Generalized thermo-piezoelectric problems with temperature-dependent properties

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Abstract

The model of the equation of generalized thermo-piezoelectricity in an isotropic elastic medium with temperature-dependent mechanical properties is established. The modulus of elasticity is taken as a linear function of reference temperature. The state-space approach is adopted for the solution of one-dimensional problems in the absence or presence of heat sources. A numerical technique is employed to obtain the solution in the physical domain. The results are given and illustrated graphically. A comparison is made with results obtained in case of temperature-independent modulus of elasticity.

Keywords: Generalized thermo-piezoelectric; State-space approach; Temperature-dependent properties

1. Introduction

Most of investigations in thermoelasticity were done under the assumption of the temperature-independent material properties, which limit the applicability of the solutions obtained to certain ranges of temperature. At high temperature the material characteristics such as the modulus of elasticity, Poisson’s ratio, the coefficient of thermal expansion and the thermal conductivity are no longer constants (Lomarkin, 1976). In recent years due to the progress in various fields in science and technology the necessity of taking into consideration the real behavior of the material characteristics became actual. Temperature-dependent measurements of Young’s modulus were performed for the first time on black and transparent bulk material of chemical vapor deposited diamond by a dynamic three point bending method in a temperature range from −150 to 850 °C (Szuecs et al., 1999). The temperature dependencies of shear elasticity of some liquids have been investigated by Budaev et al. (2003). It was found that the shear modulus decreases with increasing temperature. This decrease may be explained by the increase of the fluctuation free volume (Budaev et al., 2003). The dynamic resonance method was used by Rishin et al. (1973) to determine the temperature dependence of the modulus of elasticity of some plasma-sprayed materials. Rising in test temperature was found to cause a monotonic decrease in the modulus of elasticity.

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The theory of thermo-piezoelectricity was first proposed by Mindlin (1961). He also derived governing equations of a thermo-piezoelectric plate (Mindlin, 1979). The physical laws for the thermo-piezoelectric materials have been explored by Nowacki (1978, 1979). Chandrasekharaiah (1984) has generalized Mindlin’s theory of thermo-piezoelectricity to account for the finite speed of propagation of thermal disturbances.

In the classical theory of thermoelasticity the velocity of heat propagation is assumed to be infinitely large. In the last decade different generalizations of the classical theory of thermoelasticity are developed to eliminate this paradox. The first theory was developed by Lord and Shulman (1967). In this theory a modified law of heat conduction including both the heat flux and its time derivative replaces the conventional Fourier’s law. The heat equation associated with this a hyperbolic one and, hence, automatically eliminates the paradox of infinite speeds of propagation inherent in both the uncoupled and the coupled theories of thermoelasticity. However, for many problems involving steep heat gradients, and when short time effects are sought, this theory gives markedly different values than those predicted by any of the other theories. This is the case encountered in many problems in industry especially inside nuclear reactors where very high heat gradients act for very short times.

The investigations of propagation of thermoelastic waves in piezoelectric materials are much fewer than that in elastic materials. Majhi (1995) studied the transient thermal response of a semi-infinite piezoelectric rod subjected to a heat source. However, the obtained numerical temperature field is unrealistic due to some unreasonable material parameters in the numerical example. The Laplace transforms and state-space method were used by He et al. (2002) to solve the problem based on the Green–Lindsay theory.

In all papers quoted above, investigations are formulated on the basis of the generalized thermoelastic theories with temperature-independent mechanical properties. The aim of this article is to study the effects of temperature dependence of the modulus of elasticity on the behavior of solutions in generalized thermo-piezoelectricity. Considering the Lord and Shulman theory, the solution is obtained using a state-space approach. The first writers to introduce the state-space approach were Bahar and Hetnarski (1978). Their work dealt with coupled thermoelasticity in the absence of heat sources. The present work is an attempt to generalize the results to include the effects of heat sources when the modulus of elasticity depends on the reference temperature in the context of generalized piezoelectric thermoelasticity. Recently, Ezzat et al. (2004) have investigated the temperature dependencies of the modulus of elasticity in generalized thermoelasticity. This problem can be reduced as a special case of our study.

This article is a continuation of the work of Aouadi and El-Karamany (2003, 2004), Ezzat et al. (2004), and Aouadi (2005) to include the effect of reference temperature on thermal stress distribution.

