J. Math. Anal. Appl. 393 (2012) 66–79

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Small data global existence for the semilinear wave equation with space–time dependent damping

Yuta Wakasugi

Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka, 560-0043, Japan

ARTICLE INFO

Article history:
Received 16 November 2011
Available online 13 April 2012
Submitted by Kenji Nishihara

Keywords:
Damped wave equation
Space–time dependent coefficient
Sourcing semilinear term
Critical exponent
Small data global existence

ABSTRACT

In this paper we consider the critical exponent problem for the semilinear wave equation with space–time dependent damping. When the damping is effective, it is expected that the critical exponent agrees with that of only the space dependent coefficient case. We shall prove that there exists a unique global solution for small data if the power of nonlinearity is larger than the expected exponent. Moreover, we do not assume that the data are compactly supported. However, it is still open whether there exists a blow-up solution if the power of nonlinearity is smaller than the expected exponent.

1. Introduction

We consider the Cauchy problem for the semilinear damped wave equation

$$\begin{cases} u_{tt} - \Delta u + a(x)b(t)u_t = f(u), & (t,x) \in (0,\infty) \times \mathbf{R}^n, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.1}$$

where the coefficients of damping are

$$a(x) = a_0 \langle x \rangle^{-\alpha}, \qquad b(t) = (1+t)^{-\beta}, \quad \text{with } a_0 > 0,\ \alpha, \beta \geq 0,\ \alpha + \beta < 1,$$

where $\langle x \rangle = (1+|x|^2)^{1/2}$. Here u is a real-valued unknown function and (u_0, u_1) is in $H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$. We note that u_0 and u_1 need not be compactly supported. The nonlinear term $f(u)$ is given by

$$f(u) = \pm |u|^p \quad \text{or} \quad |u|^{p-1}u$$

and the power p satisfies

$$1 < p \leq \frac{n}{n-2} \quad (n \geq 3), \qquad 1 < p < \infty \quad (n = 1, 2).$$

Our aim is to determine the critical exponent p_c, which is a number defined by the following property:

If $p_c < p$, all small data solutions of (1.1) are global; if $1 < p \leq p_c$, the time-local solution cannot be extended time-globally for some data.

It is expected that the critical exponent of (1.1) is given by

$$p_c = 1 + \frac{2}{n - \alpha}.$$

E-mail address: y-wakasugi@cr.math.sci.osaka-u.ac.jp.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2012.03.035

In this paper we shall prove the existence of global solutions with small data when $p > 1 + 2/(n - \alpha)$. However, it is still open whether there exists a blow-up solution when $1 < p \le 1 + 2/(n - \alpha)$.

When the damping term is missing and $f(u) = |u|^p$, that is

$$\begin{cases} u_{tt} - \Delta u = |u|^p, & (t, x) \in (0, \infty) \times \mathbf{R}^n, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.2}$$

it is well known that the critical exponent $p_w(n)$ is the positive root of $(n-1)p^2 - (n+1)p - 2 = 0$ for $n \ge 2$ ($p_w(1) = \infty$). This is the famous Strauss conjecture and the proof was completed by the effort of many mathematicians (see [1–10]).

For the linear wave equation with a damping term

$$\begin{cases} u_{tt} - \Delta u + c(t, x)u_t = 0, & (t, x) \in (0, \infty) \times \mathbf{R}^n, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), & x \in \mathbf{R}^n, \end{cases} \tag{1.3}$$

there are many results about the asymptotic behavior of the solution. When $c(t, x) = c_0 > 0$ and $(u_0, u_1) \in (H^1 \cap L^1) \times (L^2 \cap L^1)$, Matsumura [11] showed that the energy of solutions decays at the same rate as the corresponding heat equation. When the space dimension is 3, using the exact expression of the solution, Nishihara [12] discovered that the solution of (1.3) with $c(t, x) = 1$ is expressed asymptotically by

$$u(t, x) \sim v(t, x) + e^{-t/2} w(t, x),$$

where $v(t, x)$ is the solution of the corresponding heat equation

$$\begin{cases} v_t - \Delta v = 0, & (t, x) \in (0, \infty) \times \mathbf{R}^3, \\ v(0, x) = u_0(x) + u_1(x), & x \in \mathbf{R}^3 \end{cases}$$

and $w(t, x)$ is the solution of the free wave equation

$$\begin{cases} w_{tt} - \Delta w = 0, & (t, x) \in (0, \infty) \times \mathbf{R}^3, \\ w(0, x) = u_0(x), \quad w_t(0, x) = u_1(x), & \in \mathbf{R}^3. \end{cases}$$

These results indicate a diffusive structure of damped wave equations. On the other hand, Mochizuki [13] showed that if $0 \le c(t, x) \le C(1 + |x|)^{-1-\delta}$, where $\delta > 0$, then the energy of solutions of (1.3) does not decay to 0 for nonzero data and the solution is asymptotically free. We can interpret this result as (1.3) loses its "parabolicity" and recover its "hyperbolicity". Wirth [14,15] treated the time-dependent damping case, that is $c(t, x) = b(t)$ in (1.3). By the Fourier transform method, he got several sharp $L^p - L^q$ estimates of the solution and showed that there exists a diffusive structure for general $b(t)$ including $b(t) = b_0(1+t)^{-\beta}(-1 < \beta < 1)$. Todorova and Yordanov [16] considered the case $c(t, x) = a(x) = a_0\langle x\rangle^{-\alpha}$ with $\alpha \in [0, 1)$ and Kenigson and Kenigson [17] considered space–time dependent coefficient case $c(t, x) = a(x)b(t)$, $a(x) = a_0\langle x\rangle^{-\alpha}$, $b(t) = (1+t)^{-\beta}$, $(0 \le \alpha+\beta < 1)$. They established the energy decay estimate that also implies diffusive structure even in the decaying coefficient cases. From these results, the decay rate -1 of the coefficient of the damping term is the threshold of parabolicity. This is the reason why we assume $\alpha + \beta < 1$ for (1.1). We mention that recently, Ikehata et al. [18] treated the case $c(t, x) = a_0\langle x\rangle^{-1}$ and obtained almost optimal decay estimates.

There are also many results for the semilinear damped wave equation with absorbing semilinear term:

$$\begin{cases} u_{tt} - \Delta u + a(x)b(t)u_t + |u|^{p-1}u = 0, & (t, x) \in (0, \infty) \times \mathbf{R}^n, \\ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), & x \in \mathbf{R}^n. \end{cases} \tag{1.4}$$

It is well known that there exists a unique global solution even for large initial data. When $a(x)b(t) = 1$, that is the constant coefficient case, Kawashima et al. [19], Karch [20], Hayashi et al. [21], Ikehata et al. [22] and Nishihara [23] showed the global existence of solutions and that their asymptotic profile is given by a constant multiple of the Gauss kernel for $1 + 2/n < p$ and $n \le 4$. For $1 < p \le 1 + 2/n$, Nishihara and Zhao [24], Ikehata et al. [22], Nishihara [23] proved that the decay rate of the solution agrees with that of a self-similar solution of the corresponding heat equation. Hayashi et al. [25–27,21] proved the large time asymptotic formulas in terms of the weighted Sobolev spaces. These results indicate the critical exponent for (1.4) with $a(x)b(t) = 1$ is given by $p_c = 1 + \frac{2}{n}$. In this case the critical exponent means the turning point of the asymptotic behavior of the solution. When $b(t) = 1$, $a(x) = \langle x\rangle^{-\alpha}(0 \le \alpha < 1)$, namely space-dependent damping case, Nishihara [28] established decay estimates of solutions and conjectured the critical exponent is given by $p_c = 1 + 2/(n - \alpha)$. When $a(x) = 1$, $b(t) = (1+t)^{-\beta}(-1 < \beta < 1)$, Nishihara and Zhai [29] proved decay estimates of solutions and conjectured the critical exponent is $p_c = 1 + 2/n$. Finally in the case $a(x) = \langle x\rangle^{-\alpha}$, $b(t) = (1+t)^{-\beta}(0 \le \alpha+\beta < 1)$, Lin et al. [30,31] showed decay estimates of the solution and conjectured the critical exponent is $p_c = 1 + 2/(n - \alpha)$. They used a weighted energy method, which was originally developed by Todorova and Yordanov [32,33]. In this paper we shall essentially use the techniques and methods that they used.

