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Realizing the chromatic numbers of triangulations of surfaces

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Abstract

Given an orientable or nonorientable closed surface S and an integer n not less than 3 and not greater than the chromatic number of S, we construct a graph admitting a triangular embedding in S and having chromatic number n.

1. Introduction

The following two problems are basic in the theory of graph colorings: (1) Given a graph, determine its chromatic number, (2) The same for a given surface.

Computer scientists are interested mainly in Problem 1. With the proof of the Four-Color Theorem in 1976, Problem 2 was completely solved. The number of algorithms regarding Problem 1 continues to increase. Here we solve constructively the problem inverse to Problem 1, with the additional restriction that the graph must triangulate a given surface. The construction is interesting in itself, and readily allows modifications to obtain a graph with given chromatic number and admitting a polygonal embedding in a given surface, with additional restrictions on the numbers of m-gons in the embedding.

In general we follow the terminology and notation of [3].

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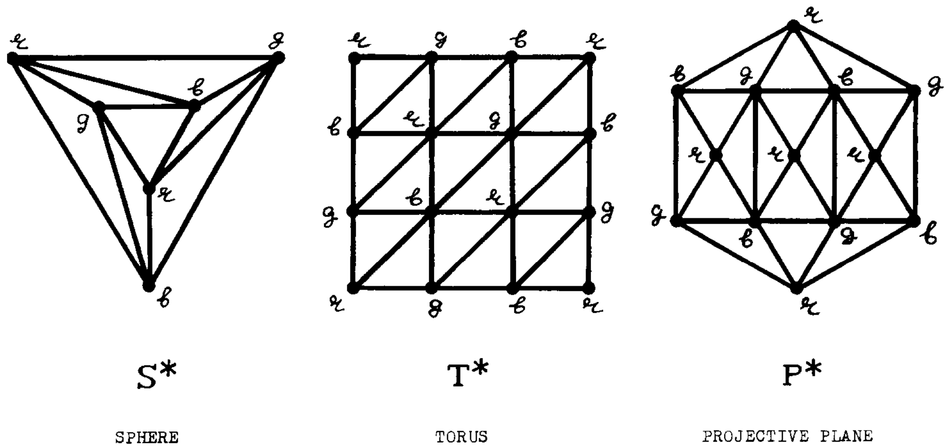


Fig. 1. Triangulations of a sphere, a torus, and a projective plane having chromatic number three.

An embedding $h: G \to S$ of a graph G in a 2-dimensional surface S is called a *2-cell embedding* provided that each component of $S - h(G)$ is homeomorphic to an open 2-disk. These components are called the *regions* of h. In particular, a 2-cell embedding $T: G \to S$ of a 3-connected graph G in S is called a *triangulation of S with the graph G* when each region of T is triangular.

A *coloring* of G is an assignment of colors to its vertices so that adjacent vertices have different colors. The *chromatic number $\chi(G)$ of a graph G* is the smallest number of colors needed for a coloring. By the *chromatic number $\chi(T)$ of a triangulation T* we mean the chromatic number of its graph G. The *chromatic number $\chi(S)$ of a surface S* is the maximum chromatic number of a graph that can be embedded in S.

We depict the torus as a square, opposite sides of which are identified in pairs, and the projective plane as a regular hexagon with each pair of its antipodal boundary points identified. Then Fig. 1 presents examples of a planar triangulation S^*, a toroidal triangulation T^*, and a projective-planar triangulation P^* having chromatic number three with their vertices colored red r, green g, and blue b.

Our construction is expounded in Section 5, for the orientable case $S = S_p$ of the sphere with $p \geqslant 0$ handles, and in Section 6, for the nonorientable case $S = N_k$ of the sphere with $k > 0$ crosscaps.

The next three sections develop technical lemmas.

2. Boundary walks

Let $h: G \to S$ be a 2-cell embedding of a graph G in a surface S. By a *boundary walk* of h we mean a closed walk which is a cyclic sequence of vertices and edges of G traced when walking inside a region of h around its boundary and keeping one hand on that boundary.

A closed walk is called a *cycle* when it has no repeated vertices. We note that a boundary walk may have repeated vertices and even repeated edges since the region closures are not required to be homeomorphic to compact disks. For example, draw two disjoint cycles — c and c' — on the sphere and then join, by an edge e, a vertex of c to one of c'; we thus obtain a 2-cell embedding with one of its boundary walks traversing e twice. Thus a boundary walk need not be a cycle.

The *length* of a boundary walk is the number of its edges, each edge being counted once or twice, according to the number of times it occurs in the walk, once or twice. Note that a boundary walk cannot pass along the same edge more than twice, and that the number of different vertices of a boundary walk cannot exceed the length of the boundary walk.