2. Formulation of the problem

The equations governing linear piezoelectric thermoelastic interactions in homogenous anisotropic medium are (He et al., 2002; Sharma and Kumar, 2000):

(a) Strain–displacement relations:

\[ e_{ij} = \frac{1}{2}(u_{ij} + u_{ji}), \quad i, j = 1, 2, 3. \]  \hfill (2.1)

(b) Stress–strain-temperature:

\[ \sigma_{ij} = c_{ijkl}e_{kl} - e_{ij}D_k - \beta_j T, \quad i, j, k, l = 1, 2, 3. \]  \hfill (2.2)

(c) Equation of motion:

\[ \sigma_{ij} = \rho \frac{\partial^2 u_i}{\partial t^2}, \quad i, j = 1, 2, 3. \]  \hfill (2.3)

(d) Gauss equation and electric field relations:

\[ D_{i,i} = 0, \quad E_i = e_{ijk}e_{jk} + e_{ij}D_j - p_i T, \quad i, j, k, l = 1, 2, 3 \]

\[ E_i = -\phi_i \] is the electric field and \( D_i \) the electric displacement.

In Eqs. (2.2)–(2.4), \( \rho \) is the mass density, \( u_i \) the mechanical displacement, \( e_{ij} \) the strain tensor, \( \sigma_{ij} \) the stress tensor, \( c_{ijkl} \) the isothermal elastic parameters tensor, \( e_{ijk} \) the piezoelectric moduli, \( e_{ij} \) the dielectric moduli,
\( p \), the pyroelectric moduli, \( T \) the temperature change of a material particle, and \( \beta_{ij} \) the thermal elastic coupling tensor.

(e) Heat conduction equation:

\[
\kappa_{ij}T_{,ij} = \left(1 + \tau_0 \frac{\partial}{\partial t}\right) \left(\rho c_E \frac{\partial T}{\partial t} + T_0 \beta_{ij} \frac{\partial \Phi_k}{\partial t} - T_0 \rho_k \frac{\partial \Phi_k}{\partial t} - Q\right). \tag{2.5}
\]

where \( T_0 \) the reference uniform of the body chosen such that \(|(T - T_0)/T_0| \ll 1\), \( \kappa_{ij} \) the heat conduction tensor, \( c_E \) the specific heat at constant strain, \( \tau_0 \) is a relaxation time, and \( Q \) is the strength of the applied heat source per unit mass.

Now we consider a thin isotropic piezoelectric rod. Let the piezoelectric rod polarization direction be parallel with the axial direction. For one-dimensional problem we assume displacement components of the form

\[
u_x = u(x, t), \quad u_y = u_z = 0. \tag{2.6}
\]

From Gauss’s law, since there is no free charge inside the piezoelectric rod, we have

\[
\text{div} D = 0, \tag{2.7}
\]

which, for one-dimensional case, transforms to the following equation:

\[
\frac{\partial D}{\partial x} = 0, \text{ which gives } D = \text{const}. \tag{2.8}
\]

Substituting from Eqs. (2.8) and (2.6) into (2.2)–(2.5), we obtain the equations for the one-dimensional problem

\[
\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + 2\mu) \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial T}{\partial x}, \tag{2.9}
\]

\[
\sigma_{xx} = \sigma = (\lambda + 2\mu) \frac{\partial u}{\partial x} - \beta T - eD, \tag{2.10}
\]

\[
\frac{\kappa}{c_E} \frac{\partial^2 T}{\partial x^2} = \left(1 + \tau_0 \frac{\partial}{\partial t}\right) \left(\rho c_E \frac{\partial T}{\partial t} + T_0 \beta \frac{\partial^2 u}{\partial t \partial x} - Q\right). \tag{2.11}
\]

where \( \lambda, \mu \) are Lamé’s constants, \( \beta = (3\lambda + 2\mu)/\lambda \), \( \epsilon_0 \) the coefficient of linear thermal expansion, \( e \) is the piezoelectric constant, and \( \kappa \) is the coefficient of thermal conductivity.

Our goal is to investigate the effect of temperature dependency of modulus of elasticity keeping the other elastic and thermal parameters constants, therefore we assume

\[
E = E_0 f(T), \quad \lambda = E_0 \lambda_0 f(T), \quad \mu = E_0 \mu_0 f(T), \quad \beta = E_0 \beta_0 f(T),
\]

where \( E_0 \) and \( \lambda_0 \) are considered constants, \( f(T) \) is a given non-dimensional function of temperature, in case of temperature-independent modulus of elasticity \( f(T) \equiv 1 \), and \( E = E_0 \).