Li and Zhou [34] considered the semilinear damped wave equation

$$u_{tt} - \Delta u + u_t = |u|^p. \tag{1.5}$$

They proved that if $n \le 2$, $1 < p \le 1 + \frac{2}{n}$ and the data are positive on average, then the local solution of (1.5) must blow up in a finite time. Todorova and Yordanov [32,33] developed a weighted energy method using the function which has the form $e^{2\psi}$ and determined that the critical exponent of (1.5) is

$$p_c = 1 + \frac{2}{n},$$

which is well known as Fujita's critical exponent for the heat equation $u_t - \Delta u = u^p$ (see [35]). More precisely, they proved small data global existence in the case $p > 1 + 2/n$ and blow-up for all solutions of (1.5) with positive on average data in the case $1 < p < 1 + 2/n$. Later on Zhang [36] showed that the critical exponent $p = 1 + 2/n$ belongs to the blow-up region. We mention that Todorova and Yordanov [32,33] assumed data have compact support and essentially used this property. However, Ikehata and Tanizawa [37] removed this assumption. Ikehata et al. [38] investigated the space-dependent coefficient case:

$$u_{tt} - \Delta u + a(x)u_t = |u|^p, \tag{1.6}$$

where

$$a(x) \sim a_0 \langle x \rangle^{-\alpha}, \quad |x| \to \infty, \qquad \text{radially symmetric and } 0 \le \alpha < 1.$$

They proved that the critical exponent of (1.5) is given by

$$p_c = 1 + \frac{2}{n - \alpha}$$

by using a refined multiplier method. Their method also depends on the finite propagation speed property. Recently, Nishihara [39] and Lin et al. [31] considered the semilinear wave equation with time-dependent damping

$$u_{tt} - \Delta u + b(t)u_t = |u|^p, \tag{1.7}$$

where

$$b(t) = b_0(1 + t)^{-\beta}, \quad \beta \in (-1, 1).$$

They proved that the critical exponent of (1.7) is

$$p_c = 1 + \frac{2}{n}.$$

This shows that, roughly speaking, time-dependent coefficients of damping term do not influence the critical exponent. Therefore we expect that the critical exponent of the semilinear wave equation (1.1) is

$$p_c = 1 + \frac{2}{n - \alpha}.$$

To state our results, we introduce an auxiliary function

$$\psi(t, x) := A\frac{\langle x \rangle^{2-\alpha}}{(1 + t)^{1+\beta}} \tag{1.8}$$

with

$$A = \frac{(1 + \beta)a_0}{(2 - \alpha)^2(2 + \delta)}, \quad \delta > 0. \tag{1.9}$$

This type of weight function was first introduced by Ikehata and Tanizawa [37]. We have the following result:

Theorem 1.1. *If*

$$p > 1 + \frac{2}{n - \alpha},$$

then there exists a small positive number $\delta_0 > 0$ such that for any $0 < \delta \le \delta_0$ the following holds: If

$$I_0^2 := \int_{\mathbf{R}^n} e^{2\psi(0,x)}(u_1^2 + |\nabla u_0|^2 + |u_0|^2)dx$$

is sufficiently small, then there exists a unique solution $u \in C([0, \infty); H^1(\mathbf{R}^n)) \cap C^1([0, \infty); L^2(\mathbf{R}^n))$ to (1.1) satisfying

$$\int_{\mathbf{R}^n} e^{2\psi(t,x)}|u(t, x)|^2dx \le C_\delta(1 + t)^{-(1+\beta)\frac{n-2\alpha}{2-\alpha}+\varepsilon}, \tag{1.10}$$

$$\int_{\mathbf{R}^n} e^{2\psi(t,x)}(|u_t(t, x)|^2 + |\nabla u(t, x)|^2)dx \le C_\delta(1 + t)^{-(1+\beta)(\frac{n-\alpha}{2-\alpha}+1)+\varepsilon},$$

where

$$\varepsilon = \varepsilon(\delta) := \frac{3(1+\beta)(n-\alpha)}{2(2-\alpha)(2+\delta)}\delta \tag{1.11}$$

and C_δ is a constant depending on δ.

Remark 1.2. When $1 < p \le 1 + 2/(n-\alpha)$, it is expected that no matter how small the data are, if the data have some shape, then the corresponding local solution blows up in finite time. However, we have no result.

Remark 1.3. We do not assume that the data are compactly supported. Hence our result is an extension of the results of Ikehata et al. [38] to noncompactly supported data cases. However, we prove only the case $a(x) = a_0\langle x\rangle^{-\alpha}$.

As a consequence of the main theorem, we have an exponential decay estimate outside a parabolic region.

Corollary 1.4. *If*

$$p > 1 + \frac{2}{n-\alpha},$$

then there exists a small positive number $\delta_0 > 0$ such that for any $0 < \delta \le \delta_0$ the following holds: Take ρ and μ so small that

$$0 < \rho < 1 - \alpha - \beta, \quad \text{and} \quad 0 < \mu < 2A,$$

and put

$$\Omega_\rho(t) := \{x \in \mathbf{R}^n;\ \langle x\rangle^{2-\alpha} \ge (1+t)^{1+\beta+\rho}\}.$$

Then, for the global solution u in Theorem 1.1, we have the following estimate

$$\int_{\Omega_\rho(t)} (u_t^2 + |\nabla u|^2 + u^2)dx \le C_{\delta,\rho,\mu}(1+t)^{-\frac{(1+\beta)(n-2\alpha)}{2-\alpha}+\varepsilon} e^{-(2A-\mu)(1+t)^\rho}, \tag{1.12}$$

here ε is defined by (1.11) and $C_{\delta,\rho,\mu}$ is a constant depending on δ, ρ and μ.

Namely, the decay rate of solution in the region $\Omega_\rho(t)$ is exponential. We note that the support of $u(t)$ and the region $\Omega_\rho(t)$ can intersect even if the data are compactly supported. This phenomenon was first discovered by Todorova and Yordanov [33]. We can interpret this result as follows: The support of the solution is strongly suppressed by damping, so that the solution is concentrated in the parabolic region much smaller than the light cone.

2. Proof of Theorem 1.1

In this section we prove our main result. At first we prepare some notation and terminology. We put

$$\|f\|_{L^p(\mathbf{R}^n)} := \left(\int_{\mathbf{R}^n} |f(x)|^p dx\right)^{1/p}, \qquad \|u\| := \|u\|_{L^2(\mathbf{R}^n)}.$$

By $H^1(\mathbf{R}^n)$ we denote the usual Sobolev space. For an interval I and a Banach space X, we define $C^r(I;X)$ as the Banach space whose element is an r-times continuously differentiable mapping from I to X with respect to the topology in X. The letter C indicates the generic constant, which may change from one line to the next line.

