Existence and uniqueness of solutions for third-order nonlinear boundary value problems

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Abstract

In this paper, we present some general results of existence and uniqueness of solutions of nonlinear two-point boundary value problems for third-order nonlinear differential equations by using the Shooting method. As applications we give certain concrete sufficient conditions for the existence and uniqueness. © 2006 Elsevier Inc. All rights reserved.

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1. Introduction

Third-order two-point boundary value problems were discussed in many papers in recent years, for instance, see [1–3, 5–11, 13] and reference therein. However, the boundary conditions in the above mentioned references are all simple, linear or nonlinear separated boundary conditions. In this paper, we mainly discuss more general third-order two-point boundary value problem, that is, third-order nonlinear differential equation

\[ y''' = f(x, y, y', y'') \] (1.1)

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with the nonlinear two-point boundary conditions

\[
\begin{align*}
  k(y(0), y'(0)) &= 0, \\
g(y'(0), y''(0)) &= 0, \\
h(y(0), y'(0), y''(0); y(1), y'(1), y''(1)) &= 0,
\end{align*}
\]

(1.2)

where \( k(y_0, y_1) \), \( g(y_1, y_2) \) are continuous on \( \mathbb{R}^2 \) and \( h(y_0, y_1, y_2; z_0, z_1, z_2) \) is continuous on \( \mathbb{R}^6 \).

In Section 2, we use techniques based on the Shooting method together with the Maximum Principle and the Kneser–Hukahara Continuum Theorem to establish some general principle of existence and uniqueness of solutions of nonlinear two-point boundary value equation (1.1) with the boundary conditions (1.2).

In Section 3, applying the general results obtained in Section 2 we establish some concrete sufficient conditions for existence and uniqueness of solutions of nonlinear two-point boundary value problems for third-order nonlinear differential equation (1.1) with the boundary conditions (1.2).

We consider throughout this paper the following conditions:

\( H_1 \): \( f(x, y_0, y_1, y_2) \) is continuous on \( [0, 1] \times \mathbb{R}^3 \);

\( H_2 \): for any \( (x, y_0, y_1, y_2), (x, \bar{y}_0, \bar{y}_1, \bar{y}_2) \in [0, 1] \times \mathbb{R}^3 \), if \( y_i \leq \bar{y}_i, i = 0, 1, \) then

\[
f(x, y_0, y_1, y_2) \leq f(x, \bar{y}_0, \bar{y}_1, \bar{y}_2);
\]

\( H_3 \): for any \( (x, y_0, y_1, y_2), (x, \bar{y}_0, \bar{y}_1, \bar{y}_2) \in [0, 1] \times \mathbb{R}^3 \),

\[
|f(x, y_0, y_1, y_2) - f(x, \bar{y}_0, \bar{y}_1, \bar{y}_2)| \leq L_0|y_0 - \bar{y}_0| + L_1|y_1 - \bar{y}_1| + L_2|y_2 - \bar{y}_2|,
\]

where \( L_i, i = 0, 1, 2 \), is nonnegative constants;

\( H'_3 \): for any \( (x, y_0, y_1, y_2), (x, \bar{y}_0, \bar{y}_1, \bar{y}_2) \in [0, 1] \times \mathbb{R}^3 \),

\[
|f(x, y_0, y_1, y_2) - f(x, y_0, y_1, \bar{y}_2)| \leq L_2|y_2 - \bar{y}_2|,
\]

where \( L_2 \) is a nonnegative constant;

\( H_4 \): \( k(y_0, y_1), g(y_1, y_2) \) are continuously differentiable on \( \mathbb{R}^2 \) and \( h(y_0, y_1, y_2; z_0, z_1, z_2) \) is continuously differentiable on \( \mathbb{R}^6 \);

\( H_5 \): \( \frac{\partial k}{\partial y_0} \geq \delta > 0, \frac{\partial k}{\partial y_1} \leq 0, \frac{\partial g}{\partial y_1} \geq \delta > 0, \frac{\partial g}{\partial y_2} \leq 0 \) on \( \mathbb{R}^2 \);

\( H_6 \): \( \frac{\partial h}{\partial y_0} \geq 0, \frac{\partial h}{\partial y_2} - \frac{\partial h}{\partial y_1} = \frac{\partial g}{\partial y_2} \geq 0 \) on \( \mathbb{R}^6 \);

\( H_7 \): \( \frac{\partial h}{\partial z_i} \geq 0, i = 0, 1, 2, \frac{\partial h}{\partial z_0} + \frac{\partial h}{\partial z_1} + \frac{\partial h}{\partial z_2} \geq \delta > 0 \) on \( \mathbb{R}^6 \);

\( H_8 \): \( \frac{\partial h}{\partial z_i} \geq 0, i = 0, 1, 2, \frac{\partial h}{\partial z_0} + \frac{\partial h}{\partial z_1} + \frac{\partial h}{\partial z_2} \geq \delta > 0 \) on \( \mathbb{R}^6 \).

In the above conditions, \( \delta \) denotes a constant.

Throughout this paper, our working assumption is that the solution of initial value problem for Eq. (1.1) is unique.
2. Main results

In order to use the Shooting method for BVP (1.1)–(1.2), we shall first investigate the initial value problem

\[
\text{IVP}(\gamma) \quad \begin{cases}
y''' &= f(x, y, y', y''), \\
k(y(0), y'(0)) &= 0, \\
g(y'(0), y''(0)) &= 0, \\
y'(0) + y''(0) &= \gamma.
\end{cases}
\]

**Lemma 2.1.** Assume that $H_4$ and $H_5$ hold. Then for each fixed $\gamma \in \mathbb{R}$, the equations

\[
k(u, v) = 0, \quad g(v, w) = 0, \quad v + w = \gamma
\]

have a unique solution in $\mathbb{R}^3$.