For simplifications we shall use the following non-dimensional variables:

\[
x' = c_0 \eta_0 x; \quad u' = c_0 \eta_0 u; \quad t' = c_0^2 \eta_0^2 t; \quad \tau_0' = c_0^2 \eta_0 \tau_0,
\]

\[
\theta = \frac{\beta (T - T_0)}{\lambda + 2\mu}; \quad \sigma_{ij}' = \sigma_{ij} \frac{\lambda}{\lambda + 2\mu}; \quad D' = \frac{eD}{\lambda + 2\mu}; \quad Q' = \frac{\beta Q}{\kappa c_0^3 \eta_0^3 (\lambda + 2\mu)},
\]

where \( \eta_0 = \rho c_E / k \) and \( c_0 = (\lambda + 2\mu) / \rho \). In terms of these non-dimensional variables, Eqs. (2.9)–(2.11) take the following form (dropping the asterisks for convenience):

\[
\frac{\partial^2 u}{\partial t^2} = f(\theta) \left(\frac{\partial^2 u}{\partial x^2} - \frac{\partial \theta}{\partial x}\right) - \left(\frac{\partial u}{\partial x} - \theta\right) \frac{\partial f}{\partial x}, \tag{2.12}
\]

\[
\sigma = f(\theta) \left(\frac{\partial u}{\partial x} - \theta\right) - D, \tag{2.13}
\]

\[
\frac{\partial^2 \theta}{\partial x^2} = \left(1 + \tau_0 \frac{\partial}{\partial t}\right) \left(\frac{\partial \theta}{\partial t} + \epsilon f(\theta) \frac{\partial^2 u}{\partial x \partial t} - Q\right). \tag{2.14}
\]
In generalized thermoelasticity, as well as in the coupled theory only the infinitesimal temperature deviations from reference temperature are considered. Therefore \( f(\theta) \) can be taken in the form \( f(\theta) = 1 - \alpha' T_0 \), where \( \alpha' \) is an empirical material constant \((1/K)\). The last system of equations is linearized and reduces to the linear system:

\[
\begin{align*}
\frac{\partial^2}{\partial x^2} \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial t^2} - \frac{\partial \theta}{\partial x}, \\
\alpha \sigma &= \frac{\partial u}{\partial x} - \theta - \alpha D, \\
\frac{\partial^2 \theta}{\partial x^2} &= \left(1 + \tau_0 \right) \left(\frac{\partial \theta}{\partial t} + \varepsilon_0 f(\theta) \frac{\partial^2 u}{\partial x \partial t} - Q\right),
\end{align*}
\]

where

\[
\alpha = \frac{1}{1 - \alpha' T_0}, \quad \varepsilon = \frac{T_0 \beta^2}{\rho c_E (\lambda + 2\mu)}, \quad \varepsilon_0 = \frac{\varepsilon}{\alpha},
\]

\[\tag{2.18}\]

3. State-space approach

Taking the Laplace transform \( \tilde{f}(x,s) \) of a function \( f(x,t) \) defined by the relation:

\[
\tilde{f}(x,s) = \mathcal{L}(f(x,t)) = \int_0^\infty f(x,t)e^{-st} \, dt, \quad \Re(s) > 0.
\]

Performing the Laplace transform (3.1) over Eqs. (2.15)–(2.17) we get

\[
\begin{align*}
\frac{\partial}{\partial x^2} \left(\frac{\partial^2}{\partial s^2} \right) \tilde{u} &= \frac{\partial \tilde{\theta}}{\partial x}, \\
\frac{\partial}{\partial x^2} \left(\frac{\partial^2}{\partial s^2} \right) \tilde{\theta} &= \varepsilon_0 s (1 + \tau_0 s) \frac{\partial u}{\partial x} - (1 + \tau_0 s) \tilde{Q}, \\
\alpha \sigma &= \frac{\partial u}{\partial x} - \tilde{\theta} - \alpha \frac{D}{s}.
\end{align*}
\]

Eqs. (3.2)–(3.4) can be written in matrix form as follows:

\[
\frac{d\mathcal{V}}{dx} (x,s) = \mathcal{A}(s) \mathcal{V}(x,s) + \mathcal{B}(x,s),
\]

where

\[
\mathcal{A}(s) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ s + \tau_0 s^2 & 0 & \varepsilon_0 (s + \tau_0 s^2) & 0 \\ 0 & \alpha s^2 & 1 & 0 \end{bmatrix}, \quad \mathcal{V}(x,s) = \begin{bmatrix} \tilde{\theta} \\ \tilde{u} \end{bmatrix}, \quad \mathcal{B}(x,s) = -(1 + \tau_0 s) \tilde{Q} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix},
\]

where the prime denotes derivative with respect to \( x \). The formal solution of Eq. (3.5) is given by

\[
\mathcal{V}(x,s) = \exp[\mathcal{A}(s)x] \begin{bmatrix} \mathcal{V}(0,s) + \int_0^x \exp(-\mathcal{A}(s)z)\mathcal{B}(z,s) \, dz \end{bmatrix}.
\]

\[\tag{3.6}\]

In the special case when there are no heat source, Eq. (3.6) simplifies to

\[
\mathcal{V}(x,s) = \exp[\mathcal{A}(s)x] \mathcal{V}(0,s).
\]

\[\tag{3.7}\]

