Subdigraphs with orthogonal factorizations of digraphs

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ABSTRACT

Let \( G = (V, E) \) be a digraph and let \( g \) and \( f \) be two pairs of integer-valued functions defined on \( V \) such that \( n \leq g(x) \leq f(x) \) for every \( x \in V \). Let \( H_1, H_2, \ldots, H_n \) be arc-disjoint \( k \)-subdigraphs of \( G \). In this article, we prove that every \((mg + k - 1, mf' - k + 1)\)-digraph \( G \) contains a subdigraph \( R \) such that \( R \) has a \((g, f)\)-factorization orthogonal to \( H_i \) \((1 \leq i \leq n)\), where \( m \) and \( k \) are positive integers with \( 1 \leq k \leq m \).

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1. Introduction

Many real-world networks can be modeled by graphs or networks. An important example of such a network is a communication network with nodes and links modeling cities and communication channels, respectively. Other examples include a railroad network with nodes and links representing railroad stations and railways between two stations, respectively, or the World Wide Web with nodes representing Web pages, and links corresponding to hyperlinks between Web pages. Orthogonal factorizations in networks are very useful in combinatorial design, network design, circuit layout, and so on [2], and attract a great deal of attention [4,6,8,9,11,7,14,16]. All graphs considered in this article are finite directed graphs (digraphs) with no loops or parallel arcs. For a general reference on graph theory, the reader is directed to [3,15].

Let \( G \) be a digraph with vertex set \( V \) and arc set \( E \). For any vertex \( x \in V \), we denote the indegree and outdegree of \( x \) by \( \deg^-_G(x) \) and \( \deg^+_G(x) \), respectively. We use \( uv \) to denote the arc with tail \( u \) and head \( v \). Let \( g = (g^-, g^+) \) and \( f = (f^-, f^+) \) be pairs of positive integer-valued functions defined on \( V \) such that \( g^-(x) \leq f^-(x) \) and \( g^+(x) \leq f^+(x) \) for every \( x \in V \). A digraph \( G \) is called a \((g, f)\)-digraph if \( g^-(x) \leq \deg^-_G(x) \leq f^-(x) \) and \( g^+(x) \leq \deg^+_G(x) \leq f^+(x) \) for every \( x \in V \). A spanning subdigraph \( F \) of \( G \) is called a \((g, f)\)-factor of \( G \) if \( F \) itself is a \((g, f)\)-digraph. A subdigraph \( H \) of \( G \) is called an \( m \)-subdigraph if \( H \) has \( m \) arcs. For convenience, we write \( g \leq f \) if \( g^-(x) \leq f^-(x) \) and \( g^+(x) \leq f^+(x) \) for every \( x \in V \), and we write \( mf + n \) for \((mf^- + n, mf^+ + n)\). A \((g, f)\)-factorization \( \mathcal{F} = \{F_1, \ldots, F_m\} \) of \( G \) is a partition of \( E \) into arc-disjoint \((g, f)\)-factors \( F_1, \ldots, F_m \). Let \( H \) be an \( m \)-subdigraph of \( G \), and let \( k \geq 1 \) be a fixed

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integer. A factorization $\mathcal{F} = \{F_1, \ldots, F_m\}$ of $G$ is called $k$-orthogonal to $H$ if $|E(F_i) \cap E(H)| = k$ for $i = 1, \ldots, m$. In particular, 1-orthogonal is abbreviated as orthogonal.

Alspach et al. [2] posed the following problem: Given a subgraph $H$ of $G$, does there exist a factorization $\mathcal{F}$ of $G$ with a given property orthogonal to $H$?

Orthogonal $(g, f)$-factorization of an $(mg + m - 1, mf - m + 1)$-graph has been studied in, for instance, [6,8,9,11]. Gallai [5] and Tutte [13] considered the $(g, f)$-factors of digraphs. Liu [10] studied the $k$-orthogonal $(g, f)$-factorization of an $(mg + m - 1, mf - m + 1)$-digraph. For a comprehensive introduction to graph factors and factorization, the reader is directed to [1,12].

