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CLASSIFYING SPACES RELATED TO FOLIATIONS

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INTRODUCTION

THE MAIN purpose of this paper is to prove two theorems.

1. The classifying space $BM(\mathbf{R}^q)$ of the discrete monoid $M(\mathbf{R}^q)$ of smooth embedding $\mathbf{R}^q \to \mathbf{R}^q$ is weakly homotopy equivalent to Haefliger's [4] classifying space $B\Gamma_q$ for smooth foliations of codimension q.

2. If X is a smooth manifold which is the interior of a smooth manifold with boundary the inclusion Diff $(X) \to M_a(X)$ of the discrete group of diffeomorphisms of X into the discrete monoid of smooth embeddings $X \to X$ which are isotopic to diffeomorphisms induces a homology equivalence of classifying spaces.

But I think that the methods of the paper may be of more interest than the results themselves, and so I have included a fuller development of the techniques than is actually needed to prove the theorems. The ideas were originally worked out in order to prove Thurston's theorem [11] relating $B\,\mathrm{Diff}_c\,(\mathbf{R}^q)$ to $B\Gamma_q$, where $\mathrm{Diff}_c\,(\mathbf{R}^q)$ denotes the diffeomorphisms of $\mathbf{R}^q$ with compact support. That programme is carried out in [6]; but I have included a slight generalization of the simplest case $q = 1$, which is due to Mather [5]:

3. If X is a compact manifold there is a homology equivalence

$$B\,\mathrm{Diff}_c\,(X \times \mathbf{R}) \longrightarrow \Omega BM_0(X \times \mathbf{R}),$$

where $M_0(X \times \mathbf{R})$ is the submonoid of $M(X \times \mathbf{R})$ consisting of embeddings which are isotopic to the identity, and Ω denotes the loop space.

In the foregoing statements one does not need to know what, if anything, the classifying spaces classify; but for some purposes it is useful and interesting to know, so I have included a general discussion of the question. From the point of view of foliations the most interesting cases seem to be the monoids $M(X)$ and $M_a(X)$ already mentioned, and the monoid $M_h(X)$ of smooth embeddings $X \to X$ which are homotopy equivalences. Thus

$$M_a(X) \subset M_h(X) \subset M(X).$$

The results concerning these are:

4. For a smooth manifold Y homotopy classes of maps $Y \to BM(X)$ can be identified with concordance classes of objects $Y' \xrightarrow{p} Y$, where

(i) Y' is a smooth manifold with a foliation,
(ii) p is a smooth map, and
(iii) in a neighborhood of each fibre of p $Y' \to Y$ is isomorphic (as foliated manifold over Y) to $Y \times X \xrightarrow{pr} Y$ foliated by $\{Y \times x\}_{x \in X}$.

5. If $M(X)$ is replaced by $M_h(X)$ the objects classified are those for which $Y' \to Y$ is a quasi-fibration; and if $M(X)$ is replaced by $M_a(X)$ the objects are those for which $Y' \to Y$ is a smooth fibre bundle (disregarding the foliation).

If one strengthens condition (iii) still more, and requires that each point of Y has a neighborhood U such that $p^{-1}(U) \cong X \times U$ as foliated manifold, then the classifying space is of course $B\,\mathrm{Diff}(X)$. That sheds a little light on the nature of Theorem (2) above.

The plan of the work is as follows.

§1 proves Theorem (1) by direct consideration of Haefliger's classifying space.

§2 develops the homotopy theory of discrete monoids, with the proof of Mather's theorem (3) as an application, as well as the comparison of $M(X)$ and $M(\bar{X})$ when X is the interior of a manifold with boundary $\bar{X}$.

§3 proves Theorem (2), and is essentially independent of the rest of the paper.

§4 describes the kind of objects for which the realization of a given simplicial space is the classifying space. As an application I show that $BM(\mathbf{R}^q)$ classifies foliated smooth microbundles. This provides an alternative proof of Theorem (1).

§5 develops a theory of principal bundles for discrete monoids, and applies it to prove theorems (4) and (5). In fact this section provides a third proof of Theorem (1) too.

The Appendix establishes a criterion for a map with contractible fibres to be a homotopy equivalence: this is used repeatedly in the body of the work.

The paper would not have been written without the stimulus of Dusa McDuff, and I am very grateful to her for the interest she has taken in my work, and her encouragement. That applies especially to the second theorem above, my proof of which I found while thinking about a partial result of hers of the same kind. In the present argument the elegant proof of surjectivity in Lemma (3.7) is due to her; and she has also pointed out innumerable small errors of detail, to the great profit of the whole.

§1. HAEFLIGER'S CLASSIFYING SPACE

Let $\mathscr{C}_q$ be the topological category [9] whose objects are all pairs (x, U), where U is an open subset of $\mathbf{R}^q$ and $x \in U$, and whose morphisms $(x_0, U_0) \to (x_1, U_1)$ are all smooth embeddings $f: U_0 \to U_1$ such that $f(x_0) = x_1$. The space of objects of $\mathscr{C}_q$ is topologized as the sum (i.e. disjoint union) $\amalg U$ of all the open sets U of $\mathbf{R}^q$, and the space of morphisms is the sum $\coprod_f U_0^f$, where $f: U_0^f \to U_1^f$ runs through all embeddings.

Haefliger's category Γ_q is obtained from $\mathscr{C}_q$ by introducing an equivalence relation $\sim$ on the spaces of objects and morphisms. One defines $(x, U) \sim (x', U')$ if $x = x'$; and $f: (x_0, U_0) \to (x_1, U_1)$ is equivalent to $f': (x_0', U_0') \to (x_1', U_1')$ if $x_0 = x_0'$ and f and f' coincide in a neighborhood of x_0. This means that the space of objects of Γ_q is simply $\mathbf{R}^q$; and a morphism in Γ_q from x_0 to x_1 is the germ at x_0 of a diffeomorphism of $\mathbf{R}^q$ taking x_0 to x_1. Topologically the space of morphisms is a subspace of the total space of the sheaf of continuous $\mathbf{R}^q$-valued functions on $\mathbf{R}^q$.

The classifying space $B\Gamma_q$ of Γ_q is by definition (cf. [1, 4]) the realization, in the thickened sense of [10] App. A, of the simplicial space whose space of n-simplexes $\Gamma_{q,n}$ is the space of sequences $(\gamma_1, \ldots, \gamma_n)$ of composable morphisms in $\Gamma_{q,1}$.

PROPOSITION (1.1). *The functor $\mathscr{C}_q \to \Gamma_q$ induces a weak homotopy equivalence $B\mathscr{C}_q \to B\Gamma_q$.*

Proof. The space of n-simplexes $\Gamma_{q,n}$ can be described in the following way. Let S_n be the set of sequences $U_\cdot = (U_0 \xrightarrow{f_1} U_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} U_n)$ of smooth embeddings of open subsets of $\mathbf{R}^q$. Order the set S_n by extension, i.e. by prescribing:

$$U_\cdot \le U'_\cdot \quad \Leftrightarrow \quad U_i \subset U'_i \qquad \text{for } 0 \le i \le n, \qquad \text{and } f'_i | U_i = f_i.$$

Then $\Gamma_{q,n} = \varinjlim_{U_\cdot \in S_n} U_0$.

Now let S_{nm} be the set of chains $(U_{\cdot 0} \le \cdots \le U_{\cdot m})$ in S_n, and let A be the bisimplicial space whose space of (n, m)-simplexes is

$$A_{nm} = \coprod_{U_{\cdot\cdot} \in S_{nm}} U_{00}.$$

One can form the realization of A either by first realizing each row $A_{n.}$ separately and then realizing the simplicial space whose nth term is $|A_{n.}|$, or else by forming first the realizations $|A_{.m}|$ of the columns, and then realizing $[m]\mapsto|A_{.m}|$.

The realization $|A_{.0}|$ is precisely the space of the category $\mathscr{C}_q$. Similarly $|A_{.1}|$ is the space of the category $\mathscr{C}_{q,1}$ whose objects are all triples (x, U_0, U_1), where $U_0\subset U_1$ are open sets of $\mathbf{R}^q$ and $x\in U_0$, and whose morphisms from (x, U_0, U_1) to (x', U_0', U_1') are all embeddings $f\colon U_1\to U_1'$ such that $f(x)=x'$ and $f(U_0)\subset U_0'$. There are two forgetful functors $\mathscr{C}_{q,1}\to\mathscr{C}_q$, and a functor $\mathscr{C}_q\to\mathscr{C}_{q,1}$ which takes (x, U) to (x, U, U). Both composites $\mathscr{C}_q\to\mathscr{C}_{q,1}\to\mathscr{C}_q$ are the identify; and each composite $\mathscr{C}_{q,1}\to\mathscr{C}_q\to\mathscr{C}_{q,1}$ is connected to the identity by a transformation of functors. So by [8] (2.1) all three functors induce homotopy equivalences of the realization.