The characteristic equation of the matrix can be written as

\[
k^4 - [\alpha s^2 + (s + \tau_0 s^2)(1 + \varepsilon_0)] k^2 + \alpha s^3 (1 + \tau_0 s) = 0.
\]

\[\tag{3.8}\]
The squares of the roots of this equation, namely $k_1^2$ and $k_2^2$ satisfy the relations

$$k_1^2 + k_2^2 = 2s^2 + (s + \tau_0 s^2)(1 + \nu_0), \quad k_1^2 k_2^2 = 2s^3(1 + \tau_0 s).$$

(3.9)

The Taylor series expansion of the matrix exponential has the form

$$\exp[\overline{\mathbf{A}}(x)] = \sum_{n=0}^{\infty} \frac{[\overline{\mathbf{A}}(x)]^n}{n!}.$$  

(3.10)

Using Cayley–Hamilton theorem, the finite series representing the matrix exponential can be truncated to the following form:

$$\exp[\overline{\mathbf{A}}(x)] = \overline{\mathbf{I}}(x, s) = a_0 \overline{\mathbf{I}} + a_1 \overline{\mathbf{A}} + a_2 \overline{\mathbf{A}}^2 + a_3 \overline{\mathbf{A}}^3,$$

(3.11)

where $\overline{\mathbf{I}}$ is the unit matrix of order 4 and $a_0, \ldots, a_3$ are some parameters depending on $x$ and $s$. By the Cayley–Hamilton theorem, the characteristic roots $\pm k_1, \pm k_2$ and of the matrix $\overline{\mathbf{A}}$ must satisfy the equations

$$\exp(k_i x) = a_0 + a_1 k_i + a_2 k_i^2 + a_3 k_i^3, \quad \exp(-k_i x) = a_0 - a_1 k_i + a_2 k_i^2 - a_3 k_i^3.$$  

The solution of the above system is given by

$$a_0 = \frac{k_1^3 \cosh(k_2 x) - k_2^3 \cosh(k_1 x)}{k_1^3 - k_2^3}, \quad a_1 = \frac{k_1^3 \sinh(k_2 x) - k_2^3 \sinh(k_1 x)}{k_1 k_2(k_1^2 - k_2^2)},$$

$$a_2 = \frac{\cosh(k_1 x) - \cosh(k_2 x)}{(k_1^3 - k_2^3)}; \quad a_3 = \frac{k_2 \sinh(k_1 x) - k_1 \sinh(k_2 x)}{k_1 k_2(k_1^2 - k_2^2)}.$$  

Substituting the above expressions into (3.11) and computing $\overline{\mathbf{A}}^2$ and $\overline{\mathbf{A}}^3$ we obtain the elements $(\ell_{ij}, i, j = 1, 2, 3, 4)$ of the matrix $\overline{\mathbf{I}}(x, s)$ as

$$\ell_{11} = \frac{1}{(k_1^3 - k_2^3)}[(k_1^3 - s(1 + \tau_0 s)) \cosh(k_2 x) - (k_2^3 - s(1 + \tau_0 s)) \cosh(k_1 x)],$$

$$\ell_{12} = \frac{\omega_0 s^3(1 + \tau_0 s)}{k_1 k_2(k_1^2 - k_2^2)}[k_2 \sinh(k_1 x) - k_1 \sinh(k_2 x)],$$

$$\ell_{13} = \frac{1}{k_1 k_2(k_1^2 - k_2^2)}[k_2(k_1^2 - 2s^2) \sinh(k_1 x) - k_1(k_2^2 - 2s^2) \sinh(k_2 x)],$$

$$\ell_{14} = \frac{\nu_0 s(1 + \tau_0 s)}{k_1^3 - k_2^3}[\cosh(k_1 x) - \cosh(k_2 x)],$$

$$\ell_{21} = \frac{s(1 + \tau_0 s)}{k_1 k_2(k_1^2 - k_2^2)}[k_2 \sinh(k_1 x) - k_1 \sinh(k_2 x)],$$

$$\ell_{22} = \frac{1}{k_1^3 - k_2^3}[(k_1^3 - 2s^2) \cosh(k_2 x) - (k_2^3 - 2s^2) \cosh(k_1 x)],$$

$$\ell_{23} = \frac{1}{k_1^3 - k_2^3}[-\cosh(k_1 x) - \cosh(k_2 x)],$$

$$\ell_{24} = \frac{1}{k_1 k_2(k_1^2 - k_2^2)}[k_2(k_1^3 - 2s(1 + \tau_0 s)) \sinh(k_1 x) - k_1(k_2^3 - 2s(1 + \tau_0 s)) \sinh(k_2 x)],$$

$$\ell_{31} = s(1 + \tau_0 s)\ell_{13},$$

$$\ell_{32} = \omega_0 s^3(1 + \tau_0 s)\ell_{23},$$

$$\ell_{33} = \frac{1}{k_1^3 - k_2^3}[(k_1^3 - 2s^2) \cosh(k_1 x) - (k_2^3 - 2s^2) \cosh(k_2 x)],$$