The existence of a subgraph with orthogonal factorization has been investigated in [7,14,16], and the following result was recently proved in [14].

**Theorem 1** ([14]). Let $G$ be an $(mg + k, mf - k)$-graph, and let $H_1, \ldots, H_n$ be vertex-disjoint subgraphs of $G$ with $k$ edges, where $1 \leq k < m$ and $n \leq g(x) \leq f(x)$ for every $x \in V$. Then there exists a subgraph $R$ of $G$ such that $R$ has a $(g, f)$-factorization orthogonal to every $H_i$, $i = 1, \ldots, n$.

In the present article, we study the orthogonal factorizations in digraphs. We obtain an analogue of Theorem 1 for digraphs, which strengthens a result in [10] (see Corollary 10). The main result of this article is the following.

**Theorem 2.** Let $G$ be an $(mg + k - 1, mf - k + 1)$-digraph, and let $H_1, \ldots, H_n$ be arc-disjoint $k$-subdigraphs of $G$, where $1 \leq k \leq m$ and $n \leq g(x) \leq f(x)$ for every $x \in V$. Then there exists a subdigraph $R$ of $G$ such that $R$ has a $(g, f)$-factorization orthogonal to every $H_i$, $i = 1, \ldots, n$.

The rest of the article is organized as follows. All lemmas are presented in Section 2, and the proof of Theorem 2 can be found in Section 3.

## 2. Lemmas

Let $G = (V, E)$ be a digraph. For any function $f$ defined on $V$ and $S \subseteq V$, we write $f(S)$ for $\sum_{x \in S} f(x)$ and $f(\emptyset) = 0$. For two subsets $S$ and $T$ of $V$, we write $E_G(S, T)$ for the set $\{uv : uv \in E, u \in S, v \in T\}$, and let $e_G(S, T) = |E_G(S, T)|$. Define

$$\gamma_{1G}(S, T) = f^+(S) - g^-(T) + e_G(V - S, T),$$

and

$$\gamma_{2G}(S, T) = f^-(T) - g^+(S) + e_G(S, V - T).$$

Gallai [5] obtained the following necessary and sufficient condition for the existence of a $(g, f)$-factor in a digraph.

**Lemma 3** ([5]). Let $G$ be a digraph, and let $g = (g^-, g^+)$ and $f = (f^-, f^+)$ be pairs of positive integer-valued functions defined on $V$ such that $g(x) \leq f(x)$ for every $x \in V$. Then $G$ has a $(g, f)$-factor if and only if $\gamma_{1G}(S, T) \geq 0$ and $\gamma_{2G}(S, T) \geq 0$ for all subsets $S$ and $T$ of vertices.

Let $E_1$ and $E_2$ be two disjoint subsets of $E$, and let $S$ and $T$ be two subsets of $V$. Define, for $i = 1, 2$,

$$E_{iS} = E_i \cap E(S, V - T), \quad E_{iT} = E_i \cap E(V - S, T),$$

and set

$$\alpha_S = |E_{1S}|, \quad \alpha_T = |E_{1T}|, \quad \beta_S = |E_{2S}|, \quad \beta_T = |E_{2T}|.$$

The following lemmas, whose proofs can be found in [10], are useful for proving Theorem 2.

**Lemma 4** ([10]). Let $G$ be a digraph, and let $g = (g^-, g^+)$ and $f = (f^-, f^+)$ be pairs of positive integer-valued functions defined on $V$ such that $g(x) \leq f(x)$ for every $x \in V$. Let $E_1$ and $E_2$ be two disjoint subsets of $E$. Then $G$ has a $(g, f)$-factor $F$ such that $E_1 \subseteq E(F)$ and $E_2 \cap E(F) = \emptyset$ if and only if for all subsets $S$ and $T$ of vertices, $\gamma_{1G}(S, T) \geq \alpha_S + \beta_T$ and $\gamma_{2G}(S, T) \geq \alpha_T + \beta_S$. 
Lemma 5 ([10]). Every \((mg, mf)\)-digraph \(G\) admits a \((g, f)\)-factorization.