Continuing in this way, we see that $|A_{.m}|$ is the space of a category $\mathscr{C}_{q,m}$ of objects $(x\in U_0\subset U_1\subset\cdots\subset U_m)$, and that all simplicial operations $|A_{.m}|\to|A_{.k}|$ are homotopy equivalences. Hence $|A|\simeq|A_{.0}|\simeq B\mathscr{C}_q$.

Looking at $|A|$ from the other point of view one observes that $|A_{n.}|$ is the realization of a category of pointed open subsets of $\Gamma_{q,n}$ and inclusions. Thus it has a natural projection on to $\Gamma_{q,n}$, and the fibre at $\gamma\in\Gamma_{q,n}$ is the space of the discrete category of the open sets which contain γ. This is a filtering category (in fact the collection of open sets is closed under intersections), so the fibres of $|A_{n.}|\to\Gamma_{q,n}$ are all contractible. By the theorem proved in the appendix we conclude that $|A_{n.}|\to\Gamma_{q,n}$ is a weak homotopy equivalence, and hence that $|A|\to|B\Gamma_q|$ is a weak homotopy equivalence. This completes the proof of (1.1).

Now suppose that one replaces the category $\mathscr{C}_q$ in the preceding discussion with the full subcategory $\mathscr{D}_q$ spanned by the pointed open sets (x, U) such that U is a disk in $\mathbf{R}^q$. The only change needed in the argument comes from the fact that the collection of open sets of $\Gamma_{q,n}$ will no longer be closed under intersections. But the family containing a given point of $\Gamma_{q,n}$ will still be filtering, so one has

PROPOSITION (1.2). *$B\mathscr{D}_q\to B\Gamma_q$ is a weak homotopy equivalence.*

From this result the first theorem of the introduction follows very simply. For if $\tilde{\mathscr{D}}_q$ is the discrete category of open disks in $\mathbf{R}^q$ and smooth embeddings then the forgetful functor $\mathscr{D}_q\to\tilde{\mathscr{D}}_q$ induces a homotopy equivalence on the spaces of simplexes of each dimension. (In fact the inverse image of each simplex of $\tilde{\mathscr{D}}_q$ is a disk.) So $B\mathscr{D}_q\to B\tilde{\mathscr{D}}_q$ is a homotopy equivalence. On the other hand, any disk in $\mathbf{R}^q$ is diffeomorphic to $\mathbf{R}^q$, so the category $\tilde{\mathscr{D}}_q$ is equivalent to the category with the single object $\mathbf{R}^q$ and all embeddings $\mathbf{R}^q\to\mathbf{R}^q$ as morphisms. Thus $\tilde{\mathscr{D}}_q$ is equivalent to the discrete monoid $M(\mathbf{R}^q)$ of embeddings of $\mathbf{R}^q$, and $B\tilde{\mathscr{D}}_q\simeq BM(\mathbf{R}^q)$. Thus

PROPOSITION (1.3). *There is a weak homotopy equivalence between $B\Gamma_q$ and $BM(\mathbf{R}^q)$.*

§2. THE HOMOTOPY THEORY OF MONOIDS

When a topological monoid M acts on a space X one can construct in a natural way a space over the classifying space BM with each fibre isomorphic to X. It is usually denoted by X_M (cf. [7]), but here I shall use the notation $X/\!\!/M$. If M is a group it is the fibre bundle $X\times_M EM$ associated to the universal principal bundle $EM\to BM$. In general it is defined as the realization of a simplicial space whose space of p-simplexes is $X\times M^p$. Let us recall the following properties from [7].

PROPOSITION (2.1).

(a) *(point)$/\!\!/M = BM$, and $M/\!\!/M = EM$, which is contractible.*

(b) *If an M-map $X\to X'$ is a homotopy (resp. homology) equivalence then so is $X/\!\!/M\to X'/\!\!/M$.*

We shall frequently use the fact that if X is a space with a left M_1-action and a right M_2-action then $M_1\backslash\!\!\backslash(X/\!\!/M_2)$ is the same space as $(M_1\backslash\!\!\backslash X)/\!\!/M_2$. For example

PROPOSITION (2.2). *A homomorphism $M \to M'$ of monoids induces a homotopy (resp. homology) equivalence $BM \to BM'$ if $M' /\!\!/ M$ is contractible (resp. acyclic).*

Proof. There is a commutative diagram

$$\begin{array}{ccc} M' \backslash\!\!\backslash M' /\!\!/ M & \longrightarrow & M' \backslash\!\!\backslash M' /\!\!/ M' \\ \downarrow & & \downarrow \\ BM & \longrightarrow & BM'. \end{array}$$

But the top row is an equivalence because $M' /\!\!/ M \xrightarrow{\sim} M' /\!\!/ M'$.

The main theorem about the construction is [7]:

PROPOSITION (2.3). *$X /\!\!/ M \to BM$ is a quasifibration (resp. homology fibration) if each m in M acts on X by a homotopy (resp. homology) equivalence.*

An immediate consequence of this is

PROPOSITION (2.4). *If $\pi: M \to Q$ is a homomorphism of monoids with kernel $K = \pi^{-1}(1)$ then*

$$BK \longrightarrow BM \longrightarrow BQ$$

is a homotopy (resp. homology) fibration sequence providing

(i) *$M /\!\!/ K \to Q$ is an equivalence, and*

(ii) *each q in Q acts by an equivalence on $M \backslash\!\!\backslash Q$.*

Proof. If $M /\!\!/ K \sim Q$ then $M \backslash\!\!\backslash Q \sim M \backslash\!\!\backslash M /\!\!/ K \simeq BK$. So $BK \to BM \to BQ$ is equivalent to

$$M \backslash\!\!\backslash Q \longrightarrow (M \backslash\!\!\backslash Q) /\!\!/ Q \longrightarrow BQ,$$

which is a fibration sequence by (2.3).

The preceding results become simpler and more explicit in a number of particular cases. For example if X and M are discrete then $X /\!\!/ M$ is the space of the category whose set of objects is X and whose morphisms from x to x' are $\{m \in M : xm = x'\}$. If the category is filtering its space is contractible, so (2.2) specializes to

PROPOSITION (2.5). *If N is a submonoid of a discrete monoid M with left cancellation then $BN \xrightarrow{\sim} BM$ if for any m_1, m_2 in M there is an m in M and n_1, n_2 in N such that $mn_1 = m_1$ and $mn_2 = m_2$.*

For discrete monoids the hypothesis (i) of (2.4) becomes: $\pi^{-1}(q) /\!\!/ K$ is contractible for all q in Q. This is certainly satisfied if $\pi^{-1}(q) \cong K$ as K-set, so one has

PROPOSITION (2.6). *Suppose that $1 \to K \to M \xrightarrow{\pi} Q \to 1$ is an exact sequence of discrete monoids, and that*

(i) *for each q in Q there is an m_q in $\pi^{-1}(q)$ such that $k \mapsto m_q k$ is a bijection $K \to \pi^{-1}(q)$, and*

(ii) *for each q in Q the endomorphism $c_q: K \to K$ defined by $km_q = m_q c(k)$ is a homotopy (resp. homology) equivalence.*

Then $BK \to BM \to BQ$ is a homotopy (resp. homology) fibration sequence.

Note. Here and in the sequel I shall say that a homomorphism of discrete monoids is a homotopy or homology equivalence if the induced map of classifying spaces is one.

Proof. One has only to check that multiplication by q induces an equivalence of $Q /\!\!/ M$. But the square

$$\begin{array}{ccc} (\text{point}) /\!\!/ K & \xrightarrow{c_q} & (\text{point}) /\!\!/ K \\ \cap & & \cap \\ Q /\!\!/ M & \xrightarrow{q\times} & Q /\!\!/ M \end{array}$$

commutes up to the transformation of functors from $i \circ c_q$ to $q \circ i$ induced by m_q, and hence up to homotopy by [9] (2.1).

Monoids of embeddings

We shall study the following situation. Let Y be the interior of a compact manifold $\bar{Y}$ with boundary. Let $\bar{A} \cong \partial\bar{Y} \times [0, 1]$ be a closed collar neighborhood of the boundary of $\bar{Y}$, and let $X = \bar{Y} - \bar{A}$. Then X is an open relatively compact submanifold of Y which is diffeomorphic to Y.

Let $M(Y)$ be the monoid of smooth embeddings $\varphi: Y \to Y$, and $M(Y, X)$ the submonoid of φ such that $\varphi(X) \subset X$.

PROPOSITION (2.7). *The inclusion and restriction*

$$M(Y) \xleftarrow{i} M(Y, X) \xrightarrow{r} M(X)$$

are homotopy equivalences.

Proof. (i) The inclusion $M(Y, X) \to M(Y)$ satisfies the conditions of (2.5). In fact for any φ_1, φ_2 in $M(Y)$ one can find an automorphism φ of Y such that $\varphi(X) \supset \varphi_1(X) \cup \varphi_2(X)$. Then $\varphi^{-1} \circ \varphi_i \in M(Y, X)$, and $\varphi_i = \varphi \circ (\varphi^{-1} \circ \varphi_i)$.