**Proof.** Substituting $\gamma - w$ for $v$ in $g(v, w)$, we let

\[
G(w) := g(\gamma - w, w).
\]

By $H_4$ and $H_5$, $G(w)$ has the derivative

\[
G'(w) = -\frac{\partial g}{\partial y_1} + \frac{\partial g}{\partial y_2} \leq -\delta < 0.
\]

Hence $G(w)$ is strictly decreasing on $\mathbb{R}$ with slope ($\leq -\delta$) which implies the range of $G(w)$ is $\mathbb{R}$. Consequently for each fixed $\gamma \in \mathbb{R}$, there exists a unique $w \in \mathbb{R}$ for which $g(\gamma - w, w) = 0$. Since $v = \gamma - w$, $v$ is uniquely determined. Similarly there exists a unique $u \in \mathbb{R}$ for which $k(u, v) = 0$. \[\square\]

**Remark 2.1.** Lemma 2.1 shows that the initial values $y^{(i)}(0, \gamma)$, $i = 0, 1, 2$, of IVP($\gamma$) are uniquely determined for each $\gamma \in \mathbb{R}$.

By using the implicit function theorem, we get easily:

**Lemma 2.2.** Assume that $H_4$ and $H_5$ hold. Then the initial values $y^{(i)}(0, \gamma)$, $i = 0, 1, 2$, are continuously differentiable with respect to $\gamma$ and

\[
\frac{dy(0, \gamma)}{d\gamma} \geq 0, \quad \frac{dy'(0, \gamma)}{d\gamma} \geq 0, \quad \frac{dy''(0, \gamma)}{d\gamma} > 0.
\]

**Lemma 2.3** (Maximum Principle [14]). Let $u = u(x)$ be a nonconstant solution of the differential inequality

\[
u'' + \alpha(x)u' + \beta(x)u \geq 0 \quad \text{in } I = (a, b),
\]

where $\alpha(x)$ and $\beta(x)$ are bounded functions in $I$, and $\beta(x) \leq 0$ in $I$. Then a nonnegative maximum of $u = u(x)$ can only occur on $\partial I$, and the outward derivative $\frac{du}{dn} > 0$ there.

**Lemma 2.4.** Assume that $H_1$, $H_2$ and $H'_3$ hold. Let $\phi_1(x)$, $\phi_2(x)$ be solutions of the differential equation (1.1) on some interval $[a, b] \subset [0, 1]$ satisfying

\[
\phi_1^{(i)}(a) \leq \phi_2^{(i)}(a), \quad i = 0, 1, 2,
\]
and

\[ \phi_1'(a) + \phi_2''(a) < \phi_2'(a) + \phi_2''(a). \]

Then \( \phi_1'(x) \leq \phi_2''(x) \) for \( x \in [a, b) \).

**Proof.** Let \( \psi(x) = \phi_2'(x) - \phi_1'(x) \). Our hypotheses imply the existence of \( a_1 \in (a, b) \) for which \( \psi(a_1) > 0 \) and \( \psi'(a_1) \geq 0 \) on \( [a, a_1] \). Suppose that \( \psi'(b_1) < 0 \) and \( \psi(x) > 0 \) on \( [a_1, b_1] \). By using the assumption \( H_2 \), it is easy to check that \( \psi(x) \) is a solution of the differential inequality

\[ u'' + \alpha(x)u' + \beta(x)u \geq 0 \quad \text{in} \quad I = (a_1, b_1), \tag{2.1} \]

where

\[
\alpha(x) = \begin{cases} 
- \frac{f(x, \phi_2(x), \phi_2'(x) - f(x, \phi_2(x), \phi_1'(x)), \phi_1''(x))}{\phi_2'(x) - \phi_1'(x)}, & \phi_2''(x) \neq \phi_1''(x), \\
0, & \phi_2''(x) = \phi_1''(x),
\end{cases}
\]

and

\[
\beta(x) = - \frac{f(x, \phi_1(x), \phi_2'(x), \phi_1''(x)) - f(x, \phi_1(x), \phi_1'(x), \phi_1''(x))}{\phi_2'(x) - \phi_1'(x)}.
\]

Assumption \( H_3' \) guarantees that \( \alpha(x), \beta(x) \) are bounded on \( (a_1, b_1) \) and by assumption \( H_2 \) we have \( \beta(x) \leq 0 \) on \( (a_1, b_1) \). Consequently, by Lemma 2.3 the positive maximum of \( \psi(x) \) can only occur on \( \partial I = [a_1, b_1] \) and \( \frac{d\psi}{dx} > 0 \) there. Since \( \psi'(b_1) < 0 \), the maximum must occur at \( a_1 \) and 

\[
\frac{d\psi}{dx} \bigg|_{x=a_1} = -\psi'(a_1) > 0, \text{i.e.} \quad \psi'(a_1) < 0, \text{which is a contradiction to} \quad \psi'(a_1) \geq 0. \]

Thus \( \psi'(x) \geq 0 \) for \( a \leq x < b \), i.e. \( \phi_2''(x) \leq \phi_1''(x) \) for \( a \leq x < b \). \( \square \)

Now, we introduce some notations:

\[ \mathcal{F} := \{ \phi \colon \phi \text{ is the solution of} \ IVPr(\gamma), \ \gamma \in \mathbb{R} \}. \]

It is easy to see that if \( H_1, H_4 \) and \( H_5 \) hold, then \( \mathcal{F} \) is an infinite set.

