E)$. We have

$$f^{[\frac{m}{2}+\alpha]+1} - \varphi = \left(f^{[\frac{m}{2}+\alpha]+1} - f^{[\frac{m}{2}+\alpha]+1} * \check{u}\right) + \nu * \check{u}.$$

As $\operatorname{supp}(u) \subset V_\varepsilon \subset W_\varepsilon$, and $\|f^{[\frac{m}{2}+\alpha]+1} - \rho(x)f^{[\frac{m}{2}+\alpha]+1}\|_{A_\omega(G)} \leqslant K\varepsilon$ for all $x \in W_\varepsilon$ and some constant $K = K(m) > 0$ which is independent of ε, it follows that

$$\begin{aligned}
&\left\| f^{[\frac{m}{2}+\alpha]+1} - f^{[\frac{m}{2}+\alpha]+1} * \check{u} \right\|_{A_\omega(G)} \\
&\quad= \left\| \int_G \left(f^{[\frac{m}{2}+\alpha]+1} - \rho(x)f^{[\frac{m}{2}+\alpha]+1}\right)u(x)\,dx \right\|_{A_\omega(G)} \\
&\quad\leqslant \int_G \left\| f^{[\frac{m}{2}+\alpha]+1} - \rho(x)f^{[\frac{m}{2}+\alpha]+1} \right\|_{A_\omega(G)} u(x)\,dx \leqslant K\varepsilon.
\end{aligned}$$

We have also $\|\nu * \check{u}\|_{A_\omega(G)} \leqslant \|\nu\|_2 \cdot \|u\|_{2,\omega^2}$. As $\|u\|_{2,\omega^2} \leqslant D|\varepsilon|^{-n/2-\alpha}$, $D > 0$, we obtain

$$\begin{aligned}
\operatorname{dist}\left(f^{[\frac{m}{2}+\alpha]+1}, J_A(E)\right) &\leqslant \left\| f^{[\frac{m}{2}+\alpha]+1} - \varphi \right\|_{A_\omega(G)} \\
&\leqslant K\varepsilon + D\varepsilon^{-n/2-\alpha}\left(\int_{EV_\varepsilon} \left|f^{[\frac{m}{2}+\alpha]+1}(x)\right|^2 dx\right)^{1/2} \\
&\leqslant K\varepsilon + D\varepsilon^{-n/2-\alpha} \sup_{x \in EV_\varepsilon} \left|f^{[\frac{m}{2}+\alpha]+1}(x)\right| |EV_\varepsilon|^{1/2} \\
&\leqslant K\varepsilon + D\frac{\varepsilon^{-n/2-\alpha}}{C_2^{1/2}} \varepsilon^{[\frac{m}{2}+\alpha]+1} |EV_\varepsilon|^{1/2}.
\end{aligned}$$

The following estimate of $|EV_\varepsilon|$ was obtained in [14]: for every small $\varepsilon > 0$

$$|EV_\varepsilon| = C\varepsilon^{n-m}.$$

Hence, for $\varepsilon > 0$ small enough,

$$\begin{aligned}
\operatorname{dist}\left(f^{[\frac{m}{2}+\alpha]+1}, J_\omega(E)\right) &\leqslant K\varepsilon + C'\varepsilon^{-n/2-\alpha}\varepsilon^{[\frac{m}{2}+\alpha]+1}\varepsilon^{(n-m)/2} \\
&= K\varepsilon + C'\varepsilon^{[\frac{m}{2}+\alpha]+1-m/2-\alpha}
\end{aligned}$$

for a new constant C' which does not depend on ε. Thus $f^{[\frac{m}{2}+\alpha]+1} \in J_\omega(E)$. It follows now by standard arguments that $J_{\mathcal{D}}(E)^{[\frac{m}{2}+\alpha]+1} \subset J_\omega(E)$. □

Corollary 6.3. *Let E be a compact subset of a smooth m-dimensional sub-manifold of the Lie group G. If E is a set of smooth synthesis, then E is of weak synthesis with $I_\omega(E)^{[m/2+\alpha]+1} = J_\omega(E)$.*

The following corollary is a generalisation of the Beurling–Pollard theorem for $A(\mathbb{T})$ (see [9]) and its weighted analog $A_\omega(\mathbb{T}^n)$, where ω is a weight on $\mathbb{Z}^n$ given by $\omega(k) = (1+|k|)^\alpha$, $\alpha > 0$ (see [18]).