(ii) Let θ: $Y \to X$ be a diffeomorphism, and define T: $M(Y) \to M(X)$ by $T(\varphi) = \theta \circ \varphi \circ \theta^{-1}$. The composition $T \circ i$: $M(Y, X) \to M(X)$ is a homotopy equivalence by (i), but it is homotopic to r because $T \circ i$ and r are intertwined by $\theta \circ j$ in $M(X)$, where j: $X \to Y$ is the inclusion. In fact for any φ in $M(Y, X)$ one has $(\theta \circ \varphi \circ \theta^{-1}) \circ (\theta \circ j) = (\theta \circ j) \circ r(\varphi)$.

Proposition (2.7) allows us to compare the monoid $M(X)$ of self-embeddings of X with that of its closure $\bar{X}$, and also with the monoid $\hat{M}(\bar{X})$ of *germs* of embeddings of $\bar{X}$ in itself. An element of $\hat{M}(\bar{X})$ is defined as an equivalence class of embeddings $\varphi: (U, \bar{X}) \to (Y, \bar{X})$, where U is a neighborhood of $\bar{X}$ in Y, two such being equivalent if they coincide in a neighborhood of $\bar{X}$. Obviously $\hat{M}(\bar{X})$ depends only on $\bar{X}$, and not on the choice of a Y containing it.

There are restriction homomorphisms $\hat{M}(\bar{X}) \to M(\bar{X}) \to M(X)$, the first being surjective, the second injective.

PROPOSITION (2.8). *The homomorphisms* $\hat{M}(\bar{X}) \to M(\bar{X}) \to M(X)$ *are homotopy equivalences.*

Proof. The argument of (2.7) proves that $M(\bar{Y}, \bar{X}) \to M(\bar{X})$ and $\hat{M}(\bar{Y}, \bar{X}) \to \hat{M}(\bar{X})$ are homotopy equivalences as well as $M(Y, X) \to M(X)$. But the last map factorizes through $\hat{M}(\bar{X})$ and $M(\bar{X})$, so one has a chain

$$\hat{M}(\bar{Y}, \bar{X}) \longrightarrow M(\bar{Y}, \bar{X}) \longrightarrow M(Y, X) \longrightarrow \hat{M}(\bar{X}) \longrightarrow M(\bar{X}) \longrightarrow M(X)$$

in which the composite of any three successive maps is an equivalence. That proves (2.8).

As an application of (2.6) it may be worth pointing out the following proposition. In the foregoing situation $\bar{X} \subset \bar{Y}$ let $K(Y)$ denote the group of diffeomorphisms of Y with compact support (i.e. which are the identity outside a compact subset of Y), and $K(Y, X)$ the submonoid $\{\varphi \in K(Y): \varphi(X) \subset X\}$. There is a restriction homomorphism $K(Y, X) \to \hat{M}(\bar{X})$ whose kernel is $K(A)$, where $A = Y - \bar{X}$. Its image is (by the isotopy extension theorem) the submonoid $\hat{M}_\partial(\bar{X})$ consisting of all φ in $\hat{M}(\bar{X})$ such that $\varphi|\partial X$ is isotopic to the identity. We have

PROPOSITION (2.9). (i) *The inclusion* $K(Y, X) \to K(Y)$ *is a homotopy equivalence.*

(ii) *The sequence* $K(A) \to K(Y, X) \to \hat{M}_\partial(X)$ *induces a homology fibration sequence of classifying spaces.*

Proof. (i) The argument of (2.7) applies.

(ii) Using (2.6) it suffices to check that if $\varphi \in K(Y, X)$ then c_φ: $K(A) \to K(A)$, defined by $c_\varphi(\psi) = \varphi^{-1}\psi\varphi$, induces the identity on homology. But $\varphi^{-1}|\bar{A}$ belongs to $\hat{M}_\partial(\bar{A})$, so we can apply the following lemma:

LEMMA (2.10). *For any manifold $\bar{X}$ with boundary the conjugation action of $M_\partial(\bar{X})$ on $K(X)$ defined by $(\varphi, \psi) \mapsto \varphi\psi\varphi^{-1}$ induces the identity on homology.*

Proof. Any homology class of $K(X)$ comes from some subgroup $K_C(X)$ consisting of maps with support in a compact subset C of X. But $\varphi|C = \tilde{\varphi}|C$ for some $\tilde{\varphi}$ in $K(X)$, so c_φ coincides with the inner automorphism $c_{\tilde{\varphi}}$ on the image of $H_*(BK_C(X))$, and so induces the identity on homology.

To conclude this section I shall prove a version of Mather's theorem[5].

PROPOSITION (2.11). *If X is a compact manifold there is a homology equivalence*

$$BK(X \times \mathbf{R}) \longrightarrow \Omega BM_0(X \times \mathbf{R}),$$

where $M_0(X \times \mathbf{R})$ is the submonoid of φ in $M(X \times \mathbf{R})$ which are isotopic to the identity, and Ω denotes the loop-space.

Proof. Let us write $Y = X \times \mathbf{R} = Y_+ \cup Y_0 \cup Y_-$, where $Y_+ = X \times [1, \infty)$, $Y_0 = X \times (-1, 1)$, $Y_- = X \times (-\infty, -1]$. There is an exact sequence of monoids

$$K(Y_0) \longrightarrow M(Y, Y_+; \text{rel } Y_-) \longrightarrow \hat{M}_0(Y_+), \qquad (*)$$

where $M(Y, Y_+; \text{rel } Y_-)$ denotes the submonoid of φ in $M(Y, Y_+)$ such that φ is the identity in a neighborhood of Y_-.

The argument of (2.7), which we have already used several times, shows that $\hat{M}_0(Y_+)$ is equivalent to M_0 (interior of Y_+), and the interior of Y_+ is a copy of $X \times \mathbf{R}$, so to prove (2.11) it is enough to show that

(i) $BY(Y, Y_+; \text{rel } Y_-)$ is contractible, and

(ii) the exact sequence $(*)$ induces a homology fibration sequence of classifying spaces.

As to (i), the inclusion $M(Y, Y_+; \text{rel } Y_-) \to M(Y; \text{rel } Y_-)$ is a homotopy equivalence by (2.5). But the monoid $M = M(Y; \text{rel } Y_-)$, regarded as a category with one object, is filtering, i.e. for any φ_1, φ_2 in M there is a φ in M such that $\varphi_1\varphi = \varphi_2\varphi$. (Choose φ so that $\varphi(Y)$ is contained in a suitable neighborhood of Y_-.) So BM is contractible.

As to (ii), we must show that the conditions of (2.6) are satisfied. The first holds because if φ_1 and φ_2 in $M(Y, Y_+; \text{rel } Y_-)$ have the same image in $\hat{M}(Y_+)$ then $\varphi_1(Y) = \varphi_2(Y)$, and one can write $\varphi_2 = \varphi_1(\varphi_1^{-1}\varphi_2)$ with $\varphi_1^{-1}\varphi_2$ in $K(Y_0)$. The second condition is a consequence of (2.10), for if φ belongs to $M(Y, Y_+; \text{rel } Y_-)$ then $\varphi(\bar{Y}_0) \supset \bar{Y}_0$, and $\varphi^{-1}|\bar{Y}_0$ belongs to $M_\partial(\bar{Y}_0)$.

§3. THE DIFFEOMORPHISM GROUP

Suppose that X is the interior of a compact manifold with boundary. Let Diff (X) be the group of diffeomorphisms of X (with the discrete topology), and $M_a(X)$ the monoid of self-embeddings of X which are isotopic to diffeomorphisms. The object of this section is to prove

PROPOSITION (3.1). *The inclusion* Diff $(X) \to M_a(X)$ *is a homology equivalence.*

The proof consists of a number of steps. The first is to consider the exact sequence of monoids

$$\text{Diff}\,(Y \text{ rel } X) \longrightarrow \text{Diff}\,(Y, X) \longrightarrow \hat{M}_a(\bar{X}),$$

where:

Y is an enlargement of X as in §2,

Diff $(Y, X) = \{\varphi \in \text{Diff}\,(Y)\colon \varphi(X) \subset X\}$,

Diff $(Y \text{ rel } X) = \{\varphi \in \text{Diff}\,(Y)\colon \varphi$ is the identity in a neighborhood of $\bar{X}\}$.

By the argument of (2.7) the maps

$$\mathrm{Diff}\,(Y) \longleftarrow \mathrm{Diff}\,(Y, X) \longrightarrow \mathrm{Diff}\,(X)$$

are homotopy equivalences, and so is $\hat{M}_a(\bar{X}) \to M_a(X)$ by (2.8). So the desired result will follow from (2.6) if we prove

PROPOSITION (3.2). *$G = \mathrm{Diff}\,(Y \text{ rel } X)$ is acyclic.*

Now let us identify $Y - X$ with $Z \times \mathbf{R}_+$, where $Z = \partial X$. Consider sequences $S \subset (0, \infty)$ which are strictly increasing and tend to infinity. For each such sequence S let G_S consist of all φ in G which are the identity in a neighborhood of $Z \times X$. Thus

(i) $G_S \subset G_T \Leftrightarrow S \supset T$, and

(ii) $G_S \cap G_T = G_{S \cup T}$.