We define a relation "\( \prec \)" on \( \mathcal{F} \) as follows:

\[ \phi \prec \psi \iff \phi^{(i)}(x) \leq \psi^{(i)}(x), \quad i = 0, 1, 2 \text{ for all } x \in \mathcal{D}(\phi) \cap \mathcal{D}(\psi), \]

where \( \mathcal{D}(\phi) \) and \( \mathcal{D}(\psi) \) denote the intersection of \( [0, 1] \) with the maximum intervals of existence of \( \phi \) and \( \psi \), respectively.

**Lemma 2.5.** Assume that \( H_1, H_2, H_3', H_4 \) and \( H_5 \) hold. Then \( \mathcal{F} \) is totally ordered by the relation "\( \prec \)."

**Proof.** By the assumption, we note \( \mathcal{F} \neq \emptyset \). It is easy to show the reflexiveness, the antisymmetry and the transitivity.

Thus we need to show that any two members of \( \mathcal{F} \) are comparable. In fact, for any \( \phi, \psi \in \mathcal{F} \), there exist constants \( \gamma_\phi, \gamma_\psi \in \mathbb{R} \) such that \( \phi \) and \( \psi \) are solutions of \( IVPr(\gamma_\phi) \) and \( IVPr(\gamma_\psi) \), respectively. Without loss of generality assume that \( \gamma_\phi \leq \gamma_\psi \). Then by the uniqueness of solution of initial value problem, we have \( \phi \equiv \psi \) in the case \( \gamma_\phi = \gamma_\psi \). If \( \gamma_\phi < \gamma_\psi \), then by Lemma 2.2 \( \phi^{(i)}(0) \leq \psi^{(i)}(0), i = 0, 1 \), and \( \phi''(0) < \psi''(0) \). Consequently by Lemma 2.4, we have \( \phi \prec \psi \). Hence \( \mathcal{F} \) is totally ordered. \( \square \)
Remark 2.2. In the proof of Lemma 2.5, it is easy to see that under the hypotheses $H_1$, $H_2$, $H_3'$, $H_4$ and $H_5$, for the solution $\phi(x)$ and $\psi(x)$ of IVP$(\gamma \phi)$ and IVP$(\gamma \psi)$, respectively, we get $\phi < \psi$ if $\gamma \phi \leq \gamma \psi$.

Lemma 2.6 (Kneser–Hukahara Continuum Theorem [4,12]). Consider the system $y' = f(x, y)$, $y \in \mathbb{R}^n$. Suppose that the function $f(x, y)$ is continuous and bounded on $D = \{ (x, y) : \alpha \leq x \leq \beta, y \in \mathbb{R}^n \}$. Let $C$ be a compact and connected subset of $D$ and $\mathcal{F}(C)$ be the set of solutions which start in $C$. Then $\mathcal{F}(C)$ is a compact and connected subset of $C^2([\alpha, \beta], \mathbb{R}^n)$.

Lemma 2.7. Assume that $H_1$, $H_2$, $H_3'$, $H_4$ and $H_5$ hold. Suppose also that there exist $\phi_1, \phi_2 \in \mathcal{F}$ which are defined on $[0, b] \subset [0, 1]$ and $\phi_1 < \phi_2$. Then

$$F = \{ \phi |_{[0, b]} : \phi \in \mathcal{F}, \phi_1 < \phi < \phi_2 \}$$

is a compact and connected subset of $C^2([0, b])$.

Proof. Let $\phi_i, i = 1, 2$, be the solutions of IVP$(\gamma \phi_i)$, $i = 1, 2$, respectively, and $\gamma \phi_1 \leq \gamma \phi_2$. For any $\gamma \in \{ \gamma \phi_1, \gamma \phi_2 \}$, if $\phi(x, \gamma)$ is the solution of IVP$(\gamma)$ defined on $[0, b]$, then by Remark 2.2 $\phi_1 < \phi < \phi_2$. Furthermore, we have

$$F = \{ \phi(x, \gamma)[|_{[0, b]} : \gamma \phi_1 \leq \gamma \leq \gamma \phi_2 \}.$$

Now, let $y_0 = y, y_1 = y_0', y_2 = y_1'$. Then IVP$(\gamma)$ is equivalent to the following initial value problem:

$$\begin{align*}
\frac{dY}{dx} &= G(x, y_0, y_1, y_2), \\
k(y_0(0), y_1(0)) &= 0, \\
g(y_1(0), y_2(0)) &= 0, \\
y_1(0) + y_2(0) &= \gamma,
\end{align*}$$

(2.2)

where $Y = (y_0, y_1, y_2), G(x, y_0, y_1, y_2) = (y_1, y_2, f(x, y_0, y_1, y_2))$. Consider a set of solutions of initial value problem (2.2) denoted by $S$ as follows:

$$S := \{(y_0(x, \gamma), y_1(x, \gamma), y_2(x, \gamma))[|_{[0, b]} : \gamma \phi_1 \leq \gamma \leq \gamma \phi_2 \}.$$

Since $\phi_1 < \phi < \phi_2$, there exists $M > 0$ such that

$$|y_i(x, \gamma)| \leq M, \quad i = 0, 1, 2, \quad x \in [0, b], \quad \gamma \phi_1 \leq \gamma \leq \gamma \phi_2.$$