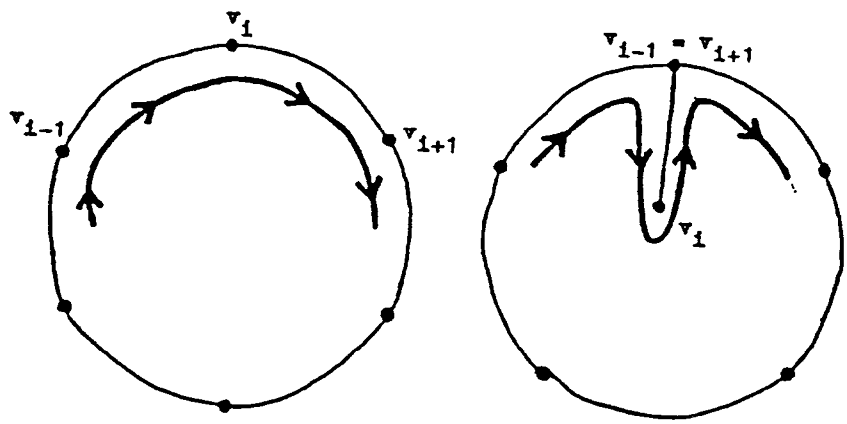
Lemma 1. *Let G be a 2-connected graph with v_{i-1}, v_i, v_{i+1} denoting three vertices of G in the order they occur in a boundary walk of a 2-cell embedding of G as in Fig.* 2(a). *Then these three vertices are distinct.*

Proof. It is enough to prove that $v_{i-1} \neq v_{i+1}$ as it is clear that $v_i \neq v_{i-1}, v_{i+1}$. Suppose to the contrary that $v_{i-1} = v_{i+1}$. Then one of the following two alternatives must arise.

(1) The edges $v_i v_{i-1}$ and $v_i v_{i+1}$ are multiedges.

(2) $v_i v_{i-1} = v_i v_{i+1}$.

The first is impossible because there are no multiedges in a graph. The second would entail that the boundary walk passes along the edge $v_i v_{i-1} = v_i v_{i+1}$ twice, in opposite



(a) (b)

Fig. 2. Boundary walk of a 2-cell embedding; for a 2-connected graph, (b) is impossible.

directions; see Fig. 2(b); so that the vertex v_i is adjacent to only one vertex: $v_{i-1}=v_{i+1}$. This however contradicts the hypothesis that G is 2-connected. □

3. Special embeddings of complete graphs

The *genus* $\gamma(G)$ of a graph G is the minimum number $p\geqslant 0$ of handles needed on the sphere to achieve embeddability $G\to S_p$.

Consider an embedding of the complete graph $K_n\to S_{\gamma(K_n)}$. It is necessarily a 2-cell embedding, see [7]. Denote by r_m the number of its regions bounded by boundary walks of length m. The following formula [6, Theorem 9-1] is a consequence of the well-known Euler–Poincaré formula.

$$\gamma(K_n)=\frac{(n-3)(n-4)}{12}+\frac{1}{6}\sum_{m\geqslant 3}(m-3)r_m \qquad n\geqslant 3. \tag{1}$$

Much more sophisticated [5] is the classical formula of Ringel and Youngs:

$$\gamma(K_n)=\left\lceil\frac{(n-3)(n-4)}{12}\right\rceil \qquad n\geqslant 3. \tag{2}$$

Lemma 2. *For each $n\geqslant 4$, there exists a 2-cell embedding $K_n\to S_{\gamma(K_n)}$ such that no region has all n vertices on its boundary.*

Proof. For $n=4$, such an embedding is presented by the tetrahedron. For $n\in\{5,6,7\}$, take the embeddings of K_5, K_6, and K_7 in the torus S_1 shown in Fig. 3. For either $n=8$ or $n\geqslant 9$ take any embedding $K_n\to S_{\gamma(K_n)}$. Then from (1) and (2) we derive $\sum_{m\geqslant 3}(m-3)r_m=2$ or $\leqslant 5$, respectively. In either case, we have $r_m=0$, for $m\geqslant n$. □

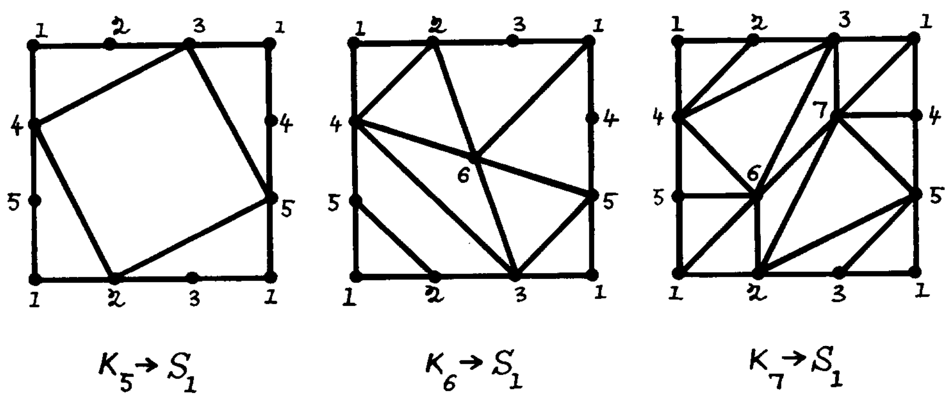


Fig. 3. Embeddings of three complete graphs in a torus.