Let $B_0G = \bigcup_S BG_S \subset BG$. The next step is to prove

PROPOSITION (3.3). *The inclusion $B_0G \to BG$ is a homotopy equivalence.*

Proof. Think of BG as constructed using Milnor's join realization. Then $BG = EG/G$, where EG is the infinite join $G * G * G * \cdots$.

Let E_0G be the subspace of EG made up of all simplexes $g_0 * g_1 * \cdots * g_k$ such that for some sequence $S \subset (0, \infty)$ as above the diffeomorphisms $g_0, \ldots, g_k$ all coincide in a neighborhood of $Z \times S$. Because E_0G is G-invariant, and $E_0G/G = B_0G$, Proposition (3.2) is equivalent to the assertion that E_0G is contractible. To see that it suffices to show that if $\sigma_i = g_{i0} * \cdots * g_{ip}$ $(i = 1, \ldots, q)$ are a finite number of p-simplexes of E_0G then there exists $g \in G$ such that $\sigma_i * g$ is contained in E_0G for $i = 1, \ldots, q$. Suppose the σ_i are associated with sequences S_i. Then one can find another sequence S such that $S \cap S_i$ is infinite for each i, and a diffeomorphism $g \in G$ which coincides with g_{i0} in a neighborhood of $Z \times (S_i \cap S)$. To do so, choose the elements s_n of S successively so that they are increasing, and $s_n \in S_{\hat{n}}$, where $n \equiv \hat{n}$ (mod q), and $g_{\hat{n}0}(Z \times [0, s_n))$ contains $g_{\hat{m}0}(Z \times s_m)$ for $m < n$. Then g can be constructed so that it coincides with $g_{\hat{n}0}$ near $Z \times s_n$ for each n (by the isotopy extension theorem), and hence $\sigma_i * G \subset E_0G$.

The final step in the proof of (3.2) is to show that B_0G is acyclic. It is enough to show that the inclusion of any finite union $BG_{S_1} \cup \cdots \cup BG_{S_k}$ in B_0G is null-homologous. For this we use a trivial lemma.

LEMMA (3.4). *If a space A has subspaces B and C such that the diagonal inclusion $B \cap C \to B \times C$ is a homology equivalence, then the inclusion $B \to A$ is null-homologous.*

Proof. The hypothesis implies that $H_i(B \cap C) \to H_i(B) \oplus H_i(C)$ is surjective for $i > 0$, so $H_i(B) \oplus H_i(C) \to H_i(B \cup C)$ is zero from the Mayer–Vietoris sequence.

To apply the lemma notice that for any sequences $S_1, \ldots, S_k$, T one has

$$(BG_{S_1} \cup \cdots \cup BG_{S_k}) \cap BG_T = BG_{S_1 \cup T} \cup \cdots \cup BG_{S_k \cup T}.$$

But

LEMMA (3.5). *$BG_{S_1 \cup T} \cup \cdots \cup BG_{S_k \cup T} \to (BG_{S_1} \cup \cdots \cup BG_{S_k}) \times BG_T$ is a homology equivalence if T is disjoint from all S_i.*

Using the Mayer–Vietoris sequence this lemma follows by induction from

LEMMA (3.6). *If S and T are disjoint then*

$$BG_{S \cup T} \longrightarrow BG_S \times BG_T$$

is a homology equivalence.

To prove (3.6) let K be the group of diffeomorphisms of $Z \times \mathbf{R}$ with compact support. For any sequence S there is a homomorphism $\Sigma: G_S \to K^S$ which associates to φ in G_S its restrictions to $Z \times (0, s) \cong Z \times \mathbf{R}$ for each s in S. The diagram

$$\begin{tikzcd} G_{S\cup T} \arrow[r] \arrow[d, "\Sigma"] & G_S \times G_T \arrow[d, "\Sigma\times\Sigma"] \\ K^{S\cup T} \arrow[r, "\cong"] & K^S \times K^T \end{tikzcd}$$

commutes, so (3.5) follows from

LEMMA (3.7). *Σ: $G_S \to K^S$ is a homology equivalence for each sequence S.*

Proof. If $S = \{s_q\}$ then G_S can be identified with K^S by $\varphi \mapsto \{\varphi | Z \times (s_{q-1}, s_q)\}$. The group K has a composition law $K \times K \to K$, well-defined up to conjugation, defined by juxtaposition of diffeomorphisms in the $\mathbf{R}$-direction. I shall write it $(k_1, k_2) \mapsto k_1 * k_2$. Then Σ can be identified with the endomorphism

$$(k_1, k_2, k_3, \ldots) \mapsto (k_1, k_1 * k_2, k_1 * k_2 * k_3, \ldots)$$

of K^S.

Let A: $K^S \to K^S$ be the shift map

$$(k_1, k_2, k_3, \ldots) \mapsto (1, k_1, k_2, \ldots).$$

If there were a map t: $K^S \to K^S$ such that $t(\xi) * \xi = 1$ for ξ in K^S (where $*$ denotes componentwise juxtaposition) then Σ would be an isomorphism with inverse $\xi \mapsto tA(\xi) * \xi$, because $\Sigma\xi = A\Sigma\xi * \xi$. Of course there is no such map t; but nevertheless $*$ makes $H_*(BK^S)$ into a connected Hopf algebra, which accordingly ([7] (8.4)) has an inversion t: $H_*(BK^S) \to H_*(BK^S)$ such that $t * I = \epsilon$, where I is the identity map and ϵ: $H_*(BK^S) \to H_*(\text{point}) \to H_*(BK^S)$ is the unit. (One can assume that the coefficients of the homology are a field.) So $tA * I$: $H_*(BK^S) \to H_*(BK^S)$ is a left-inverse of Σ, because

$$(tA * I)\Sigma = tA\Sigma * \Sigma = tA\Sigma * A\Sigma * I = \epsilon * I = I.$$

Although $tA = At$ and $\Sigma A = A\Sigma$ one cannot conclude that $tA * I$ is a right-inverse to Σ, as Σ does not distribute over $*$ on the right. This was pointed out to me by Dusa McDuff, who provided the following substitute proof that $\Sigma(At * I) = I$.

Consider the diagram

$$\begin{tikzcd} H_*(BK^S) \arrow[r, "t\times I"] \arrow[d, "\epsilon\oplus I"'] & H_*(BK^S \times BK^S) \arrow[d, "\Sigma(A*I)"] \arrow[dl] \\ H_*(BK^S \times BK^S) \arrow[r, "\Sigma A*I"'] & H_*(BK^S) \end{tikzcd}$$

where the diagonal map is induced by $(\xi, \eta) \mapsto (\eta * \xi, \eta)$. The upper triangle commutes by the definition of t, and the lower one because $\Sigma(A\xi * \eta) = \Sigma A(\eta * \xi) * \eta$ for $\xi, \eta \in K^S$. (In fact if $\xi = (k_1, k_2, \ldots)$ and $\eta = (m_1, m_2, \ldots)$ then

$$\begin{aligned} \Sigma(A\xi * \eta) &= \Sigma(1 * m_1, k_1 * m_2, k_2 * m_3, \ldots) \\ &= (1 * m_1, 1 * m_1 * k_1 * m_2, 1 * m_1 * k_1 * m_2 * k_2 * m_3, \ldots) \\ &= \Sigma A(\eta * \xi) * \eta .) \end{aligned}$$

So

$$\begin{aligned} \Sigma(At * I) &= \Sigma(A * I)(t \times) = (\Sigma A * I)(\epsilon \times I) \\ &= \Sigma A\epsilon * I = \epsilon * I = I. \end{aligned}$$

§4. CLASSIFICATION THEOREMS

There is a general classification theorem, (4.3) below, which tells one how to interpret the homotopy classes of maps into the realization of a simplical space as concordance classes of bundles of an appropriate kind. As it stands it is too abstract to be useful, and to apply it one must reinterpret the definition of the bundles in the relevant context. An example of this is the interpretation of foliated microbundles in (4.7) below. One definition of $B\Gamma_q$ is as the classifying space for foliated microbundles: if one adopts it then (4.3) and (4.7) provide a new proof of the first theorem $B\Gamma_q \simeq BM(\mathbf{R}^q)$ of the introduction.

For any space X one can consider the topological category $\hat{X}$ of pointed subsets of X. Its space of objects is the disjoint union of all the subspaces of X, i.e. an object is a pair (x, U), where U is a subset of X and $x \in U$. There is a unique morphism $(x, U) \to (y, V)$ if $x = y$ and $U \subset V$, and none otherwise, i.e. $\hat{X}$ is an ordered set. But I shall regard $\hat{X}$ as a simplicial space.

If A is a simplicial space an A-bundle on X is, roughly speaking, the germ of a simplicial map $\hat{X} \to A$. This is made precise by defining (cf. [13]) a *sieve* on X as a full subcategory of $\hat{X}$ with the properties

(a) each x in X has a neighborhood U such that $(x, U) \in S$, and

(b) if $(x, U) \in S$ and $y \in V \subset U$, then $(y, V) \in S$.