Let $H = \{ (x, y_0, y_1, y_2) : 0 \leq x \leq b, \quad |y_i| \leq M + 1, \quad i = 0, 1, 2 \}$. Then $G(x, y_0, y_1, y_2)$ is continuous and bounded on $H$, and can be extended to a bounded continuous function $G^*(x, y_0, y_1, y_2)$ on $D = [0, b] \times \mathbb{R}^3$ such that

$$G^*(x, y_0, y_1, y_2) \equiv G(x, y_0, y_1, y_2) \quad \text{for} \quad (x, y_0, y_1, y_2) \in H.$$

Now, we consider an initial value problem

$$\begin{align*}
\frac{dY}{dx} &= G^*(x, y_0, y_1, y_2), \\
k(y_0(0), y_1(0)) &= 0, \\
g(y_1(0), y_2(0)) &= 0, \\
y_1(0) + y_2(0) &= \gamma.
\end{align*}$$

(2.3)
Since \( y_i(0, \gamma) = y^{(i)}(0, \gamma), i = 0, 1, 2 \), are continuous functions with respect to \( \gamma \), we have

\[
C := \left\{ (0, y_0(0, \gamma), y_1(0, \gamma), y_2(0, \gamma)) \mid \gamma_{\phi_1} \leq \gamma \leq \gamma_{\phi_2} \right\}
\]

is a compact and connected subset of \( D \), consequently by Lemma 2.6 the set of solutions of IVP(2.3)

\[
\mathfrak{S}(C) := \left\{ (y_0(x, \gamma), y_1(x, \gamma), y_2(x, \gamma)) \mid \gamma_{\phi_1} \leq \gamma \leq \gamma_{\phi_2} \right\}
\]

is a compact and connected subset of \( C([0, b], \mathbb{R}^3) \). Since \( \mathfrak{S}(C) = S \), \( F \) is a compact and connected subset of \( C^2[0, b] \). \( \square \)

**Theorem 2.1.** Assume that \( H_1, H_2, H_3', H_4 \) and \( H_5 \) hold. Suppose also that there exist \( \phi_1, \phi_2 \in \mathfrak{S} \) which are defined on \([0, 1]\) such that

\[
h(\phi_2(0), \phi_2'(0), \phi_2''(0); \phi_2(1), \phi_2'(1), \phi_2''(1)) \geq 0
\]

and

\[
h(\phi_1(0), \phi_1'(0), \phi_1''(0); \phi_1(1), \phi_1'(1), \phi_1''(1)) \leq 0.
\]

Then BVP (1.1)–(1.2) has at least one solution.

**Proof.** Assume that \( \phi_1, \phi_2 \in \mathfrak{S} \) corresponding to \( \gamma_{\phi_1}, \gamma_{\phi_2} \), respectively, and \( \gamma_{\phi_1} \leq \gamma_{\phi_2} \). Then by Lemma 2.7, the set

\[
F = \{ \phi_{|[0,1]} \mid \phi \in \mathfrak{S}, \phi_1 < \phi < \phi_2 \}
\]

is a compact and connected subset of \( C^2[0, 1] \).

Now, we define a mapping \( T : F \to \mathbb{R} \) as follows: for any \( \phi \in F \),

\[
T(\phi) = h(\phi(0), \phi'(0), \phi''(0); \phi(1), \phi'(1), \phi''(1))
\]

It is easy to see that \( T \) is continuous on \( F \). Since \( T(\phi_1) \leq 0 \) and \( T(\phi_2) \geq 0 \), we have by Bolzano’s theorem there exists \( \phi \in F \) such that

\[
T(\phi) = h(\phi(0), \phi'(0), \phi''(0); \phi(1), \phi'(1), \phi''(1)) = 0.
\]

In fact, \( \phi \) is a solution of BVP (1.1)–(1.2). \( \square \)

**Lemma 2.8.** Assume that \( H_4, H_5 \) and \( H_6 \) hold. Let \( y(x, \gamma) \) be the solution of IVP(\(\gamma\)). Then for each fixed \((z_0, z_1, z_2) \in \mathbb{R}^3\),

\[
H(\gamma) = h(y(0, \gamma), y'(0, \gamma), y''(0, \gamma); z_0, z_1, z_2)
\]

is a nondecreasing function with respect to \( \gamma \).

**Proof.** By Lemma 2.2, we have

\[
\frac{dH(\gamma)}{d\gamma} = \frac{\partial h}{\partial y_0} \cdot \frac{dy(0, \gamma)}{d\gamma} + \frac{\partial h}{\partial y_1} \cdot \frac{dy'(0, \gamma)}{d\gamma} + \frac{\partial h}{\partial y_2} \cdot \frac{dy''(0, \gamma)}{d\gamma}
\]

\[
= \left( \frac{\partial g}{\partial y_1} - \frac{\partial g}{\partial y_2} \right)^{-1} \left( \frac{\partial k}{\partial y_0} \right)^{-1} \left[ \frac{\partial k}{\partial y_1} \cdot \frac{\partial h}{\partial y_0} \cdot \frac{\partial g}{\partial y_2} + \frac{\partial k}{\partial y_2} \left( \frac{\partial h}{\partial y_1} \cdot \frac{\partial g}{\partial y_0} - \frac{\partial h}{\partial y_2} \cdot \frac{\partial g}{\partial y_1} \right) \right]
\]

\[
\geq 0
\]

which completes the proof of the lemma. \( \square \)
Theorem 2.2. Assume that $H_1$, $H_2$, $H_3$, $H_4$, $H_5$, $H_6$ and $H_7$ hold. Then BVP (1.1)–(1.2) has at most one solution.