To prove Theorem 1.1, we use a weighted energy method which was originally developed by Todorova and Yordanov [32,33]. We first describe the local existence:

Proposition 2.1. *For any $\delta > 0$, there exists $T_m \in (0, +\infty]$ depending on I_0^2 such that the Cauchy problem (1.1) has a unique solution $u \in C([0,T_m); H^1(\mathbf{R}^n)) \cap C^1([0,T_m); L^2(\mathbf{R}^n))$, and if $T_m < +\infty$ then we have*

$$\liminf_{t\to T_m} \int_{\mathbf{R}^n} e^{\psi(t,x)}(u_t^2 + |\nabla u|^2 + u^2)dx = +\infty.$$

We can prove this proposition by standard arguments (see [37]). We prove an a priori estimate for the following functional:

$$M(t) := \sup_{0\le\tau<t}\left\{(1+\tau)^{B+1-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi}(u_t^2 + |\nabla u|^2)dx + (1+\tau)^{B-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi}a(x)b(t)u^2 dx\right\}, \tag{2.1}$$

where

$$B := \frac{(1+\beta)(n-\alpha)}{2-\alpha} + \beta$$

and ε is given by (1.11). From (1.8), (1.9), it is easy to see that

$$-\psi_t = \frac{1+\beta}{1+t}\psi, \tag{2.2}$$

$$\nabla\psi = A\frac{(2-\alpha)\langle x\rangle^{-\alpha}x}{(1+t)^{1+\beta}}, \tag{2.3}$$

$$\begin{aligned}\Delta\psi &= A(2-\alpha)(n-\alpha)\frac{\langle x\rangle^{-\alpha}}{(1+t)^{1+\beta}} + A(2-\alpha)\alpha\frac{\langle x\rangle^{-2-\alpha}}{(1+t)^{1+\beta}}\\ &\geq \frac{(1+\beta)(n-\alpha)}{(2-\alpha)(2+\delta)}\frac{a(x)b(t)}{1+t}\\ &=: \left(\frac{(1+\beta)(n-\alpha)}{2(2-\alpha)} - \delta_1\right)\frac{a(x)b(t)}{1+t}.\end{aligned} \tag{2.4}$$

Here and after, $\delta_i (i = 1, 2, \ldots)$ is a positive constant depending only on δ such that

$$\delta_i \to 0^+ \quad \text{as } \delta \to 0^+.$$

We also have

$$\begin{aligned}(-\psi_t)a(x)b(t) &= Aa_0(1+\beta)\frac{\langle x\rangle^{2-2\alpha}}{(1+t)^{2+2\beta}}\\ &\geq \frac{a_0(1+\beta)}{(2-\alpha)^2 A}A^2(2-\alpha)^2\frac{\langle x\rangle^{-2\alpha}|x|^2}{(1+t)^{2+2\beta}}\\ &= (2+\delta)|\nabla\psi|^2.\end{aligned} \tag{2.5}$$

By multiplying (1.1) by $e^{2\psi}u_t$, it follows that

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\frac{e^{2\psi}}{2}(u_t^2+|\nabla u|^2)\right] - \nabla\cdot(e^{2\psi}u_t\nabla u) + e^{2\psi}\left(a(x)b(t) - \frac{|\nabla\psi|^2}{-\psi_t} - \psi_t\right)u_t^2 + \underbrace{\frac{e^{2\psi}}{-\psi_t}|\psi_t\nabla u - u_t\nabla\psi|^2}_{T_1}\\ &= \frac{\partial}{\partial t}\left[e^{2\psi}F(u)\right] + 2e^{2\psi}(-\psi_t)F(u),\end{aligned} \tag{2.6}$$

where F is the primitive of f satisfying $F(0) = 0$, namely $F'(u) = f(u)$. Using the Schwarz inequality and (2.5), we can calculate

$$\begin{aligned}T_1 &= \frac{e^{2\psi}}{-\psi_t}(\psi_t^2|\nabla u|^2 - 2\psi_t u_t\nabla u\cdot\nabla\psi + u_t^2|\nabla\psi|^2)\\ &\geq \frac{e^{2\psi}}{-\psi_t}\left(\frac{1}{5}\psi_t^2|\nabla u|^2 - \frac{1}{4}u_t^2|\nabla\psi|^2\right)\\ &\geq e^{2\psi}\left(\frac{1}{5}(-\psi_t)|\nabla u|^2 - \frac{a(x)b(t)}{4(2+\delta)}u_t^2\right).\end{aligned}$$

From this and (2.5), we obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\frac{e^{2\psi}}{2}(u_t^2+|\nabla u|^2)\right] - \nabla\cdot(e^{2\psi}u_t\nabla u) + e^{2\psi}\left\{\left(\frac{1}{4}a(x)b(t) - \psi_t\right)u_t^2 + \frac{-\psi_t}{5}|\nabla u|^2\right\}\\ &\leq \frac{\partial}{\partial t}\left[e^{2\psi}F(u)\right] + 2e^{2\psi}(-\psi_t)F(u).\end{aligned} \tag{2.7}$$

By multiplying (2.7) by $(t_0+t)^{B+1-\varepsilon}$, here $t_0 \geq 1$ is determined later, it follows that

$$\begin{aligned}&\frac{\partial}{\partial t}\left[(t_0+t)^{B+1-\varepsilon}\frac{e^{2\psi}}{2}(u_t^2+|\nabla u|^2)\right] - (B+1-\varepsilon)(t_0+t)^{B-\varepsilon}\frac{e^{2\psi}}{2}(u_t^2+|\nabla u|^2)\\ &\quad - \nabla\cdot((t_0+t)^{B+1-\varepsilon}e^{2\psi}u_t\nabla u) + e^{2\psi}(t_0+t)^{B+1-\varepsilon}\left\{\left(\frac{1}{4}a(x)b(t) - \psi_t\right)u_t^2 + \frac{-\psi_t}{5}|\nabla u|^2\right\}\\ &\leq \frac{\partial}{\partial t}\left[(t_0+t)^{B+1-\varepsilon}e^{2\psi}F(u)\right] - (B+1-\varepsilon)(t_0+t)^{B-\varepsilon}e^{2\psi}F(u) + 2(t_0+t)^{B+1-\varepsilon}e^{2\psi}(-\psi_t)F(u).\end{aligned} \tag{2.8}$$

We put

$$E(t) := \int_{\mathbf{R}^n} e^{2\psi}(u_t^2 + |\nabla u|^2)dx, \qquad E_\psi(t) := \int_{\mathbf{R}^n} e^{2\psi}(-\psi_t)(u_t^2 + |\nabla u|^2)dx,$$
$$J(t;g) := \int_{\mathbf{R}^n} e^{2\psi} g dx, \qquad J_\psi(t;g) := \int_{\mathbf{R}^n} e^{2\psi}(-\psi_t) g dx.$$

Integrating (2.8) over the whole space, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[(t_0+t)^{B+1-\varepsilon}E(t)\right] - \frac{1}{2}(B+1-\varepsilon)(t_0+t)^{B-\varepsilon}E(t) \\
&\quad + \frac{1}{4}(t_0+t)^{B+1-\varepsilon}J(t, a(x)b(t)u_t^2) + \frac{1}{5}(t_0+t)^{B+1-\varepsilon}E_\psi(t) \\
&\le \frac{d}{dt}\left[(t_0+t)^{B+1-\varepsilon}\int e^{2\psi}F(u)dx\right] + C(t_0+t)^{B+1-\varepsilon}J_\psi(t;|u|^{p+1}) + C(t_0+t)^{B-\varepsilon}J(t;|u|^{p+1}).
\end{aligned} \tag{2.9}$$

Therefore, we integrate (2.9) on the interval $[0, t]$ and obtain the estimate for $(t_0+t)^{B+1-\varepsilon}E(t)$, which is the first term of $M(t)$:

$$\begin{aligned}
&(t_0+t)^{B+1-\varepsilon}E(t) - C\int_0^t (t_0+\tau)^{B-\varepsilon}E(\tau)d\tau + \int_0^t (t_0+\tau)^{B+1-\varepsilon}J(\tau; a(x)b(t)u_t^2) + (t_0+\tau)^{B+1-\varepsilon}E_\psi(\tau)d\tau \\
&\le CI_0^2 + C(t_0+t)^{B+1-\varepsilon}J(t;|u|^{p+1}) + C\int_0^t (t_0+\tau)^{B+1-\varepsilon}J_\psi(\tau;|u|^{p+1})d\tau \\
&\quad + C\int_0^t (t_0+t)^{B-\varepsilon}J(\tau;|u|^{p+1})d\tau.
\end{aligned} \tag{2.10}$$

In order to complete the a priori estimate, however, we have to manage the second term of the inequality above whose sign is negative, and we also have to estimate the second term of $M(t)$. The following argument, which is little more complicated, can settle both these problems.