Clearly the intersection of two sieves is a sieve.

Definition (4.1). An A-bundle on X is an equivalence class of simplicial maps $\alpha: S \to A$, where S is a sieve on X. Two such maps are equivalent if they coincide on a common subsieve.

For any map $f: X' \to X$ an A-bundle α on X induces an A-bundle f^*A on X'. For f induces a simplicial map $\hat{X} \to \hat{X}'$, and the inverse image of a sieve is a sieve. In particular, A-bundles can be restricted to subspaces. Two A-bundles on X are *concordant* if they are the restrictions of an A-bundle on $X \times [0, 1]$ to $X \times 0$ and $X \times 1$.

I shall need the following lemma (cf. [9] (2.1)).

LEMMA (4.2). *A simplicial homotopy between two A-bundles α_0, α_1: $S \to A$ induces a concordance between them.*

Here a simplicial homotopy means a simplicial map $\alpha: S \times [1] \to A$, where $[1]$ is the ordered set $\{0, 1\}$ regarded as a simplicial set. If A is a topological category a simplicial homotopy from α_0 to α_1 is the same thing as a transformation of functors.

Proof. Given a sieve S on X let $\tilde{S}$ be its inverse image on $X \times [0, 1]$ under the projection pr: $X \times [0, 1] \to X$. Define an order-preserving map j: $\tilde{S} \to S \times [1]$ by

$$j(y, Y) = \begin{cases} (pr(y), pr(U), 0) & \text{if } U \text{ does not meet } X \times 1, \\ (pr(y), pr(U), 1) & \text{if } U \text{ meets } X \times 1. \end{cases}$$

Then $\alpha \circ j$ is the desired concordance.

The join realization

I shall prove that concordance classes of A-bundles on X coincide with homotopy classes of maps of X into the realization of A. But before doing so let us recall that the realization $|A|$ of a general simplicial space does not have good homotopy-theoretic properties, and is better replaced by some thickened version (cf. [10] Appendix 1). For the present the best version to use is the join-realization. (This is the $|A_{\mathbf{N}}|$ of [9]; it is discussed in more detail in [12]. If A arises from a topological group it is Milnor's infinite-join model of the classifying space.)

To define $\langle A \rangle$, let Δ^∞ be the infinite simplex with vertices $\mathbf{N}$. Then $\langle A \rangle$ is the subspace of $|A| \times \Delta^\infty$ consisting of all simplexes whose projection on Δ^∞ is nondegenerate. Under the map $\langle A \rangle \to \Delta^\infty$ the inverse-image of a point in the interior of a

k-dimensional face of Δ^∞ is A_k, the space of k-simplexes of A. If A is a good simplicial space in the sense of [10] then $\langle A\rangle \to |A|$ is a homotopy equivalence. In any case $\langle A\rangle$ is homotopy equivalent to any of the thickened realizations discussed in [10].

The reason for using the join realization is that there is a canonical A-bundle ϵ_A: $S_A \to A$ on $\langle A\rangle$. To see that, let $V_{i_0\ldots i_p}$ be the subset of Δ^∞ consisting of points whose $i_0{}^{th}, \ldots, i_p{}^{th}$ coordinates are all non-zero, and let $\langle A\rangle_{i_0\ldots i_p}$ be the inverse-image of $V_{i_0\ldots i_p}$ in $\langle A\rangle$. The canonical sieve S_A on $\langle A\rangle$ consists of all (a, U) such that U is contained in $\langle A\rangle_i$ for some i. A simplex of S_A can be denoted $(a; U_0, U_1, \ldots, U_p)$, where $a \in U_0 \subset \cdots \subset U_p \subset \langle A\rangle$. The canonical bundle ϵ_A: $S_A \to A$ is defined by

$$\epsilon_A(a; U_0, \ldots, U_p) = a_{i_0\ldots i_p},$$

where i_k is the smallest integer such that $U_k \subset \langle A\rangle_{i_k}$, and $a \mapsto a_{i_0\ldots i_p}$ is the obvious map $\langle A\rangle_{i_0\ldots i_p} \to A_p$.

Let $A(X)$ denote the set of concordance classes of A-bundles on X. There is a transformation $[X; \langle A\rangle] \to A(X)$ defined by $f \mapsto f^*\epsilon_A$.

PROPOSITION (4.3). *For any paracompact space X the transformation $A(X) \to [X; \langle A\rangle]$ is a bijection.*

Proof. If α: $S \to A$ is an A-bundle on X one has $\langle\alpha\rangle$: $\langle S\rangle \to \langle A\rangle$. We shall see in a moment ((4.4) below) that the projection π_s: $\langle S\rangle \to X$ is a homotopy equivalence, so $\langle\alpha\rangle$ induces a map χ_α: $X \to \langle A\rangle$ well-defined up to homotopy. Clearly $\alpha \mapsto \chi_\alpha$ defines a natural transformation $A(X) \to [X; \langle A\rangle]$.

To show that the composition $A(X) \to [X; \langle A\rangle] \to A(X)$ is the identity it suffices (because $A(X) \xrightarrow{\cong} A(\langle S\rangle)$ by (4.4)) to show that $\pi_S^*\alpha$ is concordant to $\langle\alpha\rangle^*\epsilon_A$ for each A-bundle α on X. But the commutative diagram

$$\begin{array}{ccccc} \langle S\rangle & \longleftarrow & S_S & \xrightarrow{\epsilon_S} & S \\ \downarrow{\scriptstyle\langle\alpha\rangle} & & \downarrow & & \downarrow{\scriptstyle\alpha} \\ \langle A\rangle & \longleftarrow & S_A & \xrightarrow[\epsilon_A]{} & A \end{array}$$

is a simplicial homotopy from $\alpha \circ \epsilon_S$ to $\langle\alpha\rangle^*\epsilon_A$, and

$$\begin{array}{ccccc} \langle S\rangle & \longleftarrow & S_S & \xrightarrow{\epsilon_S} & S \\ \downarrow{\scriptstyle\pi_S} & & \downarrow{\scriptstyle\epsilon_S} & & \downarrow{\scriptstyle\alpha} \\ X & \longleftarrow & S & \xrightarrow[\alpha]{} & A \end{array}$$

is a simplicial homotopy from $\alpha \circ \epsilon_S$ to $\pi_S^*\alpha$. So $\pi_S^*\alpha$ is concordant to $\langle\alpha\rangle^*\epsilon_A$ by (4.2).

Finally, the composition $[X; \langle A\rangle] \to A(X) \to [X; \langle A\rangle]$ takes f to $f \circ \chi_{\epsilon_A}$. To complete the proof it is enough to show that the composition is an isomorphism, i.e. that χ_{ϵ_A} is an equivalence. But $\chi_{\epsilon_A} = \pi_{S_A}^{-1} \circ \langle\epsilon_A\rangle$; and ϵ_A is an equivalence by (4.5) below.

It remains to prove two lemmas:

LEMMA (4.4). *If S is a sieve on a paracompact space X then π_S: $\langle S\rangle \to X$ is a homotopy equivalence.*

LEMMA (4.5). *For any simplicial space A the map $\langle\epsilon_A\rangle$: $\langle S_A\rangle \to \langle A\rangle$ is a homotopy equivalence.*

Proof of (4.4). Choose a locally finite open covering $\{U_\alpha\}_{\alpha\in\Sigma}$ of X such that $(x, U_\alpha) \in S$ for all x in U_α. Let $\check{S}$ be the topological subcategory of S formed from the

U_α and their finite intersections. The projection $|\check{S}| \to X$ is a homotopy equivalence by (4.1) of [8], and hence so is $\langle \check{S} \rangle \to X$. But there is a retraction $\rho: S \to \check{S}$ defined by $\rho(x, U) = (x, U_\sigma)$, where $\sigma = \{\alpha \in \Sigma : U \subset U_\alpha\}$. Because of the natural transformation $(x, U) \to \rho(x, U)$ one concludes that $\langle \check{S} \rangle$ is a deformation retract of $\langle S \rangle$ by (2.1) of [9].

Proof of (4.5). As in the preceding proof consider the subcategory $\check{S}_A$ of S_A formed from the open covering $\{\langle A \rangle_i\}$. Because $\langle A \rangle_{i_0 \ldots i_p} \to A_p$ is a homotopy equivalence for all $i_0 \le i_1 \le \cdots \le i_p$ the simplicial space $\check{S}_A$ is equivalent to $A \times \mathbf{N}$ (where the ordered set $\mathbf{N}$ is regarded as a simplicial set). The map $\epsilon_A: \check{S}_A \to A$ becomes the projection of $A \times \mathbf{N}$ on to its first factor, which is an equivalence because $\langle \mathbf{N} \rangle$ is contractible.