Proof. Suppose $\phi_1(x)$, $\phi_2(x)$ are distinct solutions of BVP (1.1)–(1.2). Let $\gamma_i = \phi_i'(0) + \phi_i''(0)$, $i = 1, 2$. Without loss of generality, we assume that $\gamma_1 < \gamma_2$. Then by Remark 2.2 we have $\phi_1 < \phi_2$, i.e.

$$\phi_2^{(i)}(x) - \phi_1^{(i)}(x) \geq 0, \quad i = 0, 1, 2 \text{ for } 0 \leq x \leq 1. \quad (2.4)$$

Now, we consider two cases to prove.

Case 1. $\phi_2'(x) - \phi_1'(x)$ is a constant on $[0, 1]$. In this case, $\phi_2''(x) - \phi_1''(x) \equiv 0$ on $[0, 1]$, in particular $\phi_2''(0) = \phi_1''(0)$. Since

$$0 = g(\phi_2'(0), \phi_2''(0)) - g(\phi_1'(0), \phi_1''(0))$$

$$= \frac{\partial g}{\partial y_1} \cdot [\phi_2'(0) - \phi_1'(0)] + \frac{\partial g}{\partial y_2} \cdot [\phi_2''(0) - \phi_1''(0)],$$

we have $\phi_2'(0) = \phi_1'(0)$ by $H_5$. Thus $\phi_2'(x) \equiv \phi_1'(x)$ on $[0, 1]$. Similarly, we get $\phi_2''(x) \equiv \phi_1''(x)$ on $[0, 1]$.

Case 2. $\phi_2'(x) - \phi_1'(x)$ is not a constant on $[0, 1]$. In this case by (2.4), $\phi_2'(x) - \phi_1'(x)$ has a positive maximum at $x = 1$, and there exists $a \in [0, 1)$ such that $\phi_2'(a) - \phi_1'(a) \geq 0$ and $\phi_2'(x) - \phi_1'(x) > 0$, $x \in (a, 1]$. It is easy to check that $\phi_2'(x) - \phi_1'(x)$ satisfies differential inequality (2.1) on $(a, 1]$. Hence by Lemma 2.3, we have $\phi_2''(1) - \phi_1''(1) > 0$. Thus $\phi_2^{(i)}(1) - \phi_1^{(i)}(1) > 0$, $i = 0, 1$. Consequently by Lemma 2.8 and the Mean value theorem, we obtain

$$0 = h(\phi_2'(0), \phi_2''(0); \phi_2(1), \phi_2''(1))$$

$$\geq h(\phi_1'(0), \phi_1''(0); \phi_2(1), \phi_2''(1))$$

$$= h(\phi_1(0), \phi_1'(0), \phi_1''(0); \phi_2(1), \phi_2'(1), \phi_2'''(1))$$

$$- h(\phi_1(0), \phi_1'(0), \phi_1''(0); \phi_1(1), \phi_1'(1), \phi_1''(1))$$

$$= \sum_{i=0}^2 \frac{\partial h}{\partial z_i} \cdot [\phi_2^{(i)}(1) - \phi_1^{(i)}(1)] > 0$$

which is a contradiction. Hence the theorem is proved. \qed

3. Applications

In this section as the applications of the main results obtained in Section 2.2, we shall give some specific sufficient conditions of the existence and uniqueness of solutions of nonlinear two-point boundary value problems for third-order Lipschitz equation (1.1) with the boundary conditions (1.2). To do this, we give some lemmas.

Lemma 3.1. Assume that $H_1$, $H_2$ and $H_3'$ hold. Let $\phi(x)$ be a solution of (1.1) defined on $[x_1, x_2] \subset [0, 1]$. Suppose further that

(i) $\phi''(x_1) = \sigma > 0$, $\phi''(x_2) = \frac{1}{2} \sigma$, $\frac{1}{2} \sigma \leq \phi''(x) \leq \sigma$ for $x_1 \leq x \leq x_2$;

(ii) $\phi^{(i)}(x_1) \geq c_i$, $i = 0, 1$;
(iii) there exists a constant $K \leq 0$ such that
\[ K \leq \inf_{0 \leq x \leq 1} \left\{ f(x, c_0 + c_1(x - x_1), c_1, 0) \right\}. \]

Then
\[ x_2 - x_1 \geq \frac{1}{2} \cdot \frac{\sigma}{\sigma L_2 - K}. \]

Proof. By assumption (i), we have $0 < \phi''(x) \leq \sigma$ for $x_1 \leq x \leq x_2$. From $H'_3$, we have for $x_1 \leq x \leq x_2$
\[ f(x, \phi(x), \phi'(x), \phi''(x)) - f(x, \phi(x), \phi'(x), 0) \geq -L_2 \phi''(x) \geq -L_2 \sigma. \]

On the other hand, using assumptions (i)–(iii), we get
\[ \phi'(x) \geq c_1, \quad \phi(x) \geq c_0 + c_1(x - x_1), \quad x \in [x_1, x_2]. \]

By $H_2$, we have for $x_1 \leq x \leq x_2$
\[ f(x, \phi(x), \phi'(x), 0) \geq f(x, c_0 + c_1(x - x_1), c_1, 0) \geq K. \]

Thus we have
\[
-\frac{1}{2} \sigma = \phi''(x_2) - \phi''(x_1) = \int_{x_1}^{x_2} f(x, \phi(x), \phi'(x), \phi''(x)) \, dx
\]
\[ = \int_{x_1}^{x_2} \left[ f(x, \phi(x), \phi'(x), \phi''(x)) - f(x, \phi(x), \phi'(x), 0) \right] \, dx
\]
\[ + \int_{x_1}^{x_2} f(x, \phi(x), \phi'(x), 0) \, dx
\]
\[ \geq -L_2 \sigma (x_2 - x_1) + K (x_2 - x_1) = (-L_2 \sigma + K)(x_2 - x_1). \]