At first, we multiply (1.1) by $e^{2\psi}u$ and have

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[e^{2\psi}\left(uu_t + \frac{a(x)b(t)}{2}u^2\right)\right] - \nabla\cdot(e^{2\psi}u\nabla u) \\
&\quad + e^{2\psi}\left\{|\nabla u|^2 + \left(-\psi_t + \frac{\beta}{2(1+t)}\right)a(x)b(t)u^2 + \underbrace{2u\nabla\psi\cdot\nabla u}_{T_2} - 2\psi_t uu_t - u_t^2\right\} \\
&= e^{2\psi}uf(u).
\end{aligned} \tag{2.11}$$

We calculate

$$\begin{aligned}
e^{2\psi}T_2 &= 4e^{2\psi}u\nabla\psi\cdot\nabla u - 2e^{2\psi}u\nabla\psi\cdot\nabla u \\
&= 4e^{2\psi}u\nabla\psi\cdot\nabla u - \nabla\cdot(e^{2\psi}u^2\nabla\psi) + 2e^{2\psi}u^2|\nabla\psi|^2 + e^{2\psi}(\Delta\psi)u^2
\end{aligned}$$

and by (2.4) we can rewrite (2.11) to

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[e^{2\psi}\left(uu_t + \frac{a(x)b(t)}{2}u^2\right)\right] - \nabla\cdot(e^{2\psi}(u\nabla u + u^2\nabla\psi)) \\
&\quad + e^{2\psi}\left\{\underbrace{|\nabla u|^2 + 4u\nabla u\cdot\nabla\psi + ((-\psi_t)a(x)b(t) + 2|\nabla\psi|^2)u^2}_{T_3}\right. \\
&\quad \left. + (B-2\delta_1)\frac{a(x)b(t)}{2(1+t)}u^2 - 2\psi_t uu_t - u_t^2\right\} \le e^{2\psi}uf(u).
\end{aligned} \tag{2.12}$$

It follows from (2.5) that

$$\begin{aligned}
T_3 &= |\nabla u|^2 + 4u\nabla u\cdot\nabla\psi + \left\{\left(1 - \frac{\delta}{3}\right)(-\psi_t)a(x)b(t) + 2|\nabla\psi|^2\right\}u^2 + \frac{\delta}{3}(-\psi_t)a(x)b(t)u^2 \\
&\ge |\nabla u|^2 + 4u\nabla u\cdot\nabla\psi + \left(4 + \frac{\delta}{3} - \frac{\delta^2}{3}\right)|\nabla\psi|^2u^2 + \frac{\delta}{3}(-\psi_t)a(x)b(t)u^2
\end{aligned}$$

$$= \left(1 - \frac{4}{4+\delta_2}\right)|\nabla u|^2 + \delta_2|\nabla\psi|^2u^2 + \left|\frac{2}{\sqrt{4+\delta_2}}\nabla u + \sqrt{4+\delta_2}u\nabla\psi\right|^2 + \frac{\delta}{3}(-\psi_t)a(x)b(t)u^2$$
$$\geq \delta_3(|\nabla u|^2 + |\nabla\psi|^2u^2) + \frac{\delta}{3}(-\psi_t)a(x)b(t)u^2,$$

where

$$\delta_2 := \frac{\delta}{6} - \frac{\delta^2}{6}, \qquad \delta_3 := \min\left(1 - \frac{4}{4+\delta_2}, \delta_2\right).$$

Thus, we obtain

$$\frac{\partial}{\partial t}\left[e^{2\psi}\left(uu_t + \frac{a(x)b(t)}{2}u^2\right)\right] - \nabla\cdot(e^{2\psi}(u\nabla u + u^2\nabla\psi)) + e^{2\psi}\delta_3|\nabla u|^2$$
$$+ e^{2\psi}\left(\delta_3|\nabla\psi|^2 + \frac{\delta}{3}(-\psi_t)a(x)b(t) + (B - 2\delta_1)\frac{a(x)b(t)}{2(1+t)}\right)u^2 + e^{2\psi}(-2\psi_t uu_t - u_t^2)$$
$$\leq e^{2\psi}uf(u). \tag{2.13}$$

Following Nishihara [30], related to the size of $1 + |x|^2$ and the size of $(1+t)^2$, we divide the space $\mathbf{R}^n$ into two different zones $\Omega(t; K, t_0)$ and $\Omega^c(t; K, t_0)$, where

$$\Omega = \Omega(t; K, t_0) := \{x \in \mathbf{R}^n; (t_0 + t)^2 \geq K + |x|^2\},$$

and $\Omega^c = \Omega^c(t; K, t_0) := \mathbf{R}^n \setminus \Omega(t; K, t_0)$ with $K \geq 1$ determined later. Since $a(x)b(t) \geq a_0(t + t_0)^{-(\alpha+\beta)}$ in the domain Ω, we multiply (2.7) by $(t_0 + t)^{\alpha+\beta}$ and obtain

$$\frac{\partial}{\partial t}\left[\frac{e^{2\psi}}{2}(t_0+t)^{\alpha+\beta}(u_t^2 + |\nabla u|^2)\right] - \nabla\cdot(e^{2\psi}(t_0+t)^{\alpha+\beta}u_t\nabla u) + e^{2\psi}\left[\left(\frac{a_0}{4} - \frac{\alpha+\beta}{2(t_0+t)^{1-\alpha-\beta}}\right)\right.$$
$$\left. + (t_0+t)^{\alpha+\beta}(-\psi_t)\right]u_t^2 + e^{2\psi}\left[\frac{-\psi_t}{5}(t_0+t)^{\alpha+\beta} - \frac{\alpha+\beta}{2(t_0+t)^{1-\alpha-\beta}}\right]|\nabla u|^2$$
$$\leq \frac{\partial}{\partial t}[(t_0+t)^{\alpha+\beta}e^{2\psi}F(u)] - \frac{\alpha+\beta}{(t_0+t)^{1-\alpha-\beta}}e^{2\psi}F(u) + 2(t_0+t)^{\alpha+\beta}e^{2\psi}(-\psi_t)F(u). \tag{2.14}$$