In the following, we assume that \(G\) is an \((mg + k - 1, mf - k + 1)\)-graph, where \(m\) and \(k\) are integers with \(1 \leq k \leq m\). Define

\[
p^-(x) = \max\{g^-(x), \deg^-_G(x) - (m - 1)f^-(x) + k - 2\},
\]

\[
p^+(x) = \max\{g^+(x), \deg^+_G(x) - (m - 1)f^+(x) + k - 2\},
\]

\[
q^-(x) = \min\{f^-(x), \deg^-_G(x) - (m - 1)g^-(x) - k + 2\},
\]

and

\[
q^+(x) = \min\{f^+(x), \deg^+_G(x) - (m - 1)g^+(x) - k + 2\}.
\]

Let \(p(x) = (p^-(x), p^+(x))\) and \(q(x) = (q^-(x), q^+(x))\). By the definition of \(p(x)\) and \(q(x)\), the following result is an easy exercise, so it is left to the reader.

Lemma 6. For every \(x \in V\),

\[
g(x) \leq p(x) < q(x) \leq f(x).
\]

Set

\[
\Delta^{-}_1(x) = \frac{1}{m} \deg^-_G(x) - p^-(x), \quad \Delta^{+}_1(x) = \frac{1}{m} \deg^+_G(x) - p^+(x),
\]

\[
\Delta^{-}_2(x) = q^-(x) - \frac{1}{m} \deg^-_G(x), \quad \Delta^{+}_2(x) = q^+(x) - \frac{1}{m} \deg^+_G(x),
\]

and let \(\Delta_1(x) = (\Delta^{-}_1(x), \Delta^{+}_1(x))\) and \(\Delta_2(x) = (\Delta^{-}_2(x), \Delta^{+}_2(x))\).

Lemma 7. For every \(x \in V\) and \(m \geq k \geq 2\),

(i) \(\Delta^{-}_1(x) \geq \begin{cases} 
\frac{k - 1}{m}, & \text{if } p^-(x) = g^-(x); \\
1 - \frac{k - 1}{m}, & \text{otherwise}.
\end{cases}\)

(ii) \(\Delta^{+}_1(x) \geq \begin{cases} 
\frac{k - 1}{m}, & \text{if } p^+(x) = g^+(x); \\
1 - \frac{k - 1}{m}, & \text{otherwise}.
\end{cases}\)

(iii) \(\Delta^{-}_2(x) \geq \begin{cases} 
\frac{m}{k - 1}, & \text{if } q^-(x) = f^-(x); \\
1 - \frac{m}{k - 1}, & \text{otherwise}.
\end{cases}\)

(iv) \(\Delta^{+}_2(x) \geq \begin{cases} 
\frac{m}{k - 1}, & \text{if } q^+(x) = f^+(x); \\
1 - \frac{m}{k - 1}, & \text{otherwise}.
\end{cases}\)

In particular, \(\Delta_1(x) \geq \frac{1}{m}\) and \(\Delta_2(x) \geq \frac{1}{m}\) for every \(x \in V\).

Proof. We only prove (i), as the proofs of (ii)–(iv) are similar. If \(p^-(x) = g^-(x)\), then

\[
\Delta^{-}_1(x) = \frac{1}{m} \deg^-_G(x) - p^-(x)
\]

\[
\geq \frac{1}{m}(mg^-(x) + k - 1) - g^-(x)
\]

\[
= \frac{k - 1}{m}.
\]
Otherwise, by the definition of \( p^- (x) \), we have that

\[
p^- (x) = \deg^-_G (x) - (m - 1)f^- (x) + k - 2.
\]

So,

\[
\Delta^-_1 (x) = \frac{1}{m} \deg^-_G (x) - (\deg^-_G (x) - (m - 1)f^- (x) + k - 2)
\]

\[
= \frac{1 - m}{m} \deg^-_G (x) + (m - 1)f^- (x) - k + 2
\]

\[
\geq \frac{1 - m}{m} (mf^- (x) - k + 1) + (m - 1)f^- (x) - k + 2
\]

\[
= 1 - \frac{k - 1}{m}.
\]