Corollary 6.4. *Let E be a compact subset of a smooth m-dimensional sub-manifold of the Lie group G. Suppose that $f \in A_\omega(G)$ satisfies the condition*

$$\left|f(x)\right| \leqslant K \inf\left\{\|X\|\colon\ X \in \mathfrak{g},\ x\exp(-X) \in E\right\}^r \tag{6.2}$$

for some fixed norm $\|\cdot\|$ on $\mathfrak{g}$ and $K > 0$. Then f admits spectral synthesis for $A_\omega(G)$ if $r > m/2+\alpha$.

Proof. If f satisfies (6.2) then f vanishes on E. Let $V_\varepsilon = \exp B_\varepsilon$, where B_ε is the ball of radius ε of center 0 in $\mathfrak{g}$. Let $\nu = f$ on EV_ε and $\nu = 0$ elsewhere and let $u(x) = u_\varepsilon(x) \in \mathcal{D}(G)$ be the family of functions from the proof of Theorem 6.2. By arguments similar to the ones in that proof we obtain: $\varphi_\varepsilon = (f-\nu) * \check{u}_\varepsilon \in J_\omega(E) \subset J_\omega(\mathrm{null}(f))$, $f = \varphi_\varepsilon + (f - f * \check{u}_\varepsilon) + \nu * \check{u}_\varepsilon$ and

$$\lim_{\varepsilon\to 0} \|f - \varphi_\varepsilon\|_{A_\omega(G)} = \lim_{\varepsilon\to 0} \|\nu * \check{u}_\varepsilon\|_{A_\omega(G)}.$$

Moreover,

$$\begin{aligned}\|\nu * \check{u}_\varepsilon\|_{A_\omega(G)} &\leqslant \|\nu\|_2 \|u_\varepsilon\|_{2\omega^2} \leqslant \varepsilon^{-n/2-\alpha} \sup_{x\in EV_\varepsilon} \left|f(x)\right| |EV_\varepsilon|^{1/2} \\ &\leqslant C\varepsilon^{-n/2-\alpha}\varepsilon^r \varepsilon^{n/2-m/2}.\end{aligned}$$

Therefore if $r > m/2+\alpha$, then $\lim_{\varepsilon\to 0}\|f-\varphi_\varepsilon\|_{A_\omega(G)} = 0$ and hence $f \in J_\omega(E) \subset J_\omega(\mathrm{null}(f))$. □

Theorem 6.5. *Let G be a connected compact Lie group and ω be a symmetric weight on $\widehat{G}$ such that $\omega \leqslant C\omega_S^\alpha$ for some C, $\alpha > 0$. Let B be a group of affine transformations of G which preserves $A_\omega(G)$, i.e. $u(b(x)) \in A_\omega(G)$ for each $u(x) \in A_\omega(G)$ and each $b \in B$. Let $O \subset G$ be a closed m-dimensional B-orbit in G. Then O is a set of smooth synthesis and hence of weak synthesis with $I_\omega(O)^{[m/2+\alpha]+1} = J_\omega(O)$.*

Proof. The proof repeats the arguments of the proof of [14, Thm. 4.8 & Cor. 4.9], the affine transformations of G (see [14] for the definition) are assumed to preserve the algebra $A_\omega(G)$. □

Corollary 6.6. (i) *If $\omega \leqslant C\omega_S^\alpha$ for some C, $0 < \alpha < 1$ then each singleton is a set of spectral synthesis for $A_\omega(G)$.*

(ii) *If $\omega \geqslant C\omega_S^\alpha$ for some C, $\alpha \geqslant 1$ then no singleton is a set of spectral synthesis for $A_\omega(G)$.*

Proof. Thanks to Proposition 3.3, it is enough to prove the statements for the set $\{e\}$, where e is the identity element in G.