$$\ell_{34} = \frac{1}{k_1 k_2(k_1^2 - k_2^2)}[k_2(k_1^3 - 2s(1 + \tau_0 s)) \sinh(k_1 x) - k_1(k_2^3 - 2s(1 + \tau_0 s)) \sinh(k_2 x)].$$
\[ \ell_{34} = \frac{\varepsilon_0 s (1 + \tau_0 s)}{k_1^2 - k_2^2} [k_1 \sinh(k_1 x) - k_2 \sinh(k_2 x)], \]
\[ \ell_{41} = s(1 + \tau_0 s) \ell_{23}, \]
\[ \ell_{42} = \alpha s^2 \ell_{24}, \]
\[ \ell_{43} = \frac{1}{k_1^2 - k_2^2} [k_1 \sinh(k_1 x) - k_2 \sinh(k_2 x)], \]
\[ \ell_{44} = \frac{1}{k_1^2 - k_2^2} [(k_1^2 - s(1 + \tau_0 s)) \cosh(k_1 x) - (k_2^2 - s(1 + \tau_0 s)) \cosh(k_2 x)]. \]

4. Application

4.1. Problem 1: A thermal shock in semi-space problem

We consider a semi-space homogeneous piezoelectric medium occupying the region \( x \geq 0 \) with quiescent initial state. A thermal shock is applied to the boundary plane \( x = 0 \) in the form
\[ \theta(0, t) = \theta_0 H(t), \tag{4.1} \]
where \( \theta_0 \) is a constant, and \( H(t) \) is the Heaviside unit step function. The boundary plane \( x = 0 \) is taken to be traction free, i.e.
\[ \sigma(0, t) = 0. \tag{4.2} \]
Since the solution is unbounded at infinity, the initial conditions should be adjusted so that the infinite terms are eliminated. We now apply the state-space approach described previously to this problem. Thus, two components of the transformed initial state \((0, s)\) are known, namely
\[ \bar{\theta}(0, s) = \frac{\theta_0}{s}, \quad \bar{u}'(0, s) = \frac{\theta_0 + \alpha D}{s}. \tag{4.3} \]
To obtain the two remaining components \( \bar{\theta}'(0, s) \) and \( \bar{u}(0, s) \), we substitute \( x = 0 \) in both sides of Eq. (3.7) and taking into consideration the entries of \( \bar{T}(x, s) \), we obtain a system of linear algebraic equations in the two unknowns \( \bar{\theta}'(0, s) \) and \( \bar{u}(0, s) \), whose solution gives
\[ \bar{\theta}'(0, s) = -\frac{\theta_0 (k_1^2 + k_1 k_2 + k_2^2 - \alpha s^2) + \alpha s (1 + \tau_0 s) D}{s(k_1 + k_2)}, \quad \bar{u}(0, s) = -\frac{\theta_0 s^2 + D(k_1 k_2 + \alpha s^2)}{s^3(k_1 + k_2)}. \tag{4.4} \]
Inserting the values from Eqs. (4.3) and (4.4) into the right-hand side of Eq. (3.7) and upon using Eq. (3.9), we obtain
\[ \bar{u}(x, s) = -\frac{1}{s^3(k_1^2 - k_2^2)} \sum_{i=1}^{2} (-1)^{i+1} k_i [s^2 \theta_0 - (k_{3-i}^2 - \alpha s^2) D] e^{-k_i x}, \tag{4.5} \]
\[ \bar{\theta}(x, s) = \frac{2}{s(k_1^2 - k_2^2)} \sum_{i=1}^{2} (-1)^{i+1} (k_i^2 - \alpha s^2) \theta_0 + \alpha s (1 + \tau_0 s) D e^{-k_i x}, \tag{4.6} \]
\[ \bar{\sigma}(x, s) = \frac{1}{s(k_1^2 - k_2^2)} \sum_{i=1}^{2} (-1)^{i+1} [s^2 \theta_0 - (k_{3-i}^2 - \alpha s^2) D] e^{-k_i x} - \frac{D}{s}. \tag{4.7} \]

4.2. Problem 2: A heat sources in a piezoelectric medium

We assume that there is a plane distribution of continuous heat sources located at the plane \( x = 0 \). The intensity of the heat sources is thus given by
\[ Q(x, t) = Q_0 H(t) \delta(x), \tag{4.8} \]
where $Q_0$ is a constant heat and $\delta(x)$ is the Dirac’s delta function. In this case, Eq. (3.6) takes the form

$$\nabla(x,s) = \mathcal{L}(x,s)[\nabla(0,s) + \mathcal{F}(s)],$$

(4.9)

where

$$\mathcal{F}(s) = -\frac{Q_0(1 + \tau_0s)}{4s} \begin{bmatrix}
\frac{k_1k_2 + 2s^2}{k_1k_2(k_1 + k_2)} \\
\frac{1}{k_1 + s_2} \\
0
\end{bmatrix}.$$ 