Lemma 8 ([10]). For any two subsets \( S \) and \( T \) of vertices,

\[
\gamma_{1G} (S, T; p, q) = \Delta^-_1 (T) + \Delta^+_2 (S) + \frac{m - 1}{m} e_G (V - S, T) + \frac{1}{m} e_G (S, V - T),
\]

\[
\gamma_{2G} (S, T; p, q) = \Delta^+_1 (S) + \Delta^-_2 (T) + \frac{m - 1}{m} e_G (S, V - T) + \frac{1}{m} e_G (V - S, T).
\]

3. Proof of Theorem 2

Let \( G \) be a digraph, and let \( g = (g^-, g^+) \) and \( f = (f^-, f^+) \) be pairs of positive integer-valued functions defined on \( V \) such that \( n \leq g(x) \leq f(x) \) for every \( x \in V \). Let \( H_1, \ldots, H_n \) be arc-disjoint \( k \)-subdigraphs of \( G \). For \( i = 1, \ldots, n \), take \( u_i v_i \in E(H_i) \). Let \( E_1 = \{ u_1 v_1, \ldots, u_n v_n \} \) and \( E_2 = \bigcup_{i=1}^n (E(H_i) - \{ u_i v_i \}) \). Clearly, \( |E_1| = n \) and \( |E_2| = (k - 1)n \). For two subsets \( S \) and \( T \) of \( V \), let \( E_S, E_T (i = 1, 2), \alpha_S, \alpha_T, \beta_S \) and \( \beta_T \) be defined as in Section 2. Define \( p(x), q(x), \Delta_1(x) \) and \( \Delta_2(x) \) as before. It is obvious that

\[
\alpha_S \leq \min \{ n, |S| \} \quad \text{and} \quad \beta_S \leq \min \{ (k - 1)n, (k - 1)|S| \},
\]

\[
(3.1)
\]

and

\[
\alpha_T \leq \min \{ n, |T| \} \quad \text{and} \quad \beta_T \leq \min \{ (k - 1)n, (k - 1)|T| \}.
\]

(3.2)

Now we are ready to prove the following lemma, which is useful for proving Theorem 2.

Lemma 9. Let \( G \) be an \((mg + k - 1, mf - k + 1)\)-digraph with \( m \geq k \geq 2 \) and \( f(x) \geq g(x) \geq n \) for every \( x \in V \). Then \( G \) has a \((p, q)\)-factor \( F_1 \) such that \( E_1 \subseteq E(F_1) \) and \( E_2 \cap E(F_1) = \emptyset \).

Proof. We prove this by contradiction. Suppose that the statement “\( G \) has a \((p, q)\)-factor \( F_1 \) such that \( E_1 \subseteq E(F_1) \) and \( E_2 \cap E(F_1) = \emptyset \)” is false. Then by Lemma 4, there are two subsets \( S \) and \( T \) of \( V \) such that

\[
\gamma_{1G} (S, T) = \gamma_{1G} (S, T; p, q) < \alpha_S + \beta_T
\]

or

\[
\gamma_{2G} (S, T) = \gamma_{2G} (S, T; p, q) < \alpha_T + \beta_S.
\]

Without loss of generality, we may assume in the following that

\[
\gamma_{1G} (S, T) < \alpha_S + \beta_T.
\]

(3.3)
If $S = \emptyset$, then $\alpha_S = 0$. By Lemma 8, we have

$$\gamma_{1c}(S, T) = \Delta_1^+(T) + \frac{m - 1}{m}e_c(V, T)$$

$$\geq \frac{m - 1}{m} \deg^-(T)$$

$$\geq \frac{m - 1}{m} (mn + k - 1) |T|$$

$$\geq (m - 1)n|T|$$

$$\geq \beta_T$$

$$= \beta_T + \alpha_S,$$

which contradicts (3.3).

If $T = \emptyset$, then $\beta_T = 0$. By Lemma 8, we have

$$\gamma_{1c}(S, T) = \Delta_2^+(S) + \frac{1}{m}e_c(S, V)$$

$$\geq \frac{1}{m} \deg^+_c(S)$$

$$\geq \frac{1}{m} (mn + k - 1) |S|$$

$$\geq n|S|$$

$$\geq \alpha_S$$

$$= \alpha_S + \beta_T,$$

which contradicts (3.3) too. Hence, $S \neq \emptyset$ and $T \neq \emptyset$.