(i) We first note that $\{e\}$ is a 0-dimensional set of smooth synthesis. Indeed, $\mathcal{D}(G)$ is dense in $A_\omega(G)$ and hence for $u \in I_\omega(\{e\})$ there exists a sequence $\{u_n\} \subset \mathcal{D}(G)$ which converges to u. Letting $u_n' = u_n - u_n(e)$ we have $u_n' \in J_{\mathcal{D}}(\{e\})$ and $\|u - u_n'\|_{A_\omega(G)} \leqslant \|u - u_n\|_{A_\omega(G)} + \|u_n(e) - u(e)\|_\infty \leqslant 2\|u - u_n\|_{A_\omega(G)}$. Hence it follows form Theorem 6.2 that

$$I_\omega\big(\{e\}\big) = J_{\mathcal{D}}\big(\{e\}\big) = J_{\mathcal{D}}\big(\{e\}\big)^{[\alpha+1]} = J_\omega\big(\{e\}\big).$$

Alternatively, we note that $\{e\}$ is an orbit under the group $B = \{s \mapsto tst^{-1}\}$ of inner automorphisms, and we can appeal directly to Theorem 6.5.

(ii) Assume now $\alpha \geqslant 1$. Let $X_1, \dots, X_n$ and Ω be as in (5.1). Then for $\pi \in \widehat{G}$ we have by virtue of (5.4), and the fact that each X_i is skew-hermitian, that

$$\frac{\|\pi(X_i)\|_{\mathrm{op}}}{\omega(\pi)} \leqslant C\frac{(1+c(\pi))^{1/2}}{(1+\|\pi\|_1)^\alpha} \leqslant C'\frac{(1+\|\pi\|_1)}{(1+\|\pi\|_1)^\alpha} \leqslant C''$$

for some constants C, C', C'', and hence $X_i \in A_\omega(G)^*$. Thus $\mathfrak{g} \subset A_\omega(G)^*$, where each element of $\mathfrak{g}$ defines a bounded point derivation at e. We note for each non-zero X in $\mathfrak{g}$, $I_\omega(\{e\}) \not\subset \ker(X)$ (indeed $J_{\mathcal{D}}(\{e\}) \not\subset \ker(X)$), but $I_\omega(\{e\})^2 \subset \ker(X)$. Hence $\overline{I_\omega(\{e\})^2} \subsetneq I_\omega(\{e\})$. $\square$

Remark 6.7. (1) Corollary 6.6 is a generalisation of the result on spectral synthesis of singletons for $A_\omega(\mathbb{T}^n)$, where $\omega(k) = (1+|k|)^\alpha$, $k \in \mathbb{Z}^n$, $\alpha > 0$. Note that for such weights a stronger result holds: singletons are Ditkin sets (see [17, Ch. 6.3]).

(2) For the dimension weight $\omega(\pi) = d_\pi = \omega_S$ $(S = \{\pi_1\})$ on $G = \mathrm{SU}(2)$, the failure of spectral synthesis for $A_\omega(G)$ at $\{e\}$ was noted in [7].

6.2. *Operator synthesis and spectral synthesis for $A_\omega(G)$*

We let G denote a separable compact group with normalised Haar measure m and ω be a bounded weight. For a function $u \in A_\omega(G)$ and $t, s \in G$ define

$$(Nu)(s,t) = u\big(st^{-1}\big).$$

Consider the projective tensor product $L^2_\omega(G) \hat{\otimes} L^2_\omega(G)$. Every $\Psi = \sum_i f_i \otimes g_i \in L^2_\omega(G) \hat{\otimes} L^2_\omega(G)$ can be identified with a function $\Psi : G \times G \to \mathbb{C}$ which admits a representation

$$\Psi(s,t) = \sum_{i=1}^\infty f_i(t) g_i(s),$$

$\sum_i \|f_i\|_{2,\omega} \cdot \|g_i\|_{2,\omega} < \infty$. Such a representation defines a function marginally almost everywhere (m.a.e.), i.e. two functions which coincide everywhere apart a marginally null set are identified. Recall that a subset $E \subset G \times G$ is *marginally null* if $E \subset (M \times G) \cup (G \times N)$ and $m(M) = m(N) = 0$.

Proposition 6.8. $Nu \in L^2_\omega(G) \hat{\otimes} L^2_\omega(G)$.