(4.10)

Eq. (4.9) expresses the solution of the problem in Laplace transform domain for $x \geq 0$ in terms of the vector $\nabla(s)$, the applied heat source, and the vector $\nabla(0,s)$ representing the condition at the plane $x = 0$. To evaluate the components of this vector, we note first that that due to symmetry of the problem, the displacement vanishes at the plane source of heat, thus

$$u(0,t) = 0, \quad \text{or} \quad u(0,s) = 0.$$ 

(4.11)

Gauss’s divergence theorem will now be used to obtain the thermal condition at the plane source. We consider a cylinder of unit base whose axis is perpendicular to the plane source of heat and whose bases line on opposite sides of it. Taking the limit as height of the cylinder tends to zero and noting that there is no heat flux through the lateral surface, upon using the symmetry of the temperature field we get

$$q(0,t) = \frac{1}{2} Q_0 H(t), \quad \text{or} \quad \bar{q}(0,s) = \frac{Q_0}{2s}.$$ 

(4.12)

We shall use generalized Fourier law of condition in the non-dimensional form, namely

$$q + \tau_0 \frac{\partial q}{\partial t} = -\frac{\partial \theta}{\partial x}.$$ 

(4.13)

Following the same procedure as above, the components of the initial vector are given by

$$\tilde{\theta}(0,s) = \frac{Q_0(1 + \tau_0s)(k_1k_2 + 2s^2)}{2sk_1k_2(k_1 + k_2)}, \quad \tilde{\theta}'(0,s) = -\frac{Q_0(1 + \tau_0s)}{2s}, \quad \tilde{\theta}''(0,s) = \frac{Q_0(1 + \tau_0s)}{2s(k_1 + k_2)}.$$ 

(4.14)

As before, we have suppressed the positive exponential terms appearing in the entries of $\mathcal{L}(x,s)$. Substituting the above in the right-hand of Eq. (4.9), we obtain

$$\tilde{\theta}(x,s) = \frac{Q_0(1 + \tau_0s)}{2s(k_1 - k_2)^2} \sum_{i=1}^{2} (-1)^{i+1} \frac{k_i^2 - 2s^2}{k_i} e^{ik_i x},$$

(4.15)

$$\tilde{u}(x,s) = \frac{Q_0(1 + \tau_0s)}{2s(k_1 - k_2)} \sum_{i=1}^{2} (-1)^{i+1} e^{ik_i x},$$

(4.16)

$$\tilde{\sigma}(x,s) = \frac{Q_0s(1 + \tau_0s)}{2(k_1 - k_2)} \sum_{i=1}^{2} (-1)^{i+1} e^{ik_i x} - \frac{D}{s}.$$ 

(4.17)

In the above equations the upper (plus) sign indicates the solution in the region $x < 0$, while the lower (minus) sign indicates the solution in the region $x \geq 0$, respectively.

5. Inversion of the transforms

We shall now outline the numerical inversion method to obtain the solution of the problem in the physical domain. Let $g(s)$ be the Laplace transform of some function $g(t)$. Following Honig and Hirdes (1984), the Laplace transformed function $g(s)$ can be inverted as follow:

$$g(t) = \mathcal{L}^{-1}[g(s)] = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{s \tau} g(s) ds.$$ 

(5.1)
The numerical inversion form of the Laplace transform can be written
\[ g(s) = \int_0^\infty e^{-st} \mathbf{g}(t) [\cos(wt) - i \sin(wt)] \, dt = \Re \mathbf{g}(v + iw) + i \Im \mathbf{g}(v + iw). \] (5.2)

Substituting Eq. (5.2) into Eq. (5.1) yields
\[ g(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{st} [\cos(wt) + i \sin(wt)] [\Re \mathbf{g}(s)] + i \Im \mathbf{g}(s)] \, dw \]
\[ = \frac{e^{st}}{2\pi} \left[ \int_{-\infty}^{\infty} \Re \mathbf{g}(s) \cos(wt) - \Im \mathbf{g}(s) \sin(wt) \, dw + i \int_{-\infty}^{\infty} \Re \mathbf{g}(s) \sin(wt) + \Im \mathbf{g}(s) \cos(wt) \, dw \right]. \] (5.3)

Combining Eqs. (5.3) and (5.2) leads to
\[ g(t) = \frac{e^{st}}{\pi} \left[ \int_{-\infty}^{\infty} \Re \mathbf{g}(s) \cos(wt) - \Im \mathbf{g}(s) \sin(wt) \, dw \right]. \] (5.4)