The space of A-bundles

A more complete, though not really more useful, result can be deduced from (4.3) with little extra work. The A-bundles on a space X are the 0-dimensional component of a simplicial space $\mathscr{A}(X)$ which is characterized by the fact that for any simplicial space K the simplicial maps $K \to \mathscr{A}(X)$ correspond exactly to simplicial maps $K \times S_X \to A$ for some sieve S_X on X. Then for any other space Y the $\mathscr{A}(X)$-bundles on Y can be identified up to concordance with A-bundles on $X \times Y$, because any sieve $S_{X\times Y}$ on $X \times Y$ contains $S_X \times S_Y$ for some sieves S_X and S_Y on X and Y, and the inclusion $S_X \times S_Y \to S_{X\times Y}$ is adjoint to the projection $S_{X\times Y} \to S_X \times S_Y$. It follows that the realization of $\mathscr{A}(X)$ is homotopy equivalent to the space of maps $X \to \langle A \rangle$, i.e.

Proposition (4.3a). $\langle \mathscr{A}(X) \rangle \simeq \mathrm{Map}\,(X; \langle A \rangle)$.

In fact for any space Y one knows by (4.3) that

$$\begin{aligned}[][Y; \langle \mathscr{A}(X) \rangle] \cong \mathscr{A}(X)(Y) = A(X \times Y) &= [X \times Y; \langle A \rangle] \\ &= [Y; \mathrm{Map}\,(X; \langle A \rangle)].\end{aligned}$$

The application to microbundles

We shall apply (4.3) to the topological category C dual to the category of pointed open sets of $\mathbf{R}^n$ and smooth embeddings. (An object of C is a pair (x, U), where $x \in U$ and U is an open set of $\mathbf{R}^n$; and a morphism $(x, U) \to (y, V)$ is a smooth embedding $f: V \to U$ such that $f(y) = x$.) I shall show that a C-bundle is essentially a foliated smooth microbundle.

Definition (4.6). A foliated smooth microbundle on a space X is a diagram $X \xrightarrow{i} E \xrightarrow{\pi} X$ such that $\pi \circ i = id$, together with an atlas $\{(U_\alpha, \varphi_\alpha)\}$ for E, where $\varphi_\alpha: U_\alpha \to V_\alpha$ is a map on to an open set of $\mathbf{R}^n$ such that

(i) $\pi \times \varphi_\alpha: U_\alpha \to \pi(U_\alpha) \times V_\alpha$ is a homeomorphism, and

(ii) for each α, β there is a diffeomorphism $f_{\alpha\beta}: \varphi_\alpha(U_{\alpha\beta}) \to \varphi_\beta(U_{\alpha\beta})$ such that

$$f_{\alpha\beta} \circ \varphi_\alpha = \varphi_\beta \qquad \text{on} \qquad U_{\alpha\beta} = U_\alpha \cap U_\beta.$$

Usually the microbundle is identified with any microbundle obtained from it by replacing E by a neighborhood of $i(X)$ in E; but in any case the two are obviously concordant.

If X happens to be a smooth manifold then E is a smooth manifold, and $\pi: E \to X$ is smooth, and E has a smooth foliation transversal to the fibers of π. The zero-section i need not be smooth, but up to concordance that makes no difference.

Proposition (4.7). *Concordance classes of foliated smooth microbundles on a paracompact space X can be identified with concordance classes of C-bundles on X.*

Proof. Given a C-bundle $(S, \gamma: S \to C)$ on X write $\gamma(x, U) = (s_U(x), Y_{U,x})$, with $s_U(x) \in Y_{U,x}$. ($Y_{U,x}$ is locally constant as a function of x in U.) Then construct E_γ by attaching together the spaces

$$E_U = \bigcup_{x \in U} x \times Y_{U,x} \subset U \times \mathbf{R}^n$$

by the equivalence relation defined by $(x, i^*y) \sim (i(x), y)$ whenever $x \in U$, $y \in Y_{U', i(x)}$, and $i\colon U \hookrightarrow U'$. The maps $U \to E_U$ given by $x \mapsto (x, s_U(X))$ fit together to define $s\colon X \to E_\gamma$, and $\{E_U\}$ is an atlas making E_γ into a foliated microbundle.

Conversely, if $X \xrightarrow{i} E \xrightarrow{\pi} X$ is a foliated microbundle, select a locally finite collection $\{W_\alpha\}_{\alpha \in A}$ of open sets of E which cover $i(X)$ and have the properties

(i) $W_\alpha \supset i\pi(W_\alpha)$ for each α, and

(ii) whenever σ is a finite subset of A for which $W_\sigma = \bigcap_{\alpha \in \sigma} W_\alpha \neq \emptyset$ there is a chart $\varphi_\sigma\colon U_\sigma \to \mathbf{R}^n$ belonging to the atlas for E such that $W^\sigma = \bigcup_{\alpha \in \sigma} W_\alpha \subset U_\sigma$.

The last condition can be achieved because any open covering of a paracompact space has a star-refinement.

Define $V_\sigma = \varphi_\sigma(W^\sigma) \subset \mathbf{R}^n$. If $\sigma \subset \tau$ then $W^\sigma \subset U_\sigma \cap U_\tau$, and $f_{\sigma\tau}(V_\sigma) \subset V_\tau$, where $f_{\sigma\tau}$ is the diffeomorphism such that $f_{\sigma\tau}\varphi_\sigma = \varphi_\tau$. Let S_E be the sieve on X consisting of all (x, U) such that U is contained in $\pi(W_\alpha)$ for some α. Define $\gamma_E\colon S_E \to C$ by $\gamma_E(x, U) = (\varphi_\sigma(i(x)), V_\sigma)$, where $\sigma = \{\alpha\colon x \in \pi(W_\alpha)\}$.

One can form the microbundle E_{γ_E} from γ_E by attaching together the $\pi(W_\sigma) \times V_\sigma$. This reconstructs an open set of E, so E_{γ_E} is concordant to E. On the other hand if one starts with γ and forms γ_{E_γ} then $S_{E_\gamma} \subset S$, and the diagram

$$\begin{tikzcd} S_{E_\gamma} \arrow[d] \arrow[dr, "\gamma_{E_\gamma}"] & \\ S \arrow[r, "\gamma"'] & C \end{tikzcd}$$

commutes up to a transformation of functors, so that γ_{E_γ} and γ are simplicially homotopic, and hence concordant by (4.2).

§5. THE CASE OF DISCRETE MONOIDS

In this section I shall describe a theory of principal bundles for discrete monoids with left-cancellation. The main example is the monoid $M(F)$ of smooth embeddings of a manifold F in itself. On one level the treatment here is simply an alternative to the general theory of §4, but each approach is useful for some purposes.

Definition (5.1). If M is a discrete monoid with left-cancellation a *principal M-set* is a set S with a right action of M which is

(a) free—i.e. $sm_1 = sm_2 \Rightarrow m_1 = m_2$, and

(b) transitive in the sense that for any s_1, s_2 in S there exists s in S and m_1, m_2 in M such that $s_1 = sm_1$ and $s_2 = sm_2$.

LEMMA (5.2). *Any map of principal M-sets is injective.*

Proof. If $f(s_1) = f(s_2)$ then $f(sm_1) = f(sm_2)$, so $f(s)m_1 = f(s)m_2$, so $m_1 = m_2$ and $s_1 = s_2$.

Definition (5.3). A *principal M-bundle* on a space X is a space P over X with a fibre-preserving right action of M such that

(a) each fibre P_x is a principal M-set, and

(b) $P \to X$ is a local homeomorphism.

Let $C_M(X)$ denote the concordance classes of principal M-bundles on X. The main result of this section is

PROPOSITION (5.4). *If X is of the homotopy type of a CW-complex then*

$$C_M(X) \cong [X; BM].$$

I shall call a principal M-bundle $P \to X$ *strict* if $P_x \cong M$ as M-set for each x in X. In the course of proving (5.4) the following fact will emerge.

PROPOSITION (5.5). *A principal M-bundle P on a finite polyhedron contains a strict subbundle.*

Before proving (5.4) and (5.5) I shall give an application. If F is a smooth manifold a *foliated F-bundle* on a smooth manifold X is a smooth manifold Y with a smooth map $\pi: Y \to X$ and a smooth foliation on Y such that near each fibre Y is isomorphic (as a foliated manifold over X) to $X \times F$ foliated by $\{X \times f\}_{f \in F}$, i.e. there is a neighborhood U of Y_x in Y and a neighborhood V of $x \times F$ in $X \times F$ and a diffeomorphism $V \to U$ compatible with the foliations and the projection on to X.

I shall write $\mathrm{Fol}(X; F)$ for the set of concordance classes of foliated F-bundles on X.

The following more narrowly defined classes of bundles are also of interest:

(a) foliated F-bundles Y on X such that $Y \to X$ is a quasi-fibration, and

(b) foliated F-bundles Y on X such that $Y \to X$ is a locally trivial fibre bundle.

I shall denote the respective sets of concordance classes by $\mathrm{Fol}_h(X; F)$ and $\mathrm{Fol}_a(X; F)$.

PROPOSITION (5.6). (i) $\mathrm{Fol}(X; F) \cong C_{M(F)}(X)$,

(ii) $\mathrm{Fol}_h(X; F) \cong C_{M_h(F)}(X)$, *and*

(iii) $\mathrm{Fol}_a(X; F) \cong C_{M_a(F)}(X)$,

where $M(F)$ is the discrete monoid of smooth embeddings $F \to F$, $M_h(F)$ is the submonoid of embeddings which are homotopy equivalences, and $M_a(F)$ the submonoid of embeddings which are isotopic to diffeomorphisms.