Let ν be a small positive number depending on δ, which will be chosen later. By (2.14) + ν(2.13), we have

$$\frac{\partial}{\partial t}\left[e^{2\psi}\left(\frac{(t_0+t)^{\alpha+\beta}}{2}u_t^2 + \nu uu_t + \frac{\nu a(x)b(t)}{2}u^2 + \frac{(t_0+t)^{\alpha+\beta}}{2}|\nabla u|^2\right)\right] - \nabla\cdot(e^{2\psi}(t_0+t)^{\alpha+\beta}u_t\nabla u$$
$$+ \nu e^{2\psi}(u\nabla u + u^2\nabla\psi)) + e^{2\psi}\left[\left(\frac{a_0}{4} - \frac{\alpha+\beta}{2(t_0+t)^{1-\alpha-\beta}} - \nu\right) + (t_0+t)^{\alpha+\beta}(-\psi_t)\right]u_t^2$$
$$+ e^{2\psi}\left[\nu\delta_3 - \frac{\alpha+\beta}{2(t_0+t)^{1-\alpha-\beta}} + \frac{-\psi_t}{5}(t_0+t)^{\alpha+\beta}\right]|\nabla u|^2$$
$$+ e^{2\psi}\nu\left[\delta_3|\nabla\psi|^2 + \frac{\delta}{3}(-\psi_t)a(x)b(t) + (B - 2\delta_1)\frac{a(x)b(t)}{2(1+t)}\right]u^2 + 2\nu e^{2\psi}(-\psi_t)uu_t$$
$$\leq \frac{\partial}{\partial t}[(t_0+t)^{\alpha+\beta}e^{2\psi}F(u)] - \frac{\alpha+\beta}{(t_0+t)^{1-\alpha-\beta}}e^{2\psi}F(u) + 2(t_0+t)^{\alpha+\beta}e^{2\psi}(-\psi_t)F(u) + \nu e^{2\psi}uf(u). \tag{2.15}$$

By the Schwarz inequality, the last term of the left hand side in the above inequality can be estimated as

$$|2\nu(-\psi_t)uu_t| \leq \frac{\nu\delta}{3}(-\psi_t)a(x)b(t)u^2 + \frac{3\nu}{a_0\delta}(-\psi_t)(t_0+t)^{\alpha+\beta}u_t^2.$$

Thus, we have

$$\frac{\partial}{\partial t}\left[e^{2\psi}\left(\frac{(t_0+t)^{\alpha+\beta}}{2}u_t^2 + \nu uu_t + \frac{\nu a(x)b(t)}{2}u^2 + \frac{(t_0+t)^{\alpha+\beta}}{2}|\nabla u|^2\right)\right] - \nabla\cdot(e^{2\psi}(t_0+t)^{\alpha+\beta}u_t\nabla u$$
$$+ \nu e^{2\psi}(u\nabla u + u^2\nabla\psi)) + e^{2\psi}\left[\left(\frac{a_0}{4} - \frac{\alpha+\beta}{2(t_0+t)^{1-\alpha-\beta}} - \nu\right) + \left(1 - \frac{3\nu}{a_0\delta}\right)(t_0+t)^{\alpha+\beta}(-\psi_t)\right]u_t^2$$
$$+ e^{2\psi}\left[\nu\delta_3 - \frac{\alpha+\beta}{2(t_0+t)^{1-\alpha-\beta}} + \frac{-\psi_t}{5}(t_0+t)^{\alpha+\beta}\right]|\nabla u|^2 + e^{2\psi}\left[\nu\left(\delta_3|\nabla\psi|^2 + (B - 2\delta_1)\frac{a(x)b(t)}{2(1+t)}\right)\right]u^2$$
$$\leq \frac{\partial}{\partial t}[(t_0+t)^{\alpha+\beta}e^{2\psi}F(u)] - \frac{\alpha+\beta}{(t_0+t)^{1-\alpha-\beta}}e^{2\psi}F(u) + 2(t_0+t)^{\alpha+\beta}e^{2\psi}(-\psi_t)F(u) + \nu e^{2\psi}uf(u). \tag{2.16}$$

Now we choose the parameters ν and t_0 such that

$$\frac{a_0}{4} - \frac{\alpha+\beta}{2(t_0+t)^{1-\alpha-\beta}} - \nu \ge c_0, \qquad 1 - \frac{3\nu}{a_0\delta} \ge c_0,$$
$$\nu\delta_3 - \frac{\alpha+\beta}{2(t_0+t)^{1-\alpha-\beta}} \ge c_0, \quad \nu\delta_3 \ge c_0, \qquad \frac{1}{5} \ge c_0,$$

hold for some constant $c_0 > 0$. This is possible because we first determine ν sufficiently small depending on δ and then we choose t_0 sufficiently large depending on ν. Therefore, integrating (2.16) on Ω, we obtain the following energy inequality:

$$\frac{d}{dt}\overline{E}_\psi(t;\Omega(t;K,t_0)) - N_1(t) - M_1(t) + H_\psi(t;\Omega(t;K,t_0)) \le P_1, \tag{2.17}$$

where

$$\begin{aligned}
\overline{E}_\psi(t;\Omega) &= \overline{E}_\psi(t;\Omega(t;K,t_0)) \\
&:= \int_\Omega e^{2\psi}\left(\frac{(t_0+t)^{\alpha+\beta}}{2}u_t^2 + \nu u u_t + \frac{\nu a(x)b(t)}{2}u^2 + \frac{(t_0+t)^{\alpha+\beta}}{2}|\nabla u|^2\right)dx, \\
N_1(t) &:= \int_{\mathbf{S}^{n-1}} e^{2\psi}\left(\frac{(t_0+t)^{\alpha+\beta}}{2}u_t^2 + \nu u u_t + \frac{\nu a(x)b(t)}{2}u^2 + \frac{(t_0+t)^{\alpha+\beta}}{2}|\nabla u|^2\right)\Bigg|_{|x|=\sqrt{(t_0+t)^2-K}} \\
&\quad \times [(t_0+t)^2-K]^{(n-1)/2}d\theta \cdot \frac{d}{dt}\sqrt{(t_0+t)^2-K}, \\
M_1(t) &:= \int_{\partial\Omega}(e^{2\psi}(t_0+t)^{\alpha+\beta}u_t\nabla u + \nu e^{2\psi}(u\nabla u + u^2\nabla\psi))\cdot\vec{n}dS, \\
H_\psi(t;\Omega) &= H_\psi(t;\Omega(t;K,t_0)) \\
&:= c_0\int_\Omega e^{2\psi}(1+(t_0+t)^{\alpha+\beta}(-\psi_t))(u_t^2+|\nabla u|^2)dx + \nu(B-2\delta_1)\int_\Omega \frac{e^{2\psi}a(x)b(t)}{2(1+t)}u^2dx, \\
P_1 &:= \frac{d}{dt}\left[(t_0+t)^{\alpha+\beta}\int_\Omega e^{2\psi}F(u)dx\right] - \int_{\mathbf{S}^{n-1}}(t_0+t)^{\alpha+\beta}e^{2\psi}F(u)\Bigg|_{|x|=\sqrt{(t_0+t)^2-K}} \\
&\quad \times [(t_0+t)^2-K]^{(n-1)/2}d\theta \cdot \frac{d}{dt}\sqrt{(t_0+t)^2-K} + C\int_\Omega e^{2\psi}(1+(t_0+t)^{\alpha+\beta}(-\psi_t))|u|^{p+1}dx.
\end{aligned}$$

Here $\vec{n}$ denotes the unit outer normal vector of $\partial\Omega$. We note that by $\nu \le a_0/4$ and

$$|\nu u u_t| \le \frac{\nu a(x)b(t)}{4}u^2 + \frac{\nu(t_0+t)^{\alpha+\beta}}{a_0}u_t^2,$$

it follows that

$$\begin{aligned}
c\int_\Omega e^{2\psi}(t_0+t)^{\alpha+\beta}(u_t^2+|\nabla u|^2)dx + c\int_\Omega e^{2\psi}a(x)b(t)u^2dx &\le \overline{E}_\psi(t;\Omega(t;K,t_0)) \\
&\le C\int_\Omega e^{2\psi}(t_0+t)^{\alpha+\beta}(u_t^2+|\nabla u|^2)dx \\
&\quad + C\int_\Omega e^{2\psi}a(x)b(t)u^2dx
\end{aligned}$$

for some constants $c > 0$ and $C > 0$.