We now discuss the following four cases.

Case 1. $1 \leq |S| \leq k - 1$ and $|T| \geq 2$.

By Lemmas 7 and 8, and the hypothesis $g(x) \geq n$, we have that

$$\gamma_{1c}(S, T) = \Delta_1^-(T) + \Delta_2^+(S) + \frac{m - 1}{m}e_c(V - S, T) + \frac{1}{m}e_c(S, V - T)$$

$$\geq \frac{m - 1}{m}e_c(V - S, T)$$

$$= \frac{m - 1}{m} (\deg^-(T) - e_c(S, T))$$

$$\geq \frac{m - 1}{m} (mn + k - 1 - |S|) |T|$$

$$\geq (m - 1)n|T|$$

$$\geq 2(m - 1)n$$

$$\geq kn$$

$$\geq \alpha_S + \beta_T,$$

which again contradicts (3.3).

Case 2. $1 \leq |S| \leq k - 1$ and $|T| = 1$.

By Lemmas 7 and 8, and the hypothesis $g(x) \geq n$, we have that

$$\gamma_{1c}(S, T) = \Delta_1^-(T) + \Delta_2^+(S) + \frac{m - 1}{m}e_c(V - S, T) + \frac{1}{m}e_c(S, V - T)$$

$$\geq \frac{|S| + |T|}{m} + \frac{m - 1}{m}e_c(V - S, T) + \frac{1}{m}e_c(S, V - T)$$

which again contradicts (3.3).
a contradiction.

Case 3. $|S| \geq k$ and $|T| \leq k - 1$.

By Lemmas 7 and 8, and the hypothesis $g(x) \geq n$, we have that

$$\gamma_{\alpha}(S, T) = \Delta_1(T) + \Delta^+_2(S) + \frac{m - 1}{m} e_G(V - S, T) + \frac{1}{m} e_G(S, V - T)$$

$$\geq \frac{1}{m} e_G(S, V - T)$$

$$\geq \frac{1}{m} (mn + k - 1 - |T|) |S|$$

$$\geq n|S|$$

$$\geq kn$$

$$\geq \alpha_s + \beta_T,$$

which contradicts (3.3).

Case 4. $|S| \geq k$ and $|T| \geq k$.

We know, from the definitions of $\alpha_s$ and $\beta_T$, that

$$e_G(S, V - T) \geq \alpha_s$$ and $$e_G(V - S, T) \geq \beta_T.$$

For each $u \in S$, let $E_u = \{ uv : uv \in E_1 \}$. Take $x \in S$ such that $|E_{1x}| = \min(|E_{1u}| : u \in S)$. As $|S| \geq k$ and $|E_1| = n$, we have

$$e_G(S - \{ x \}, V - T) \geq \alpha_s - \frac{n}{k}. \tag{3.4}$$

Similarly, there exists $y \in T$ such that

$$e_G(V - S, T - \{ y \}) \geq \beta_T - \frac{(k - 1)n}{k}. \tag{3.5}$$

By Lemmas 7 and 8, (3.1), (3.2), (3.4), (3.5), and the hypothesis $g(x) \geq n$, we have that

$$\gamma_{\alpha}(S, T) = \Delta_1(T) + \Delta^+_2(S) + \frac{m - 1}{m} e_G(V - S, T) + \frac{1}{m} e_G(S, V - T)$$

$$\geq \frac{|S| + |T|}{m} + \frac{m - 1}{m} e_G(V - S, T) + \frac{1}{m} e_G(S, V - T)$$

$$\geq \frac{|S| + e_G(V - S, T)}{m} + \frac{|T| + e_G(S, V - T)}{m} + \frac{m - 2}{m} e_G(V - S, T)$$

$$= \frac{\deg_G(y) + e_G(V - S, T - \{ y \})}{m} + \frac{\deg_G^+(x) + e_G(S - \{ x \}, V - T)}{m}$$