Consequently we get
\[ x_2 - x_1 \geq \frac{1}{2} \cdot \frac{\sigma}{\sigma L_2 - K}. \]

Lemma 3.2. Assume that $H_1$, $H_2$ and $H'_3$ hold. Let $\phi(x)$ be a solution of (1.1) defined on $[x_1, x_2] \subset [0, 1]$. Suppose further that

(i) $\phi''(x_1) = \sigma_0 > 0$;
(ii) $\phi^{(i)}(x_1) \geq c_i, \ i = 0, 1$;
(iii) $\frac{\sigma_0}{2^{i+1}L_2} \geq -K$, where $L_2 \geq 1$ as in $H'_3$ and $K \leq 0$ is a constant with
\[ K \leq \inf_{0 \leq x, \bar{x} \leq 1} \left\{ f(x, m_0 + c_1(x - \bar{x}), c_1, 0) \right\}, \]

where $m_0 \leq c_0 + c_1(x - x_1)$ for $0 \leq x \leq 1$. 

Then
\[ \phi''(x) \geq -K \quad \text{for } x_1 \leq x \leq x_2. \]

**Proof.** Suppose the desired conclusion is false. Since \( \phi''(x_1) = \sigma_0 > -K \), there exists \( x_4 \in (x_1, x_2) \) such that \( \phi''(x_4) = -K \) and \( \phi''(x) > -K \) for \( x_1 \leq x < x_4 \). Let \( \phi''(x) \) have its maximum \( \sigma \) at \( x_3 \in [x_1, x_4] \), i.e.

\[ \sigma := \phi''(x_3) = \max \{ \phi''(x) : x_1 \leq x \leq x_4 \}. \]

Then \( \phi''(x) \leq \sigma \) for \( x_3 \leq x \leq x_4 \).

Now, we choose points \( a_j \) and \( b_j \) so that

\[ x_3 = a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_l < b_l \leq x_4, \]

where \( l = [4L_2 + 1] \) is the greatest integer less than or equal to \( 4L_2 + 1 \), such that for each \( j = 1, 2, \ldots, l \),

\[ \phi''(a_j) = \frac{\sigma}{2^{j-1}}, \quad \phi''(b_j) = \frac{\sigma}{2^j}, \quad \frac{\sigma}{2^j} \leq \phi''(x) \leq \frac{\sigma}{2^{j-1}} \quad \text{on } [a_j, b_j], \]

and

\[ \phi''(x) \geq \frac{\sigma}{2^j} \quad \text{on } [x_3, b_j]. \]

Apply Lemma 3.1 on each subinterval \( [a_j, b_j] \), \( j = 1, 2, \ldots, l \), to get

\[ b_j - a_j \geq \frac{1}{2} \cdot \frac{\sigma 2^{-j+1}}{\sigma 2^{-j+1} L_2 - K} \geq \frac{1}{2} \cdot \frac{\sigma 2^{-j+1}}{2\sigma 2^{-j+1} L_2} = \frac{1}{4L_2}. \]

Since \( l > 4L_2 \), then

\[ x_4 - x_3 \geq \sum_{j=1}^{l} (b_j - a_j) \geq \frac{l}{4L_2} > 1 \]

which is a contradiction. Hence the proof is completed. \( \square \)

**Lemma 3.3.** Assume that \( H_1, H_2 \) and \( H_3' \) hold. Let \( \phi_n(x) \) be a solution of (1.1) defined on some interval \([0, b_n] \subset [0, 1], n = 1, 2, \ldots \). Suppose further that

(i) \( \phi_n^{(i)}(0) \geq c_i, \ i = 0, 1, n = 1, 2, \ldots ; \)

(ii) \( \phi_n''(0) \to \infty \) as \( n \to \infty \).

Then there exists \( n_0 \in \mathbb{N} \) such that

\[ \phi_{n_0}''(x) \geq -K \quad \text{for } 0 \leq x \leq b_{n_0}, \]

where \( K \) is a nonpositive constant satisfying

\[ K \leq \inf_{0 \leq x, \tilde{x} \leq 1} \{ f(x, m_0 + c_1(x - \tilde{x}), c_1, 0) \}, \]

and \( m_0 \leq c_0 + c_1 x \) for \( 0 \leq x \leq 1 \).

**Proof.** For the given \( K \), by assumption (ii), there exists \( n_0 \in \mathbb{N} \) such that \( \sigma_0 := \phi_{n_0}''(0) > -K \) and \( \frac{\sigma_0}{2^{n_0} + 1} > -K \). Note that
Lemma 3.4. Assume that \( H_1, H_2, H_3', H_4 \) and \( H_5 \) hold. Then

1. There exists a sequence of initial values \((y(0, \gamma_n), y'(0, \gamma_n), y''(0, \gamma_n))\) of IVP(\(\gamma_n\)) such that \(\gamma_n \to \infty\) and \(y''(0, \gamma_n) \to \infty\) as \(n \to \infty\), and
\[
y'(0, \gamma_{n+1}) \geq y'(0, \gamma_n), \quad y(0, \gamma_{n+1}) \geq y(0, \gamma_n), \quad n = 1, 2, \ldots;
\]
2. For any \(c > 0\), there exists \(\phi \in \mathcal{F}\) dependent on \(c\) such that,
\[
\phi''(x) \geq c, \quad \phi'(x) \geq y'(0, \gamma_1) + cx, \quad \phi(x) \geq y(0, \gamma_1) + y'(0, \gamma_1)x + \frac{c}{2}x^2,
\]
where \(\phi''(0) = y''(0, \gamma_n)\) and \(n\) is a positive integer.