In Eq. (5.4), \( \sin(w(\tau - t)) \) is an odd function of \( w \); therefore, the second integral is zero and the equation is simplified as
\[ g(t) = \frac{e^{st}}{\pi} \left[ \int_{0}^{\infty} \Re \mathbf{g}(s) \cos(wt) - \Im \mathbf{g}(s) \sin(wt) \, dw \right]. \] (5.5)

Expanding the function \( h(t) = e^{-st} \mathbf{g}(t) \) in a Fourier series in the interval \([0, 2T]\), Durbin (1973) derived the approximation formula
\[ g(t) = \frac{e^{st}}{T} \left[ \frac{1}{2} \Re \mathbf{g}(v) + \sum_{k=0}^{\infty} \Re \left\{ \mathbf{g} \left( v + i \frac{k\pi}{T} \right) \cos \left( \frac{k\pi}{T} t \right) \right\} \right. \]
\[ \left. - \sum_{k=0}^{\infty} \Im \left\{ \mathbf{g} \left( v + i \frac{k\pi}{T} \right) \sin \left( \frac{k\pi}{T} t \right) \right\} \right] - F_1(v, t, T) \] (5.6)

where \( F_1(v, t, T) \) is the discretization error given by
\[ F_1(v, t, T) = \sum_{k=1}^{\infty} e^{-2k\pi t} \mathbf{g}(2kT + t). \] (5.7)

As the infinite series in Eq. (5.6) can only be summed up to a infinite number \( N \) of terms, a truncation error is introduced in the form of
\[ F_\infty(N, v, t, T) = \frac{e^{st}}{T} \left[ \sum_{k=N+1}^{\infty} \Re \left\{ \mathbf{g} \left( v + i \frac{k\pi}{T} \right) \cos \left( \frac{k\pi}{T} t \right) \right\} \right. \]
\[ \left. - \sum_{k=N+1}^{\infty} \Im \left\{ \mathbf{g} \left( v + i \frac{k\pi}{T} \right) \sin \left( \frac{k\pi}{T} t \right) \right\} \right]. \] (5.8)

Hence the approximation value for \( g(t) \) is
\[ g_N(t) = \frac{e^{st}}{T} \left[ \frac{1}{2} \Re \mathbf{g}(v) + \sum_{k=0}^{N} \Re \left\{ \mathbf{g} \left( v + i \frac{k\pi}{T} \right) \cos \left( \frac{k\pi}{T} t \right) \right\} \right. \]
\[ \left. - \sum_{k=0}^{N} \Im \left\{ \mathbf{g} \left( v + i \frac{k\pi}{T} \right) \sin \left( \frac{k\pi}{T} t \right) \right\} \right]. \] (5.9)

It is obvious from Eq. (5.7) that the discretization error can be made arbitrarily small if the free parameter \( vT \) is large enough. Unfortunately, the truncation error in Eq. (5.8) may diverge for large values of \( vT \).

Two methods are used to reduce the total error. First, the Korrektur method is used to reduce the discretization error. Next, the \( \varepsilon \)-algorithm is used to reduce the truncation error and hence to accelerate convergence. With Eq. (5.9), Eq. (5.6) can be written in the form
\[ g(t) = g_\infty(t) - e^{-2\pi t} g_\infty(2T + t) - F_2(v, t, T), \]
where the discretization error \(|F_2(v, t, T)| \ll |F_1(v, t, T)|\). Thus, the approximate value of \(g(t)\) becomes
\[
g_{NK}(t) = g_N(t) - e^{-2\pi r}g_{N'}(2T + t),
\]
(5.10)
where \(N'\) is an integer less \(N\). Let
\[
c_k = \frac{e^{i\pi}}{T} \left[ \Re \left( g(v + ik\pi) \right) \cos \left( \frac{k\pi}{T} \right) - \Im \left( g(v + ik\pi) \right) \sin \left( \frac{k\pi}{T} \right) \right].
\]
(5.11)
According to Eq. (5.10), Eq. (5.9) can be expressed as
\[
g_N(t) = \frac{1}{2}c_0 + \sum_{k=1}^{N} c_k.
\]
(5.12)
Now the \(\varepsilon\)-algorithm is described in the following. Let \(N\) be an odd natural number, and let
\[
s_m = \sum_{k=1}^{m} c_k
\]
be the sequence of partial sums of Eq. (5.12), we define the \(\varepsilon\)-sequence by
\[
\varepsilon_{0,m} = 0, \quad \varepsilon_{1,m} = s_m, \quad m = 1, 2, 3, \ldots
\]
and
\[
\varepsilon_{n+1,m} = \varepsilon_{n,m+1} + \frac{1}{\varepsilon_{n,m+1} - \varepsilon_{n,m}}, \quad n = m = 1, 2, 3, \ldots
\]
Honig and Hirdes (1984) shows that the sequence \(\varepsilon_{1,1}, \varepsilon_{2,1}, \ldots, \varepsilon_{N,1}\) converges to \(g_m(t) - \frac{1}{2}c_0\) faster than the sequence of partial sums \(s_m, m = 1, 2, 3, \ldots\). The actual procedure used to invert the Laplace transforms consists of using Eq. (5.12) together with the \(\varepsilon\)-algorithm. The values of \(v\) and \(T\) are chosen according the criteria outlined by Honig and Hirdes (1984).