Next, we derive an energy inequality in the domain Ω^c. We use the notation

$$\langle x\rangle_K := (K+|x|^2)^{1/2}.$$

Since $a(x)b(t) \ge a_0\langle x\rangle_K^{-(\alpha+\beta)}$ in $\Omega^c(t,;K,t_0)$, we multiply (2.7) by $\langle x\rangle_K^{\alpha+\beta}$ and obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\frac{e^{2\psi}}{2}\langle x\rangle_K^{\alpha+\beta}(u_t^2+|\nabla u|^2)\right] - \nabla\cdot(e^{2\psi}\langle x\rangle_K^{\alpha+\beta}u_t\nabla u) + e^{2\psi}\left(\frac{a_0}{4} + (-\psi_t)\langle x\rangle_K^{\alpha+\beta}\right)u_t^2 \\
&\quad + \frac{1}{5}e^{2\psi}(-\psi_t)\langle x\rangle_K^{\alpha+\beta}|\nabla u|^2 + (\alpha+\beta)e^{2\psi}\langle x\rangle_K^{\alpha+\beta-2}x\cdot u_t\nabla u \\
&\le \frac{\partial}{\partial t}[e^{2\psi}\langle x\rangle_K^{\alpha+\beta}F(u)] + 2e^{2\psi}\langle x\rangle_K^{\alpha+\beta}(-\psi_t)F(u).
\end{aligned} \tag{2.18}$$

By (2.18) + $\hat{\nu} \times$ (2.13), here $\hat{\nu}$ is a small positive parameter determined later, it follows that

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[e^{2\psi}\left(\frac{\langle x\rangle_K^{\alpha+\beta}}{2}u_t^2+\hat{\nu}uu_t+\frac{\hat{\nu}a(x)b(t)}{2}u^2+\frac{\langle x\rangle_K^{\alpha+\beta}}{2}|\nabla u|^2\right)\right]-\nabla\cdot(e^{2\psi}\langle x\rangle_K^{\alpha+\beta}u_t\nabla u\\
&\quad+\hat{\nu}e^{2\psi}(u\nabla u+u^2\nabla\psi))+e^{2\psi}\left[\frac{a_0}{4}-\hat{\nu}+(-\psi_t)\langle x\rangle_K^{\alpha+\beta}\right]u_t^2+e^{2\psi}\left[\hat{\nu}\delta_3+\frac{-\psi_t}{5}\langle x\rangle_K^{\alpha+\beta}\right]|\nabla u|^2\\
&\quad+e^{2\psi}\left[\hat{\nu}\left(\delta_3|\nabla\psi|^2+\frac{\delta}{3}(-\psi_t)a(x)b(t)+(B-2\delta_1)\frac{a(x)b(t)}{2(1+t)}\right)\right]u^2\\
&\quad+e^{2\psi}\underbrace{[(\alpha+\beta)\langle x\rangle_K^{\alpha+\beta-2}x\cdot u_t\nabla u-2\hat{\nu}\psi_t uu_t]}_{T_4}\\
&\le\frac{\partial}{\partial t}\left[e^{2\psi}\langle x\rangle_K^{\alpha+\beta}F(u)\right]+2e^{2\psi}\langle x\rangle_K^{\alpha+\beta}(-\psi_t)F(u)+\hat{\nu}e^{2\psi}uf(u). \qquad (2.19)
\end{aligned}$$

The terms T_4 can be estimated as

$$\begin{aligned}
|(\alpha+\beta)\langle x\rangle_K^{\alpha+\beta-2}x\cdot u_t\nabla u| &\le \frac{\hat{\nu}\delta_3}{2}|\nabla u|^2+\frac{(\alpha+\beta)^2}{2\hat{\nu}\delta_3K^{2(1-\alpha-\beta)}}u_t^2,\\
|2\hat{\nu}(-\psi_t)uu_t| &\le \frac{\hat{\nu}\delta}{3}(-\psi_t)a(x)b(t)u^2+\frac{3\hat{\nu}}{a_0\delta}(-\psi_t)\langle x\rangle_K^{\alpha+\beta}u_t^2.
\end{aligned}$$

From this we can rewrite (2.19) as

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[e^{2\psi}\left(\frac{\langle x\rangle_K^{\alpha+\beta}}{2}u_t^2+\hat{\nu}uu_t+\frac{\hat{\nu}a(x)b(t)}{2}u^2+\frac{\langle x\rangle_K^{\alpha+\beta}}{2}|\nabla u|^2\right)\right]-\nabla\cdot(e^{2\psi}\langle x\rangle_K^{\alpha+\beta}u_t\nabla u\\
&\quad+\hat{\nu}e^{2\psi}(u\nabla u+u^2\nabla\psi))+e^{2\psi}\left[\left(\frac{a_0}{4}-\hat{\nu}-\frac{(\alpha+\beta)^2}{2\hat{\nu}\delta_3K^{2(1-\alpha-\beta)}}\right)+\left(1-\frac{3\hat{\nu}}{a_0\delta}\right)(-\psi_t)\langle x\rangle_K^{\alpha+\beta}\right]u_t^2\\
&\quad+e^{2\psi}\left[\frac{\hat{\nu}\delta_3}{2}+\frac{-\psi_t}{5}\langle x\rangle_K^{\alpha+\beta}\right]|\nabla u|^2+e^{2\psi}\left[\hat{\nu}\left(\delta_3|\nabla\psi|^2+(B-2\delta_1)\frac{a(x)b(t)}{2(1+t)}\right)\right]u^2\\
&\le\frac{\partial}{\partial t}\left[e^{2\psi}\langle x\rangle_K^{\alpha+\beta}F(u)\right]+2e^{2\psi}\langle x\rangle_K^{\alpha+\beta}(-\psi_t)F(u)+\hat{\nu}e^{2\psi}uf(u). \qquad (2.20)
\end{aligned}$$

Now we choose the parameters $\hat{\nu}$ and K in the same manner as before. Indeed taking $\hat{\nu}$ sufficiently small depending on δ and then choosing K sufficiently large depending on $\hat{\nu}$, we can obtain

$$\frac{a_0}{4}-\hat{\nu}-\frac{(\alpha+\beta)^2}{2\hat{\nu}\delta_3K^{2(1-\alpha-\beta)}}\ge c_1,\qquad 1-\frac{3\hat{\nu}}{a_0\delta}\ge c_1,\qquad \nu\delta_3\ge c_1,\qquad \frac{1}{5}\ge c_1$$

for some constant $c_1>0$. Consequently, By integrating (2.20) on Ω^c, the energy inequality on Ω^c follows:

$$\frac{d}{dt}\overline{E}_\psi(t;\Omega^c(t;K,t_0))+N_2(t)+M_2(t)+H_\psi(t;\Omega^c(t;K,t_0))\le P_2, \qquad (2.21)$$