Proof. (1) We choose a strictly increasing sequence \(\{\gamma_n\}_{n=1}^{\infty}\) such that \(y''_n \to \infty\) as \(n \to \infty\). By \(H_5\), for each \(n\) (or \(y''_n\)), the equation \(g(s, y''_n) = 0\) has a unique solution \(s_n\) with respect to \(s\), i.e. \(g(s_n, y''_n) = 0\). Furthermore there exists a unique \(t_n\) such that \(k(t_n, s_n) = 0\). Let \(\gamma_n = s_n + y''_n\). Then by Lemma 2.1, we have
\[
y(0, \gamma_n) = t_n, \quad y'(0, \gamma_n) = s_n, \quad y''(0, \gamma_n) = y''_n.
\]
Hence \(y''(0, \gamma_n) \to \infty\) as \(n \to \infty\).

Now, from the initial conditions, we get
\[
0 = g(y'(0, \gamma_{n+1}), y''(0, \gamma_{n+1})) - g(y'(0, \gamma_n), y''(0, \gamma_n))
= \frac{\partial g}{\partial y_1} \cdot [y'(0, \gamma_{n+1}) - y'(0, \gamma_n)] + \frac{\partial g}{\partial y_2} \cdot [y''(0, \gamma_{n+1}) - y''(0, \gamma_n)].
\]
Thus by \(H_5\) we get \(y'(0, \gamma_{n+1}) - y'(0, \gamma_n) \geq 0\), i.e.
\[
y'(0, \gamma_{n+1}) \geq y'(0, \gamma_n), \quad n = 1, 2, \ldots.
\]
Similarly, we get
\[
y(0, \gamma_{n+1}) \geq y(0, \gamma_n), \quad n = 1, 2, \ldots,
\]
and clearly \(\gamma_n \to \infty\) as \(n \to \infty\).

(2) It is easy to see in the proof of (1) that for the solution \(\phi_n(x)\) of IVP(\(\gamma_n\)), we have
\[
(i') \quad \phi_n^{(i)}(0) = y^{(i)}(0, \gamma_n) \geq y^{(i)}(0, \gamma_1) := c_i, \quad i = 0, 1, n = 1, 2, \ldots;
(ii') \quad \phi''(0) \to \infty \text{ as } n \to \infty.
\]
Now, we choose a constant $K$ for which $K \leq -c < 0$ and

\[
K \leq \inf_{0 \leq x, \tilde{x} \leq 1} \{ f(x, m_0 + c_1(x - \tilde{x}), c_1, 0) \},
\]

$m_0 \leq c_0 + c_1 x$ for $0 \leq x \leq 1$. Hence by Lemma 3.3, there exists $n_0 \in \mathbb{N}$ such that

\[
\phi_{n_0}''(x) \geq -K \geq c \quad \text{for} \quad x \in \mathcal{D}(\phi_{n_0}).
\]

Integrating the inequality, one can obtain that

\[
\phi_{n_0}'(x) \geq y'(0, \gamma_1) + cx, \quad \phi_{n_0}(x) \geq y(0, \gamma_1) + y'(0, \gamma_1)x + \frac{c}{2}x^2, \quad x \in \mathcal{D}(\phi_{n_0}).
\]

For each of Lemmas 3.1–3.4, there is a dual result. For reference we state only the dual of Lemma 3.4.

**Lemma 3.4**. Assume that $H_1$, $H_2$, $H_3$, $H_4$ and $H_5$ hold. Then

1. there exists a sequence of initial values $(y(0, \gamma_n), y'(0, \gamma_n), y''(0, \gamma_n))$ of IVP$(\gamma_n)$ such that $\gamma_n \to \infty$ and $y''(0, \gamma_n) \to -\infty$ as $n \to -\infty$, and

\[
y'(0, \gamma_{n+1}) \leq y'(0, \gamma_n), \quad y(0, \gamma_{n+1}) \leq y(0, \gamma_n), \quad n = 1, 2, \ldots;
\]

2. for any $c < 0$, there exists $\phi \in \mathcal{F}$ dependent on $c$ such that,

\[
\phi''(x) \leq c, \quad \phi'(x) \leq y'(0, \gamma_1) + cx, \quad \phi(x) \leq y(0, \gamma_1) + y'(0, \gamma_1)x + \frac{c}{2}x^2, \quad x \in \mathcal{D}(\phi),
\]

where $\phi''(0) = y''(0, \gamma_n)$ and $n$ is a positive integer.

Now, let us use the following notations:

\[
\Gamma := \{ \gamma: \text{IVP}(\gamma) \text{ has a solution on } [0, 1] \}
\]

and

\[
\gamma_S := \sup \Gamma, \quad \gamma_I := \inf \Gamma.
\]

**Lemma 3.5**. Assume that $H_1$, $H_2$, $H_3$, $H_4$ and $H_5$ hold. Then for $\gamma_1, \gamma_2 \in \Gamma, \ [\gamma_1, \gamma_2] \subset \Gamma$ if $\gamma_1 < \gamma_2$.