4. The construction

Lemma 3. *For each $n \geqslant 3$, there exists a triangulation of $S_{\gamma(K_n)}$ having chromatic number n.*

Proof. For $n=3$, it is easy to verify that S^* of Fig. 1 is such a triangulation.

For $n \geqslant 4$, we fix a coloring of K_n, whose n colors will be called an *initial coloring*, and choose an embedding of K_n satisfying Lemma 2. In case the chosen embedding has a nontriangular region, denote it by R. Denote by W the boundary walk of R, and by m its length. Denote by C_i ($i=1, 2, 3, \ldots, m$) the vertices of W in the order they occur in W, and preserve the same notation for the initial colors of these vertices. The latter will cause no confusion as different vertices of a complete graph are colored differently.

The idea of the proof is to expose explicitly a procedure to triangulate R judiciously and to extend then the initial coloring to R, using only the initial colors (therefore without increasing the number of colors). Before giving the procedure, we make two motivating observations.

Observation I. *If the walk W is not a cycle then we have $C_i = C_j$, for some i and j, but anyway, by Lemma 1, the three colors $C_{i-1 \pmod m}$, C_i, and $C_{i+1 \pmod m}$ are different, for each $i=1, 2, 3, \ldots, m$.*

Observation II. *By Lemma 2, at least one of the initial colors is omitted from W.*

The procedure is expounded below and is illustrated by Fig. 4.

Step I. Inside R, adjoin a triangle to the edge $C_{i-1}C_i$ of W, for $i=2, 3, \ldots, m$, and color its new vertex with $C_{i+1 \pmod m}$. Join then the $m-1$ new vertices by a cycle in the

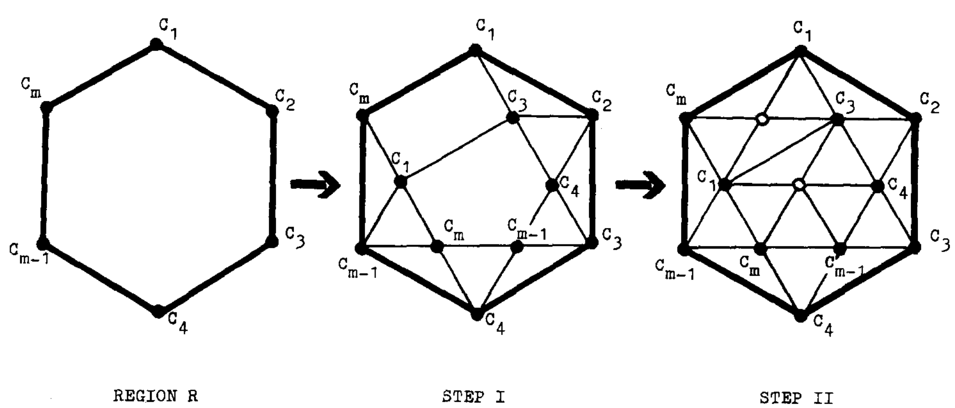


Fig. 4. Triangulation of a nontriangular region R.

cyclic order naturally inherited from W. Now R is subdivided into $2m-3$ triangles, one quadrilateral, and one $(m-1)$-gon.

Step II. Add a new vertex inside the quadrilateral and one in the $(m-1)$-gon, and then join each of the two vertices to each boundary vertex of the respective polygon. Finally, apply any initial color omitted from W to the two new vertices.

By so triangulating all nontriangular regions, we obtain a required triangulation. Note that n colors are necessary because the resulting graph has K_n as a subgraph. □

Now let T and T' be triangulations of two disjoint closed surfaces S and S'. Let us remove an arbitrary triangular region from T and one from T', and then paste their boundaries together, vertices being identified with vertices. We thus obtain a new closed surface which is known in topology as a *connected sum* of S and S' denoted by $S \# S'$. We also denote by $T \# T'$ the resulting triangulation of $S \# S'$.