Remark. If E is a closed manifold the proposition is more or less trivial, for then an F-bundle is a fibre bundle, and the foliation is a reduction of its structural group to the discrete group $M(F) = \mathrm{Diff}(F)$.

Proof. (i) Given a foliated F-bundle $Y \to X$ define $P_{Y,x}$ for x in X as the set $\mathrm{Emb}_c(F; Y_x)$ of embeddings of F in the fibre Y_x which have a relatively compact image. Clearly $M(F)$ acts on $P_{Y,x}$ on the right. Let $P_Y = \bigcup_{x \in X} P_{Y,x}$, and put a topology on P_Y so that to move continuously in it is to move the embedding $F \to Y$ "along the foliation". Then $P_Y \to X$ is a local homeomorphism. $M(F)$ acts transitively on $P_{Y,x}$ in the appropriate sense because for any two embeddings $p_1, p_2: F \to Y_x$ with relatively compact image there is an embedding $p: F \to Y_x$ with relatively compact image such that $p(F) \supset p_1(F) \cup p_2(F)$. Then $p^{-1} \circ p_i \in M(F)$, and $p \circ (p^{-1} \circ p_i) = p_i$.

Conversely, if P is a principal M-bundle on X choose a strict subbundle P_0, and define $Y_P = P_0 \times_M F$. This is certainly a foliated F-bundle. Up to concordance it does not depend on the choice of P_0, for if P_1 is another choice one can find a third one P_2 which is strictly concordant to subbundles of both P_0 and P_1, and then $P_2 \times_M F$ will be concordant to open subbundles of $P_0 \times_M F$ and $P_1 \times_M F$, and hence concordant to $P_0 \times_M F$ and $P_1 \times_M F$. (To find P_2 one simply chooses a strict subbundle of $(P_0 \times 0) \cup (P \times (0, 1)) \cup (P_1 \times 1)$ on $X \times [0, 1]$.)

If one begins with P and forms Y_P and then P_{Y_P} one obtains a subbundle of P, for the fibre of P_{Y_P} at x is $\mathrm{Emb}_c(F; Y_{P,x}) = \mathrm{Emb}_c(F; P_{0,x} \times_{M(F)} F) \subset P_{0,x}$. So P_{Y_P} is concordant to P.

Conversely, beginning with Y, one finds $Y_{P_Y} = P_{Y,0} \times_M F$, which is an open subset of Y, and hence is concordant to Y.

(ii) The proof needs no essential change in this case. One forms P_Y by taking for each x the embeddings $F \to Y_x$ with relatively compact image which are homotopy equivalences.

(iii) In this case the functor $Y \mapsto P_Y$ is as before, except that one considers embeddings $F \to Y_x$ with relatively compact image which are isotopic to diffeomorphisms.

But to construct Y_P from P requires more work. I shall write $M^c = M^c(F)$ for the topological monoid obtained by giving $M(F)$ its usual C^∞ topology, and M_a^c for the topological submonoid of M^c corresponding to $M_a(F)$. It is known[3] that the inclusion $G^c \to M_a^c$, where $G^c = \mathrm{Diff}(F)$ with the C^∞ topology, is a homotopy equivalence. That is equivalent to the following assertion. If P^c is P retopologized so that M_a^c rather than M_a acts on its fibres (i.e. $P^c = P \times_{M_a} M_a^c$) then there is an (essentially unique) principal G^c-bundle P_1^c contained in P^c. For to find such a G^c-bundle it suffices to find a section of $P^c /\!\!/ G^c$ over X. As the fibres of $P_0^c /\!\!/ G^c$ are $M^c /\!\!/ G^c$, which is contractible, that can be done by the theorem of the appendix.

Now define Y_P as $P_1^c \times_{G^c} F = (P_1^c \times_{G^c} M^c) \times_{M^c} F$. This is a smooth fibre bundle which is contained in $Y'_P = P_0^c \times_{M^c} F = P_0 \times_M F$ as an open set. The foliation of Y'_P induces a foliation on Y_P, and Y_P is an object of the desired type. P_{Y_P} is contained in P as before. We must show there is a concordance between Y_{P_Y} and Y through foliated fibre bundles. Clearly a bundle such as Y has a fibrewise shrinkage Y_0 such that $Y_{0,x} = (Y_x\text{-(closed collar)})$ for each x. Define $P_1 \subset P_Y$ by $P_{1,x} = \{\varphi \in P_{Y,x} : \varphi(F) = Y_{0,x}\}$. Then $Y_{P_Y} = P_1^c \times_{G^c} F = Y_0$, which is diffeomorphic to Y. The desired concordance is obtained by choosing an isotopy between the inclusion $Y_0 \to Y$ and a diffeomorphism, and pulling back the foliation of Y.

Proof of (5.4). If $P \to X$ is a principal M-bundle then $P /\!\!/ M$ is a space over X whose fibre at x is the contractible space $P_x /\!\!/ M$. As $P /\!\!/ M \to X$ satisfies the hypotheses of the theorem of the appendix there is a map $s: X \to P /\!\!/ M$, unique up to homotopy, which is right inverse to $P /\!\!/ M \to X$. Composing s with the projection $P /\!\!/ M \to BM$, unique up to homotopy, which is right inverse to $P /\!\!/ M \to X$. Composing s with the projection $P /\!\!/ M \to BM$ gives a classifying map for P. In fact one has a transformation $C_M(X) \to [X; BM]$.

To define a transformation in the opposite direction notice that there is a natural principal M-bundle PM on BM. It is a thickened version of EM, which is not a principal M-bundle because $EM \to BM$ is not a local homeomorphism. To define it, think of BM as the join realization of the simplicial set $M^* = \{M^p\}$, i.e. $BM \subset \Delta^\infty \times |M^*|$. A point of BM can be denoted by (σ, λ), where σ is a simplex of $\Delta^\infty \times |M_*|$ and $\lambda = \{\lambda_i\}_{i \in \mathrm{N}}$ belongs the projection of σ into Δ^∞. BM is covered by contractible open sets U_σ consisting of points (τ, λ) such that σ is a face of τ and

$$\sum_{i \in S} \lambda_i > \sum_{i \notin S} \lambda_i,$$

where $S \subset \mathrm{N}$ is the set of vertices of the projection of σ. The sets U_σ and U_τ intersect if and only if σ is a face of τ or vice-versa. The usual space EM over BM can be constructed by attaching together pieces $\sigma \times M_\sigma$ for each simplex σ of BM, where M_σ is a copy of M. When σ is a face of τ the attaching map $M_\tau \to M_\sigma$ is left multiplication by a certain element $m_{\sigma\tau}$ of M. The new space PM is constructed by attaching together pieces $U_\sigma \times M_\sigma$ for all σ by the maps

$$U_\sigma \times M_\sigma \hookleftarrow (U_\sigma \cap U_\tau) \times M_\tau \hookrightarrow U_\tau \times M_\tau$$

when σ is a face of τ. (The left-hand map is (inclusion) $\times\, m_{\sigma\tau}$.) The fibre of $PM \to BM$ at x in BM is M_σ, where σ is the smallest simplex of BM such that $x \in U_\sigma$. So PM is a strict principal M-bundle. As a space PM is contractible, for EM is contained in it as a deformation retract.

We define a transformation $[X; BM] \to C_M(X)$ by $f \mapsto f^*PM$.

To see that $C_M(X) \to [X; BM] \to C_M(X)$ is the identity it suffices to show that if a bundle P on X is lifted to $P /\!\!/ M$ it contains the pull-back of PM as a subbundle—for a bundle is concordant to any subbundle. That is, it suffices to construct a commutative diagram

$$\begin{array}{ccccc} P & \longleftarrow & Q & \longrightarrow & PM \\ \downarrow & & \downarrow & & \downarrow \\ X & \longleftarrow & P /\!\!/ M & \longrightarrow & BM \end{array}$$

where the right-hand square is Cartesian.

But $P/\!\!/M$ is covered by open sets $U_\sigma \times P$ which are the inverse images of the U_σ covering BM. So the pull-back Q of PM to $P/\!\!/M$ is the union of pieces $U_\sigma \times P \times M_\sigma$. The maps $P \times M_\sigma \to P$ coming from the right action of M on P fit together to define the desired M-map $Q \to P$.

By naturality the composite map $[X; BM] \to C_M(X) \to [X; BM]$ is $f \mapsto \chi \circ f$, where χ is the classifying map for PM. To complete the proof it is enough to show that $\chi: BM \to BM$ is an equivalence, i.e. to show that $(PM)/\!\!/M \to (\text{point})/\!\!/M$ is an equivalence. That follows from the contractibility of PM.