where

$$\begin{aligned}
\overline{E}_\psi(t;\Omega^c) &= \overline{E}_\psi(t;\Omega^c(t;K,t_0))\\
&:= \int_{\Omega^c}e^{2\psi}\left(\frac{\langle x\rangle_K^{\alpha+\beta}}{2}u_t^2+\hat{\nu}uu_t+\frac{\hat{\nu}a(x)b(t)}{2}u^2+\frac{\langle x\rangle_K^{\alpha+\beta}}{2}|\nabla u|^2\right)dx,\\
N_2(t) &:= \int_{\mathbf{S}^{n-1}}e^{2\psi}\left(\frac{\langle x\rangle_K^{\alpha+\beta}}{2}u_t^2+\hat{\nu}uu_t+\frac{\hat{\nu}a(x)b(t)}{2}u^2+\frac{\langle x\rangle_K^{\alpha+\beta}}{2}|\nabla u|^2\right)\Bigg|_{|x|=\sqrt{(t_0+t)^2-K}}\\
&\quad\times[(t_0+t)^2-K]^{(n-1)/2}d\theta\cdot\frac{d}{dt}\sqrt{(t_0+t)^2-K},\\
M_2(t) &:= \int_{\partial\Omega^c}(e^{2\psi}\langle x\rangle_K^{\alpha+\beta}u_t\nabla u+\hat{\nu}e^{2\psi}(u\nabla u+u^2\nabla\psi))\cdot\vec{n}dS,\\
H_\psi(t;\Omega^c) &= H_\psi(t;\Omega^c(t;K,t_0))\\
&:= c_1\int_\Omega e^{2\psi}(1+\langle x\rangle_K^{\alpha+\beta}(-\psi_t))(u_t^2+|\nabla u|^2)dx+\hat{\nu}(B-2\delta_1)\int_{\Omega^c}\frac{e^{2\psi}a(x)b(t)}{2(1+t)}u^2dx,
\end{aligned}$$

$$P_2 := \frac{d}{dt}\left[\int_{\Omega^c} e^{2\psi}\langle x\rangle_K^{\alpha+\beta}F(u)dx\right] + \int_{\mathbf{S}^{n-1}} \langle x\rangle_K^{\alpha+\beta} e^{2\psi}F(u)\Big|_{|x|=\sqrt{(t_0+t)^2-K}} \times [(t_0+t)^2-K]^{(n-1)/2}d\theta \cdot \frac{d}{dt}\sqrt{(t_0+t)^2-K} + C\int_{\Omega^c} e^{2\psi}(1+\langle x\rangle_K^{\alpha+\beta}(-\psi_t))|u|^{p+1}dx.$$

In a similar way as for the case in Ω, we note that

$$c\int_{\Omega^c} e^{2\psi}(t_0+t)^{\alpha+\beta}(u_t^2+|\nabla u|^2)dx + c\int_{\Omega^c} e^{2\psi}a(x)b(t)u^2dx \le \overline{E}_\psi(t;\Omega^c(t;K,t_0)) \le C\int_{\Omega^c} e^{2\psi}(t_0+t)^{\alpha+\beta}(u_t^2+|\nabla u|^2)dx + C\int_{\Omega^c} e^{2\psi}a(x)b(t)u^2dx$$

for some constants $c > 0$ and $C > 0$.

We add the energy inequalities on Ω and Ω^c. We note that replacing ν and $\hat{\nu}$ by $\nu_0 := \min\{\nu, \hat{\nu}\}$, we can still have the inequalities (2.17) and (2.21), provided that we retake t_0 and K larger.

By ((2.17) + (2.21)) × $(t_0+t)^{B-\varepsilon}$, we have

$$\frac{d}{dt}[(t_0+t)^{B-\varepsilon}(\overline{E}_\psi(t;\Omega)+\overline{E}_\psi(t;\Omega^c))] - \underbrace{(B-\varepsilon)(t_0+t)^{B-1-\varepsilon}(\overline{E}_\psi(t;\Omega)+\overline{E}_\psi(t;\Omega^c))}_{T_5} + \underbrace{(t_0+t)^{B-\varepsilon}(H_\psi(t;\Omega)+H_\psi(t;\Omega^c))}_{T_6} \le (t_0+t)^{B-\varepsilon}(P_1+P_2), \tag{2.22}$$

here we note that

$$N_1(t) = N_2(t), \qquad M_1(t) = M_2(t)$$

on $\partial\Omega$. Since

$$|\nu_0 uu_t| \le \frac{\nu_0\delta_4}{2}a(x)b(t)u^2 + \frac{\nu_0}{2\delta_4 a_0}(t_0+t)^{\alpha+\beta}u_t^2$$

on Ω and

$$|\nu_0 uu_t| \le \frac{\nu_0\delta_4}{2}a(x)b(t)u^2 + \frac{\nu_0}{2\delta_4 a_0}\langle x\rangle_K^{\alpha+\beta}u_t^2$$

on Ω^c, we have

$$-T_5 + T_6 \ge (t_0+t)^{B-\varepsilon}I_1 + (t_0+t)^{B-\varepsilon}I_2, \tag{2.23}$$

where

$$\begin{aligned} I_1 := &\int_\Omega e^{2\psi}\left\{\frac{c_0}{2}(1+(t_0+t)^{\alpha+\beta}(-\psi_t)) - \frac{B-\varepsilon}{2(t_0+t)}\left(1+\frac{2\nu_0}{\delta_4 a_0}\right)(t_0+t)^{\alpha+\beta}\right\}u_t^2 \\ &+ e^{2\psi}\left\{\frac{c_0}{2}(1+(t_0+t)^{\alpha+\beta}(-\psi_t)) - \frac{B-\varepsilon}{2(t_0+t)}(t_0+t)^{\alpha+\beta}\right\}|\nabla u|^2dx \\ &+ \int_{\Omega^c} e^{2\psi}\left\{\frac{c_1}{2}(1+\langle x\rangle_K^{\alpha+\beta}(-\psi_t)) - \frac{B-\varepsilon}{2(t_0+t)}\left(1+\frac{2\nu_0}{\delta_4 a_0}\right)\langle x\rangle_K^{\alpha+\beta}\right\}u_t^2 \\ &+ e^{2\psi}\left\{\frac{c_1}{2}(1+\langle x\rangle_K^{\alpha+\beta}(-\psi_t)) - \frac{B-\varepsilon}{2(t_0+t)}\langle x\rangle_K^{\alpha+\beta}\right\}|\nabla u|^2dx \\ =:\ & I_{11} + I_{12}, \end{aligned}$$

$$I_2 := \nu_0(B-2\delta_1-(1+\delta_4)(B-\varepsilon))\left(\int_\Omega + \int_{\Omega^c}\right)e^{2\psi}\frac{a(x)b(t)}{2(1+t)}u^2dx + \frac{c_2}{2}\int_{\mathbf{R}^n} e^{2\psi}(u_t^2+|\nabla u|^2)dx,$$

where $c_2 := \min(c_0, c_1)$. Recall the definition of ε and δ_1 (i.e. (1.11) and (2.4)). A simple calculation shows $\varepsilon = 3\delta_1$. Choosing δ_4 sufficiently small depending on ε, we have

$$(t_0+t)^{B-\varepsilon}I_2 \ge c_3(t_0+t)^{B-1-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi}a(x)b(t)u^2dx + \frac{c_2}{2}(t_0+t)^{B-\varepsilon}E(t)$$

for some constant $c_3 > 0$. Next, we prove that $I_1 \ge 0$. By noting that $\alpha+\beta < 1$, it is easy to see that $I_{11} \ge 0$ if we retake t_0 larger depending on c_0, ν_0 and δ_4. To estimate I_{12}, we further divide the region Ω^c into

$$\Omega^c(t;K,t_0) = (\Omega^c(t;K,t_0)\cap\Sigma_L) \cup (\Omega^c(t;K,t_0)\cap\Sigma_L^c),$$

where

$$\Sigma_L := \{x \in \mathbf{R}^n; \langle x\rangle^{2-\alpha} \le L(1+t)^{1+\beta}\}, \qquad \Sigma_L^c := \mathbf{R}^n \setminus \Sigma_L$$

with $L \gg 1$ determined later. First, since $K + |x|^2 \le K(1+|x|^2) \le KL^{2/(2-\alpha)}(1+t)^{2(1+\beta)/(2-\alpha)}$ on $\Omega^c \cap \Sigma_L$, we have

$$\begin{aligned}&\frac{c_1}{2}(1+\langle x\rangle_K^{\alpha+\beta}(-\psi_t)) - \frac{B-\varepsilon}{2(t_0+t)}\left(1+\frac{2\nu_0}{\delta_4 a_0}\right)\langle x\rangle_K^{\alpha+\beta}\\ &\quad\ge \frac{c_1}{2} - \frac{B-\varepsilon}{2(t_0+t)}\left(1+\frac{2\nu_0}{\delta_4 a_0}\right)K^{(\alpha+\beta)/2}L^{(\alpha+\beta)/(2-\alpha)}(1+t)^{\frac{(1+\beta)(\alpha+\beta)}{2-\alpha}}.\end{aligned}$$