Even though the combinatorial structure of $T \# T'$ is not defined uniquely, the topological type, i.e., the Euler characteristic and the orientability class, of $S \# S'$ depends neither on which regions are removed nor on how their boundaries are pasted together. Moreover, it is no problem to determine the topological type of $S \# S'$ if those of S and S' are known. In particular, it is a well-known fact that when $S = S_p$ and $S' = S_{p'}$ then we have $S \# S' = S_{p+p'}$, and thus

$$\gamma(T \# T') = \gamma(T) + \gamma(T'). \tag{3}$$

Another simple property of connected sums is that

$$\chi(T \# T') = \max\{\chi(T), \chi(T')\}. \tag{4}$$

5. The main result

Theorem 1. *For each integer pair (p, n), where $p \geqslant 0$ and $3 \leqslant n \leqslant \chi(S_p)$, there exists a triangulation of the surface S_p having chromatic number n.*

Proof. Due to the 4-Color Theorem [1, 2], the case $p = 0$ of the sphere is finished: take S^* of Fig. 1 and the tetrahedron. Now consider $p \geqslant 1$. Let T^* be the torus triangulation shown in Fig. 1. Let T_0 be a triangulation as in Lemma 3 and let $T_s = T_{s-1} \# T^*$ $(s = 1, 2, \ldots)$. Then from (3) we derive

$$\gamma(T_s) = \gamma(T_0) + s\gamma(T^*) = \gamma(K_n) + s. \tag{5}$$

Furthermore, by (4), we have

$$\chi(T_s) = \max\{\chi(T_0), \chi(T^*)\} = \max\{n, 3\} = n. \tag{6}$$

Therefore $T_{p-\gamma(K_n)}$ is a triangulation of S_p having chromatic number n. It remains to prove that $p - \gamma(K_n) \geqslant 0$. Actually, $\chi(S_p)$ is attained by the largest complete graph embeddable in S_p, see [5]. Hence $K_{\chi(S_p)}$ is embeddable in S_p, and so K_n is too, so that $\gamma(K_n) \leqslant p$. □

6. The nonorientable case

Now we turn to the nonorientable case $S = N_k$ of the sphere with $k > 0$ crosscaps. The *nonorientable genus*, $\bar{\gamma}(G)$, of a graph G is the minimum number $k \geqslant 0$ of crosscaps needed on S_0 to achieve embeddability $G \to N_k$. Although the sphere S_0 itself is orientable, it is convenient to declare by convention that $S_0 = N_0$. The formula for the nonorientable genus of K_n ($n \geqslant 3$) is known [4]:

$$\bar{\gamma}(K_7) = 3,$$

$$\bar{\gamma}(K_n) = \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil, \quad n \neq 7. \tag{7}$$

In the orientable case, the above construction is based on the fact that each embedding of K_n in the orientable surface of minimum genus is a 2-cell embedding. The nonorientable analog does not survive without modification. For example, having fitted a region of the triangulation $K_7 \to S_1$ shown in Fig. 3 with a crosscap, we obtain an embedding $K_7 \to N_3 = N_{\bar{\gamma}(K_7)}$ which is not a 2-cell embedding. Fortunately, it is the only example!

Lemma 4. *For $n \neq 7$, each embedding $K_n \to N_{\bar{\gamma}(K_n)}$ is a 2-cell embedding.*

Proof. For $n \neq 7$, we deduce from (2) and (7) that

$$2 - \bar{\gamma}(K_n) \geqslant 2 - 2\gamma(K_n).$$

Thus the surface $N_{\bar{\gamma}(K_n)}$ has the maximum Euler characteristic among the surfaces, orientable or nonorientable, in which K_n can be embedded. On the other hand, due to a result of Youngs [7], each embedding of a connected graph in a surface of highest possible Euler characteristic is necessarily a 2-cell embedding. □

Using Lemma 4, the direct nonorientable analogs of Lemmas 2 and 3 (replacing the symbols S and γ by N and $\bar{\gamma}$) for $n \neq 7$ are established similarly. For $n = 7$, to construct a triangulation of $N_{\bar{\gamma}(K_7)} = N_3$ having chromatic number 7, it is enough to take $(K_7 \to S_1) \# P^*$; see Figs. 3 and 1. Therefore, the direct nonorientable analog of Lemma 3 certainly holds.

Theorem 2. *For each integer pair (k, n), where $k > 0$ and $3 \leqslant n \leqslant \chi(N_k)$, there exists a triangulation of the surface N_k having chromatic number n.*

Proof. This is completely analogous to the above proof of Theorem 1 with the only difference that we use, instead of Lemma 3, its direct nonorientable analog and add P^* instead of T^*. □

Remark. The construction also applies if we allow multigraphs to triangulate surfaces. The class of multigraphs is easier to handle because the proof of Lemma 3 is simplified. Actually, we simply add one vertex in each nontriangular region and join it to each vertex in the boundary, with repeated vertices in the boundary walk causing no complications. Therefore, Theorems 1 and 2 are also true when by a triangulation we mean a triangular embedding of a multigraph.

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