Proof of (5.5). The universal bundle $PM \to BM$ is strict, and so any pull-back of it is strict. If $P \to X$ is an arbitrary bundle we have seen that π^*P contains a canonical strict subbundle Q, where $\pi: P/\!\!/M \to X$. Choosing a section $s: X \to P/\!\!/M$ of π we conclude that $P = s^*\pi^*P$ contains the strict subbundle s^*Q.

APPENDIX: MAPS WITH CONTRACTIBLE FIBRES

It is often useful to have a criterion allowing one to assert that a map with contractible fibres is a homotopy equivalence. Of course it is enough to know that the map is a fibration, or even a quasi-fibration. The following proposition gives a slightly different condition.

I shall say that $f: Y \to X$ is *almost locally trivial* if for any x in X there is a neighborhood U of the fibre $Y_x = f^{-1}(x)$ in Y which is homeomorphic (as a space over X) to a neighborhood of $Y_x \times \{x\}$ in $Y_x \times X$.

The property of being almost locally trivial is hereditary in the sense that if f has it so does the pull-back of f by any map. For a *proper* map f being almost locally trivial is equivalent to being locally trivial.

PROPOSITION (A.1). *If $f: Y \to X$ is almost locally trivial, and Y_x is contractible for each x in X, then f is a weak homotopy equivalence.*

Proof. Because the hypothesis is hereditary it is enough to treat the case when the base X is a finite polyhedron. Assuming that, it is then enough (cf. [2] (3.1)) to prove

LEMMA (A.2). *If $s: A \to Y$ is a partial section of f over a subpolyhedron A of X then there is a homotopy equivalence $(X', A') \to (X, A)$ and a map $s': X' \to Y$ such that*

$$\begin{tikzcd}
A' \arrow[r] \arrow[dr] & A \arrow[r] & Y \arrow[d] \\
& X' \arrow[ur] \arrow[r] & X
\end{tikzcd}$$

commutes.

The space X' occurring here will be constructed by the following technique. Suppose that X is a finite cell complex in the sense that it is the union of closed cells σ each homeomorphic to a closed disk. The *dual* $D(X)$ of X is the abstract polyhedron which is the space of the partially ordered set of cells of X (ordered by inclusion). It is the union of closed subsets $D(\sigma)$ indexed by the cells σ of X. The vertices of $D(\sigma)$ are σ and the cells contained in it: $D(\sigma)$ is the cone on the *link* $L(\sigma) = D(\partial\sigma)$ of σ.

Define X', which one might call the *explosion of* X, as the closed subset of $X \times D(X)$ which is the union of $\sigma \times D(\sigma)$ for all cells σ of X. The utility of X' is that not only can it be constructed by successively attaching pieces $\sigma \times D(\sigma)$ in order of increasing dimension of σ (by attaching-maps defined on $\partial\sigma \times D(\sigma)$), but it can also be built up dually by taking the $\sigma \times D(\sigma)$ in order of *decreasing* dimension of σ and attaching them by maps defined on $\sigma \times L(\sigma)$.

LEMMA (A.3). (i) *X' is homeomorphic to X.*
(ii) *The projection $pr: X' \to X$ on to the first factor is a homotopy equivalence.*

Proof. Statement (i) is almost obvious, but I shall not need it, and shall omit the proof. To prove (ii) one shows inductively that $pr^{-1}(Y) \xrightarrow{\sim} Y$ for all subcomplexes Y of X. For $pr^{-1}(Y \cup \sigma) = pr^{-1}(Y) \cup (\sigma \times D(\sigma))$, and $pr^{-1}(Y) \cap (\sigma \times D(\sigma)) = \partial\sigma \times D(\sigma)$, and $D(\sigma)$ is contractible.

To prove (A.2) notice that in a cell complex X one can subdivide the cells of dimension $\leq k$ without affecting the cells of higher dimension. And if a cell σ is subdivided into cells σ_i of the same dimension then each $D(\sigma_i)$ and $L(\sigma_i)$ can be identified with $D(\sigma)$ and $L(\sigma)$.

We construct the map $s': X' \to Y$ of (A.2) inductively. Suppose it has been defined on $\tau \times D(\tau)$ for all cells τ which are either in A or of dimension $> k$. Let σ be a cell of dimension k not in A. Then s' is prescribed on $(\sigma \times L(\sigma)) \cup (\sigma_A \times D(\sigma))$, where $\sigma_A = \sigma \cap A$. Thus for any $x \in \sigma$ one has a map $L(\sigma) \to Y_x$, and because Y_x is contractible it extends to $D(\sigma)$. Because f is almost locally trivial one can find an open covering $\{U\}$ of σ and maps $s'_U: U \times D(\sigma) \to Y$ over X which coincide with s' on $(U \times L(\sigma)) \cup ((U \cap A) \times D(\sigma))$. Subdivide σ into cells σ_i each contained in a set U_i of the covering. The s'_{U_i} provide the desired inductive extension of s'. That completes the proof.

The argument we have just used proves also

COROLLARY (A.4). *If $f: Y \to X$ is almost locally trivial, and Y_x is contractible for each x in X, and X is a finite polyhedron, then f has a cross-section.*

Proof. When constructing $s'\colon X' \to Y$ in the proof of (A.2) we actually found compatible maps $s_\sigma\colon U_\sigma \times D(\sigma) \to Y$ for each cell σ, where U_σ was a neighborhood of σ in X such that $U_\sigma \subset U_\tau$ whenever $\sigma \subset \tau$. That is, we constructed a map $s''\colon X'' \to Y$ over X, where $X'' = \cup U_\sigma \times D(\sigma)$ is a neighborhood of X' in $X \times D(X)$. But $pr\colon X'' \to X$, unlike $pr\colon X' \to X$, obviously has a cross-section.

Homotopic direct limits

An important application of (A.2) is to the following situation. Let C be a category, and F a functor from C to topological spaces. Then there is a space $F/\!\!/C$ which is the realization of the topological category whose objects are pairs (α, x) with $\alpha \in C$ and $x \in F(\alpha)$, and whose morphisms $(\alpha, x) \to (\beta, y)$ are the morphisms $\theta\colon \alpha \to \beta$ in C such that $\theta_*(x) = y$. The space $F/\!\!/C$ is often called the *homotopical direct limit* of the spaces $F(\alpha)$ for α in C. When C is a monoid, i.e. a category with one object, then F is a space with a C-action in the usual sense, and $F/\!\!/C$ has the same meaning as in §2.

Suppose that F is a functor from C into the category of open subsets of a space X and their inclusions. Then there is an obvious projection $\pi\colon F/\!\!/C \to X$, from the homotopic direct limit to the ordinary direct limit. For any x in X let C_x be the full subcategory of C spanned by the objects α such that $x \in F(\alpha)$.

PROPOSITION (A.5). *If $|C_x|$ is contractible for each x in X then $\pi\colon F/\!\!/C \to X$ is a weak homotopy equivalence.*

Proof. The map π is not almost locally trivial, but it is fibre-homotopy equivalent to an almost locally trivial map $Y \to X$. Let Y be the open subspace of $X \times \langle C \rangle$ which is the union of $F_\sigma \times V_\sigma$ for all simplexes σ of $\langle C \rangle$, where V_σ is the neighborhood of σ in $\langle C \rangle$ described in §5, and $F_\sigma = F(\alpha_\sigma)$, where $\alpha_\sigma \in C$ is the first vertex of σ. The projection $Y \to X$ is obviously almost locally trivial, and its fibre Y_x at x is a regular neighborhood of $\langle C_x \rangle$ in $\langle C \rangle$. So $Y \to X$ is a weak homotopy equivalence by (A.2); but $F/\!\!/C \to Y$ is obviously a homotopy equivalence.

The thickening Y of $F/\!\!/C$ used in the proof of (A.5) can be denoted $F/\!\!/\!\!/C$. Its construction makes sense whenever one has a functor from a category C to a category of spaces and open embeddings. The projection $F/\!\!/\!\!/C \to \langle C \rangle$ is always almost locally trivial. If F maps into a category of spaces étale over a space X (i.e. each $F(\alpha) \to X$ is a local homeomorphism), then $F/\!\!/\!\!/C \to X$ is almost locally trivial. Thus we have

PROPOSITION (A.6). *If F is a functor from a category C to a category of spaces étale over X, with embeddings as morphisms, then $F/\!\!/C \to X$ is a weak homotopy equivalence providing it has contractible fibres.*

One example of this is the projection $P/\!\!/M \to X$ discussed in §5, where $P \to X$ is a principal M-bundle. Another is what may be called the basic fact of étale homotopy theory:

Suppose that X is a space, and $\mathcal{U}$ is a collection of open subsets of X with the property that if U and V belong to $\mathcal{U}$ then $U \cap V$ is a union of sets of $\mathcal{U}$. Suppose that each U in $\mathcal{U}$ is a $K(\pi, 1)$, and choose a simply connected covering space $\tilde{U}$ of U for each U. Let $C_{\mathcal{U}}$ be the discrete category whose objects are the spaces $\tilde{U}$ for $U \in \mathcal{U}$, and whose morphisms are all embeddings $\tilde{U} \to \tilde{V}$ over X. Then $|C_{\mathcal{U}}|$ is weakly homotopy equivalent to X.

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