Proof. Using the Fourier inversion formula we have

$$Nu(s,t) = \sum_{\pi \in \widehat{G}} \operatorname{Tr}\bigl(\pi(s)\pi\bigl(t^{-1}\bigr)\hat{u}(\pi)\bigr)d_\pi.$$

Let $\{e_i^\pi\colon\ i = 1, \dots, d_\pi\}$ be an orthonormal basis in $\mathcal{H}_\pi$. Consider for each $\pi \in \widehat{G}$ the polar decomposition $\hat{u}(\pi) = V(\pi)|\hat{u}(\pi)|$. Then

$$\begin{aligned} Nu(s,t) &= \sum_{\pi \in \widehat{G}} \sum_{i=1}^{d_\pi} \bigl(\bigl|\hat{u}(\pi)\bigr|^{1/2}\pi(s)e_i^\pi, \bigl|\hat{u}(\pi)\bigr|^{1/2}V(\pi)^*\pi(t)e_i^\pi\bigr)d_\pi \\ &= \sum_{\pi \in \widehat{G}} \sum_{i,j=1}^{d_\pi} \bigl(\bigl|\hat{u}(\pi)\bigr|^{1/2}\pi(s)e_i^\pi, e_j^\pi\bigr)\bigl(e_j^\pi, \bigl|\hat{u}(\pi)\bigr|^{1/2}V(\pi)^*\pi(t)e_i^\pi\bigr)d_\pi. \end{aligned}$$

Let

$$\varphi_{i,j}^\pi(s) = \bigl(\bigl|\hat{u}(\pi)\bigr|^{1/2}\pi(s)e_i^\pi, e_j^\pi\bigr)d_\pi^{1/2}$$

and

$$\psi_{i,j}^\pi(t) = \bigl(e_j^\pi, \bigl|\hat{u}(\pi)\bigr|^{1/2}V(\pi)^*\pi(t)e_i^\pi\bigr)d_\pi^{1/2}.$$

In order to show the statement we have to prove that

$$\sum_{\pi \in \widehat{G}} \sum_{i,j=1}^{d_\pi} \bigl\|\varphi_{i,j}^\pi\bigr\|_{2,\omega}^2 < \infty \quad \text{and} \quad \sum_{\pi \in \widehat{G}} \sum_{i,j=1}^{d_\pi} \bigl\|\psi_{i,j}^\pi\bigr\|_{2,\omega}^2 < \infty.$$

Using the orthogonality property of matrix coefficients one can see that

$$\bigl(\hat{\varphi}_{i,j}^\pi(\rho)e_k^\pi, e_l^\pi\bigr) = \frac{1}{d_\pi^{1/2}}\bigl(e_k^\pi, \bigl|\hat{u}(\pi)\bigr|^{1/2}e_j^\pi\bigr)\delta_{il}\delta_{\pi\rho}.$$

Hence

$$\begin{aligned} \bigl\|\hat{\varphi}_{i,j}^\pi(\pi)\bigr\|_2^2 &= \sum_{k=1}^{d_\pi}\bigl\|\hat{\varphi}_{i,j}^\pi(\pi)e_k^\pi\bigr\|^2 = \sum_{k,l=1}^{d_\pi}\bigl(\hat{\varphi}_{i,j}^\pi(\pi)e_k^\pi, e_l^\pi\bigr)\bigl(e_l^\pi, \hat{\varphi}_{i,j}^\pi(\pi)e_k^\pi\bigr) \\ &= \sum_{k=1}^{d_\pi}\bigl(\hat{\varphi}_{i,j}^\pi(\pi)e_k^\pi, e_i^\pi\bigr)\bigl(e_i^\pi, \hat{\varphi}_{i,j}^\pi(\pi)e_k^\pi\bigr) \\ &= \frac{1}{d_\pi}\sum_{k=1}^{d_\pi}\bigl(e_k^\pi, \bigl|\hat{u}(\pi)\bigr|^{1/2}e_j^\pi\bigr)\bigl(\bigl|\hat{u}(\pi)\bigr|^{1/2}e_j^\pi, e_k^\pi\bigr) \\ &= \frac{1}{d_\pi}\bigl(\bigl|\hat{u}(\pi)\bigr|e_j^\pi, e_j^\pi\bigr) \end{aligned}$$