**Proof.** Let $\gamma_0 \in (\gamma_1, \gamma_2)$, and $\phi_i(x)$ denote the solution of IVP$(\gamma_i)$ for $i = 0, 1, 2$. Then from Remark 2.2, we have

\[
\phi_1^{(i)}(x) \leq \phi_0^{(i)}(x) \leq \phi_2^{(i)}(x), \quad i = 0, 1, 2 \quad \text{for} \quad x \in \mathcal{D}(\phi_1) \cap \mathcal{D}(\phi_0) \cap \mathcal{D}(\phi_2).
\]

Since $\gamma_1, \gamma_2 \in \Gamma$, $\mathcal{D}(\phi_1) \cap \mathcal{D}(\phi_0) \cap \mathcal{D}(\phi_2) = \mathcal{D}(\phi_0)$. By the extension theorem of solutions, the solution $\phi_0(x)$ can be extended to $[0, 1]$, i.e. $\mathcal{D}(\phi_0) = [0, 1]$, and so $\gamma_0 \in \Gamma$. □

**Lemma 3.6**. Assume that $H_1$, $H_2$, $H_3$, $H_4$ and $H_5$ hold, and $\gamma_S = \infty, \gamma_I = -\infty$. Then

1. there exists a sequence $\{\phi_n(x)\}$ of $\mathcal{F}$ defined on $[0, 1]$ such that

\[
\lim_{n \to \infty} \phi_n^{(i)}(1) = \infty, \quad i = 0, 1, 2, \quad \lim_{n \to \infty} \phi_n''(0) = \infty;
\]
Since $\gamma_S = \infty$ and $\gamma_I = -\infty$, by Lemma 3.5, $\mathcal{D}(\phi_n) = [0, 1]$. Thus we have
\[
\phi''(0) \geq n, \quad \phi''(1) \geq n, \quad \phi'(1) \geq y'(0, \gamma_I) + n,
\]
\[
\phi_n(1) \geq y(0, \gamma_I) + y'(0, \gamma_I) + \frac{n}{2}
\]
which implies (1). \ \Box

With the above lemmas we may now formulate our main results of this section on the existence and uniqueness of solutions for BVP (1.1)–(1.2).

**Theorem 3.1.** Assume that $H_1$, $H_2$, $H_3$, $H_4$, $H_5$ and $H_8$ hold. Then BVP (1.1)–(1.2) has at least one solution.

**Proof.** By $H_1$ and $H_3$, $\gamma_S = \infty$ and $\gamma_I = -\infty$. Now, we choose $r_1, s_1, t_1 \in \mathbb{R}$ for which $k(r_1, s_1) = 0$ and $g(s_1, t_1) = 0$. Note that if $N_1 > 0$ is large enough and $t, z_i \geq N_1$, $i = 0, 1, 2$, then we have by $H_8$,
\[
h(r_1, s_1, t; z_0, z_1, z_2)
\]
\[
= h(r_1, s_1, 0; 0, 0, 0) + \frac{\partial h}{\partial y_2} \cdot t + \frac{\partial h}{\partial z_0} \cdot z_0 + \frac{\partial h}{\partial z_1} \cdot z_1 + \frac{\partial h}{\partial z_2} \cdot z_2
\]
\[
\geq h(r_1, s_1, 0; 0, 0, 0) + N_1 \delta \geq 0.
\]

Let $\gamma_I = s_1 + t_1$ and let $\phi(x)$ be a solution of IVP($\gamma_I$). Then by Lemma 3.5, $\phi(x)$ is defined on $[0, 1]$.

Now, by Lemma 3.6(1), there exists $\tilde{\phi}(x) \in \mathcal{F}$ defined on $[0, 1]$ such that
\[
\tilde{\phi}^{(i)}(1) \geq N_1, \quad i = 0, 1, 2, \quad \text{and} \quad \tilde{\phi}''(0) \geq N_1.
\]

Let $\phi_2(x)$ be the larger one of $\phi(x)$ and $\tilde{\phi}(x)$. Then by $H_8$ we have
\[
h\left(\phi_2(0), \phi_2'(0), \phi_2''(0); \phi_2(1), \phi_2'(1), \phi_2''(1)\right)
\]
\[
\geq h\left(r_1, s_1, \phi_2''(0); \phi_2(1), \phi_2'(1), \phi_2''(1)\right) \geq 0.
\]

Similarly, we can show that there exists $\phi_1(x) \in \mathcal{F}$ defined on $[0, 1]$ such that
\[
h\left(\phi_1(0), \phi_1'(0), \phi_1''(0); \phi_1(1), \phi_1'(1), \phi_1''(1)\right) \leq 0.
\]

Hence by Theorem 2.1, BVP (1.1)–(1.2) has at least one solution. \ \Box

**Theorem 3.2.** Assume that $H_1$, $H_2$, $H_3$, $H_4$, $H_5$, $H_6$ and $H_7$ hold. Then BVP (1.1)–(1.2) has exactly one solution.
Proof. The proof of existence of solutions is similar to that of Theorem 3.1, and is omitted. The proof of uniqueness of solution is obtained immediately by Theorem 2.2. □

Corollary 3.1. Assume that $H_1$, $H_2$ and $H_3$ hold. Suppose further that $a_0a_1 \leq 0, a_0 \neq 0; \tilde{a}_1a_2 \leq 0, \tilde{a}_1 \neq 0$ and $b_0b_1 \geq 0, b_0b_2 \geq 0, b_0 + b_1 + b_2 \neq 0$. Then for any $\lambda_i \in \mathbb{R}, i = 0, 1, 2$, the two-point boundary value problem of Eq. (1.1) with linear boundary conditions

$$
\begin{align*}
& a_0 y(0) + a_1 y'(0) = \lambda_0, \\
& \tilde{a}_1 y'(0) + a_2 y''(0) = \lambda_1, \\
& b_0 y(1) + b_1 y'(1) + b_2 y''(1) = \lambda_2
\end{align*}
$$

has exactly one solution.

References