$$+ \frac{m - 2}{m} e_G(V - S, T)$$

$$\geq \frac{mn + k - 1 + \beta_T - \frac{(k - 1)n}{k}}{m} + \frac{mn + k - 1 + \alpha_s - \frac{n}{k}}{m} + \frac{m - 2}{m} \beta_T$$

$$\geq \alpha_s + \beta_T.$$

This last contradiction completes the proof. \Box
Proof of Theorem 2. We proceed by induction on \( m \) and \( k \). According to Lemma 5, the theorem is true when \( k = 1 \). Hence, we may assume that \( k \geq 2 \) in the following. For the inductive step, we assume that the theorem holds for any \((m'g+k'-1, m'f-k'+1)\)-digraph \( G' \) with \( m' < m \), \( k' < k \) and \( 1 \leq k' \leq m' \), and any arc-disjoint \( k' \)-subdigraphs \( H'_1, \ldots, H'_n \) of \( G' \). We now consider an \((mg + k - 1, mf - k + 1)\)-digraph \( G \) and any arc-disjoint \( k \)-subdigraphs \( H_1, \ldots, H_n \) of \( G \), where \( n \leq g(x) \leq f(x) \) for every \( x \in V \).

For \( i = 1, \ldots, n \), take \( u_iv_i \in E(H_i) \). Let \( E_1 = \{ u_1v_1, \ldots, u_nv_n \} \) and \( E_2 = \bigcup_{i=1}^n E(H_i) - E_1 \). By Lemma 9, \( G \) has a \((p, q)\)-factor \( F_1 \) such that \( E_1 \subseteq E(F_1) \) and \( E_2 \cap E(F_1) = \emptyset \). Clearly, \( F_1 \) is a \((g, f)\)-factor of \( G \). Let \( G' = G - E(F_1) \). Observe that, by (2.1)–(2.4),

\[
\deg_G^-(x) = \deg_G^-(x) - \deg_{F_1}^-(x) \\
\geq \deg_G^-(x) - q^-(x) \\
\geq (m - 1)g^-(x) + (k - 1) - 1,
\]

\[
\deg_G^+(x) = \deg_G^+(x) - \deg_{F_1}^+(x) \\
\leq \deg_G^+(x) - p^+(x) \\
\leq (m - 1)f^+(x) - (k - 1) + 1,
\]

\[
\deg_G^+(x) = \deg_G^+(x) - \deg_{F_1}^+(x) \\
\geq (m - 1)g^+(x) + (k - 1) - 1,
\]

and

\[
\deg_G^+(x) = \deg_G^+(x) - \deg_{F_1}^+(x) \\
\leq (m - 1)f^+(x) - (k - 1) + 1.
\]

Hence, \( G' \) is an \(( (m - 1)g + (k - 1) - 1, (m - 1)f - (k - 1) + 1) \)-digraph. Let \( H'_i = H_i - u_iv_i \) for \( i = 1, \ldots, n \). Clearly, \( H'_1, \ldots, H'_n \) are arc-disjoint \((k - 1)\)-subdigraphs of \( G' \). By the induction hypothesis, there exists a subdigraph \( R' \) of \( G' \) such that \( R' \) has a \((g, f)\)-factorization orthogonal to every \( H'_i, i = 1, \ldots, n \). Let \( R \) be the subdigraph of \( G \) induced by \( E(R') \cup E(F_1) \). Hence, \( R \) is a subdigraph of \( G \) such that \( R \) has a \((g, f)\)-factorization orthogonal to every \( H_i (i = 1, 2, \ldots, n) \).

Corollary 10 ([10]). Let \( G \) be an \((mg + m - 1, mf - m + 1)\)-digraph, and let \( g = (g^-, g^+) \) and \( f = (f^-, f^+) \) be pairs of integer-valued functions defined on \( V \) such that \( k \leq g(x) \leq f(x) \), and let \( H \) be a km-subdigraph of \( G \). Then \( G \) has a \((g, f)\)-factorization k-orthogonal to \( H \).

References