and

$$\|\varphi_{i,j}^{\pi}\|_{2,\omega}^{2} = \omega(\pi)\big(|\hat{u}(\pi)|e_j^{\pi}, e_j^{\pi}\big)$$

giving

$$\sum_{\pi\in\widehat{G}}\sum_{i,j=1}^{d_\pi}\|\varphi_{i,j}^{\pi}\|_{2,\omega}^{2} = \sum_{\pi\in\widehat{G}}\sum_{i,j=1}^{d_\pi}\omega(\pi)\big(|\hat{u}(\pi)|e_j^{\pi}, e_j^{\pi}\big)$$
$$= \sum_{\pi} d_\pi\omega(\pi)\|\hat{u}(\pi)\|_1 = \|u\|_{A_\omega(G)}.$$

Similarly, $\sum_{\pi\in\widehat{G}}\sum_{i,j=1}^{d_\pi}\|\psi_{i,j}^{\pi}\|_{2,\omega}^{2} = \|u\|_{A_\omega(G)}$. $\square$

Let $H = L^2_\omega(G)$. Then the projective tensor product $\mathcal{T}(H) = H\hat{\otimes}H$ can be identified with the trace class operators on H. Recall that $\mathcal{L}(H) = \mathcal{T}(H)^*$ via

$$\langle T, f\otimes g\rangle_{\mathcal{L}(H),\mathcal{T}(H)} = (Tf, \bar{g})_{2,\omega}.$$

Let $T = (T_\pi)_{\pi\widehat{G}} \in A_\omega(G)^*$. Define a linear bounded operator $S(T)$ on H by $\widehat{S(T)f}(\pi) = \frac{1}{\omega(\pi)}\hat{f}(\pi)T_\pi$. Since $\sup_{\pi\in\widehat{G}}\frac{\|T_\pi\|_{\mathrm{op}}}{\omega(\pi)} < \infty$, $S(T)$ is bounded.

Lemma 6.9. *Let $u\in A_\omega(G)$, $T\in A_\omega(G)^*$. Then*

$$\langle T, u\rangle = \big\langle S(T), Nu\big\rangle_{\mathcal{L}(H),\mathcal{T}(H)}.$$

Proof. It is enough to show the statement for a matrix coefficient $u(t) = (\pi(t)e_i^{\pi}, e_j^{\pi}) =: c_{ji}^{\pi}$, where $\{e_i^{\pi},\ i = 1,\dots,d_\pi\}$ is an orthonormal basis in $\mathcal{H}_\pi$. We have $\langle T, u\rangle = (T_\pi e_i^{\pi}, e_j^{\pi})$.

$$Nu(s,t) = u\big(st^{-1}\big) = \big(\pi\big(t^{-1}\big)e_i^{\pi}, \pi\big(s^{-1}\big)e_j^{\pi}\big) = \sum_{k=1}^{d_\pi}\big(\pi(s)e_k^{\pi}, e_j^{\pi}\big)\big(e_i^{\pi}, \pi(t)e_k^{\pi}\big).$$

Hence

$$\big\langle S(T), Nu\big\rangle_{\mathcal{L}(H),\mathcal{T}(H)} = \sum_{k=1}^{d_\pi}\big(S(T)c_{jk}^{\pi}, c_{ik}^{\pi}\big) = \sum_{k=1}^{d_\pi}\sum_{\rho\in\widehat{G}} d_\rho\,\mathrm{Tr}\big(\hat{c}_{jk}^{\pi}(\rho)T_\rho\hat{c}_{ik}^{\pi}(\rho)^*\big).$$

Since $(\hat{c}_{jk}^{\pi}(\rho)e_m^{\pi}, e_l^{\pi}) = \frac{1}{d_\pi}\delta_{\pi\rho}\delta_{kl}\delta_{jm}$ and

$$\sum_{k=1}^{d_\pi}\hat{c}_{ik}^{\pi}(\pi)^*\hat{c}_{jk}^{\pi}(\pi)(f) = \frac{1}{d_\pi}\big(f, e_j^{\pi}\big)e_i^{\pi}$$

we have

$$\langle S(T), Nu\rangle_{\mathcal{L}(H),\mathcal{T}(H)} = \left(T_\pi e_i^\pi, e_j^\pi\right). \qquad \square$$

Let $E \subset G$. We set

$$E^* = \{(s,t) \in G \times G\colon st^{-1} \in E\}.$$

Definition 6.10. If $S \in \mathcal{L}(L^2_\omega(G))$, we define the support of S (written $\mathrm{supp}_{\mathcal{L}}(S)$) as the set of all points $(s,t) \in G \times G$ with the following property: for any neighbourhoods U of s, V of t, there are f, g in $L^2_\omega(G)$ such that $\mathrm{supp}(f) \subset U$, $\mathrm{supp}(g) \subset V$ and $(Sf, g)_{2,\omega} \neq 0$.