6. Numerical results

The numerical values of the temperature, displacement and stress have been calculated for small time \(t = 0.1\). In the calculation process, the following constants are necessary to be known including:
\[
0 = Q_0 = 1, \quad \varepsilon = 0.003887, \quad D = 10^{-4}, \quad T_0 = 973 \text{ K}, \quad \nu = 0.0005 \text{ 1/K}.
\]
For the generalized theory with one relaxation time (LS theory), the relaxation time is taken to be \(\tau_0 = 0.03\), while for the coupled theory (CT theory) \(\tau_0 = 0\).

The temperature distributions for problem 1 (thermal shock problem) and problem 2 (heat source problem) are shown in Fig. 1a and b, respectively. For both problems, the temperature starts with its maximum value at the origin and decreases until attaining zero beyond a wavefront for the generalized theory, whereas it is continuous everywhere else for the coupled theory. This wavefront is located approximately at \(x = 0.8\) for problem 1 and at \(x = 0.6\) for problem 2. It should be noted in both problems, that there is no significant difference in the values of \(\theta\) when \(\alpha = 1.75\) compared to those when \(\alpha = 1\), and they are large for LS theory in comparison with those for CT theory.

Fig. 2a and b shows the variation of displacement \(u\) for problems 1 and 2, respectively. For problem 1, the displacement increases monotonically from a negative value at \(x = 0\) to a positive value, and in the positive domain it attains a peak value at another wavefront located at \(x = 0.1\). The position of this wavefront changes slightly with \(\alpha\), whereas the magnitude of the peak value is reduced with the increase of \(\alpha\) and the decrease of \(\tau_0\).

For problem 2, \(u\) is zero at \(x = 0\), which is in agreement with Eq. (4.11). It may be observed that the displacement is compressive in nature, and attains a peak value at the wavefront located at \(x = 0.1\).

Fig. 3a and b shows the variation of the stress \(\sigma\) for problems 1 and 2, respectively. The stress is zero at \(x = 0\) for problem 1 which is in agreement with Eq. (4.2) whereas it is negative for problem 2. For both
problems, the magnitude of the compressive stress increases rapidly as $x$ increases and it attains a peak value at $x = 0.1$, thereafter it decreases slowly with increasing $x$.
7. Concluding remarks

Important phenomena are observed in all these computations:

1. From the above figures, there exist two propagating wavefronts. Under the coupled theory it is clear that there is only an elastic wavefront. This of course due to the fact that the temperature satisfies a parabolic equation. So, the wavefront located at \( x = 0.8 \) for problem 1 and at \( x = 0.6 \) for problem 2 is the mainly thermal wave mentioned above, and the wavefront located at \( x = t = 0.1 \) is mainly elastic in nature (Aouadi, 2005). It is clear that under CT theory the thermal wavefront propagates with infinite velocity, whereas the velocity of propagation of the elastic wavefront is \( v = 1 \) which means \( v = C_0 = [(\lambda + \mu)/\rho]^{1/2} \). The thermal wavefront is faster than the elastic one, corresponds to the second sound, and results from the temperature forcing term in the displacement equations.

2. In all these figures, it is clear that the considered function vanishes identically outside a bounded region of space surrounding the heating source at a distance from it. The edge of this region is the thermal wavefront. This is not the case for the coupled theory where an infinite speed of propagation is inherent and hence all the considered functions have non-zero (although it may be very small) value for any point in the medium.

3. The values of solutions for LS theory are large in comparison with those for CT theory. In fact, the relaxation time is large for LS theory \( (\tau_0 > 0) \), so the time available for the exchange of thermal energy with the domain is large and then values of solutions are higher. All these remarks indicate that the generalized heat conduction mechanism is completely different from the classic Fourier’s in essence.

4. The magnitudes of the solutions are less when \( \alpha = 1.75 \) compared to those when \( \alpha = 1 \), except for the temperature. This phenomenon is also observed by Aouadi and El-Karamany (2003, 2004). This may be due to the fact that the temperature in thermoelasticity is the infinitesimal deviation from reference temperature.

Thus, the dependence of the modulus of elasticity on reference temperature has a significant effect on the thermal and mechanical interactions by decreasing the magnitude of solution (Ezzat et al., 2004).

References