Lemma 6.11. *Let $T \in A_\omega(G)^*$. Then* $\mathrm{supp}_{\mathcal{L}}(S(T)) \subset \mathrm{supp}(T)^*$.

Proof. The proof is similar to the one given in [19, Thm. 4.6], we include it for completeness. It follows from the definition that $(S(T)f, g)_{2,\omega} = \langle T, f * \check{\bar{g}}\rangle$ for $f, g \in L^2_\omega(G)$, where $\check{g}(s) = g(s^{-1})$.

If $(s,t) \in \mathrm{supp}_{\mathcal{L}}(S(T))$ but $st^{-1} \notin \mathrm{supp}(T)^*$, find neighbourhoods U of s and V of t such that $UV^{-1} \cap W = \emptyset$, where W is a neighbourhood of $\mathrm{supp}(T)$, and then take $f, g \in L^2_\omega(G)$ supported in U and V respectively and $(S(T)f, g)_{2,\omega} \neq 0$. As $(S(T)f, g)_{2,\omega} = \langle T, u\rangle$, where $u = f * \check{\bar{g}}$, $\mathrm{supp}(u) \subset \overline{UV^{-1}}$, so u vanishes in a neighbourhood of $\mathrm{supp}(T)$ and hence $\langle T, u\rangle = 0$ giving a contradiction. $\square$

Proposition 6.12. *Let S be an operator on $L^2_\omega(G)$ and let E, $F \subset G$ be closed. Then if $E \times F$ is disjoint from* $\mathrm{supp}_{\mathcal{L}}(S)$ *then $(Sf, g)_{2,\omega} = 0$ for any f, $g \in L^2_\omega(G)$ supported in E and F respectively.*

Proof. The proof is similar to [1, Prop. 2.2.5]. $\square$

We say that $E \subset G \times G$ is a *set of operator synthesis* with respect to the weight ω if $\langle S, F\rangle_{\mathcal{L}(H),\mathcal{T}(H)} = 0$ whenever $S \in \mathcal{L}(L^2_\omega(G))$ and $F \in L^2_\omega(G) \hat{\otimes} L^2_\omega(G)$ with $\mathrm{supp}_{\mathcal{L}} T \subset E$ and $F|_E = 0$ m.a.e.

Proposition 6.13. *Let G be a connected compact group and ω be a symmetric weight of polynomial growth. Let $E \subset G$ be closed. If E^* is a set of operator synthesis with respect to the weight ω on $\widehat{G}$ then E is a set of spectral synthesis for $A_\omega(G)$.*

Proof. By assumption, $A_\omega(G)$ is a semisimple, regular, commutative Banach algebra with unit. Let $T \in A_\omega(G)^*$, $u \in A_\omega(G)$ such that $\mathrm{supp}\, T \subset E \subset \mathrm{null}(u)$. Then $Nu = 0$ on E^* and by Lemma 6.11 $\mathrm{supp}_{\mathcal{L}}(S(T)) \subset E^*$. The statement now follows from the equality $\langle T, u\rangle = \langle S(T), Nu\rangle_{\mathcal{L}(H),\mathcal{T}(H)}$ which is due to Lemma 6.9. $\square$

Corollary 6.14. *Let G be a connected compact Lie group and let $D = \{(x,x)\colon x \in G\}$. If ω is a symmetric polynomial weight on $\widehat{G}$ such that $\omega \geqslant C\omega_S^\alpha$ for some $C > 0$, $\alpha \geqslant 1$ then D is not of operator synthesis with respect to the weight ω.*

Proof. If $E = \{e\}$ then $E^* = D$. The statement now follows from Corollary 6.6 and Proposition 6.13. □

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