We note that $-1 + \frac{(1+\beta)(\alpha+\beta)}{2-\alpha} < 0$ by $\alpha+\beta < 1$. Thus, we obtain

$$\frac{c_1}{2} - \frac{B-\varepsilon}{2(t_0+t)}\left(1+\frac{2\nu_0}{\delta_4 a_0}\right)K^{(\alpha+\beta)/2}L^{(\alpha+\beta)/(2-\alpha)}(1+t)^{\frac{(1+\beta)(\alpha+\beta)}{2-\alpha}} \ge 0$$

for large t_0 depending on L and K. Secondly, on $\Omega^c \cap \Sigma_L^c$, we have

$$\begin{aligned}&\frac{c_1}{2}(1+\langle x\rangle_K^{\alpha+\beta}(-\psi_t)) - \frac{B-\varepsilon}{2(t_0+t)}\left(1+\frac{2\nu_0}{\delta_4 a_0}\right)\langle x\rangle_K^{\alpha+\beta}\\ &\quad\ge \left\{\frac{c_1}{2}(1+\beta)\frac{\langle x\rangle^{2-\alpha}}{(1+t)^{2+\beta}} - \frac{B-\varepsilon}{2(t_0+t)}\left(1+\frac{2\nu_0}{\delta_4 a_0}\right)\right\}\langle x\rangle_K^{\alpha+\beta}\\ &\quad\ge \left\{\frac{c_1}{2}(1+\beta)\frac{L}{1+t} - \frac{B-\varepsilon}{2(t_0+t)}\left(1+\frac{2\nu_0}{\delta_4 a_0}\right)\right\}\langle x\rangle_K^{\alpha+\beta}.\end{aligned}$$

Therefore one can obtain $I_{12} \ge 0$, provided that $L \ge \frac{B-\varepsilon}{c_1(1+\beta)}(1+\frac{2\nu_0}{\delta_4 a_0})$. Consequently, we have $I_1 \ge 0$. By (2.23) and what we mentioned above, it follows that

$$-T_5 + T_6 \ge c_3(t_0+t)^{B-1-\varepsilon}\int_{\mathbf{R}^n} e^{2\psi}a(x)b(t)u^2dx + \frac{c_2}{2}(t_0+t)^{B-\varepsilon}E(t).$$

Therefore, we have

$$\begin{aligned}&\frac{d}{dt}[(t_0+t)^{B-\varepsilon}(\overline{E}_\psi(t;\Omega^c)+\overline{E}_\psi(t;\Omega^c))] + \frac{c_2}{2}(t_0+t)^{B-\varepsilon}E(t) + c_3(t_0+t)^{B-1-\varepsilon}J(t;a(x)b(t)u^2)\\ &\quad\le (t_0+t)^{B-\varepsilon}(P_1+P_2).\end{aligned} \tag{2.24}$$

Integrating (2.24) on the interval $[0,t]$, one can obtain the energy inequality on the whole space:

$$\begin{aligned}&(t_0+t)^{B-\varepsilon}(\overline{E}_\psi(t;\Omega)+\overline{E}_\psi(t;\Omega^c)) + \frac{c_2}{2}\int_0^t (t_0+\tau)^{B-\varepsilon}E(\tau)d\tau + c_3\int_0^t (t_0+\tau)^{B-1-\varepsilon}J(\tau;a(x)b(\tau)u^2)d\tau\\ &\quad\le CI_0^2 + \int_0^t (t_0+\tau)^{B-\varepsilon}(P_1+P_2)d\tau.\end{aligned} \tag{2.25}$$

By (2.25) $+\,\mu \times$ (2.10), here μ is a small positive parameter determined later, it follows that

$$\begin{aligned}&(t_0+t)^{B-\varepsilon}\overline{E}_\psi(t;\Omega) + (t_0+t)^{B-\varepsilon}\overline{E}_\psi(t;\Omega^c) + \int_0^t \frac{c_2}{2}(t_0+\tau)^{B-\varepsilon}E(\tau) - \mu C(t_0+\tau)^{B-\varepsilon}E(\tau)d\tau\\ &\qquad + c_3\int_0^t (t_0+\tau)^{B-1-\varepsilon}J(\tau;a(x)b(\tau)u^2)d\tau + \mu(t_0+t)^{B+1-\varepsilon}E(t)\\ &\qquad + \mu\int_0^t (t_0+\tau)^{B+1-\varepsilon}J(\tau;a(x)b(\tau)u_t^2) + (t_0+\tau)^{B+1-\varepsilon}E_\psi(\tau)d\tau\\ &\quad\le CI_0^2 + P + C(t_0+t)^{B+1-\varepsilon}J(t;|u|^{p+1}) + C\int_0^t (t_0+\tau)^{B+1-\varepsilon}J_\psi(\tau;|u|^{p+1})d\tau\\ &\qquad + C\int_0^t (t_0+\tau)^{B-\varepsilon}J(\tau;|u|^{p+1})d\tau,\end{aligned} \tag{2.26}$$

where

$$P = \int_0^t (t_0+\tau)^{B-\varepsilon}(P_1+P_2)d\tau.$$

Now we choose μ sufficiently small; then we can rewrite (2.26) as

$$
\begin{aligned}
(t_0+t)^{B+1-\varepsilon}E(t)+(t_0+t)^{B-\varepsilon}J(t;a(x)b(t)u^2) \le\ & CI_0^2+P+C(t_0+t)^{B+1-\varepsilon}J(t;|u|^{p+1})\\
&+C\int_0^t (t_0+\tau)^{B+1-\varepsilon}J_\psi(\tau;|u|^{p+1})d\tau\\
&+C\int_0^t (t_0+\tau)^{B-\varepsilon}J(\tau;|u|^{p+1})d\tau. \qquad (2.27)
\end{aligned}
$$

We shall estimate the right hand side of (2.27). We need the following lemma.

Lemma 2.2 (*Gagliardo–Nirenberg*). *Let $p,q,r(1\le p,q,r\le\infty)$ and $\sigma\in[0,1]$ satisfy*

$$\frac{1}{p}=\sigma\left(\frac{1}{r}-\frac{1}{n}\right)+(1-\sigma)\frac{1}{q}$$

except for $p=\infty$ or $r=n$ when $n\ge 2$. Then for some constant $C=C(p,q,r,n)>0$, the inequality

$$\|h\|_{L^p}\le C\|h\|_{L^q}^{1-\sigma}\|\nabla h\|_{L^r}^{\sigma},\quad \textit{for any } h\in C_0^1(\mathbf{R}^n)$$

holds.

We first estimate $(t_0+t)^{B+1-\varepsilon}J(t;|u|^{p+1})$. From the above lemma, we have

$$J(t;|u|^{p+1})\le C\left(\int_{\mathbf{R}^n} e^{\frac{4}{p+1}\psi}u^2dx\right)^{(1-\sigma)(p+1)/2}\times\left(\int_{\mathbf{R}^n} e^{\frac{4}{p+1}\psi}|\nabla\psi|^2u^2dx+\int_{\mathbf{R}^n} e^{\frac{4}{p+1}\psi}|\nabla u|^2dx\right)^{\sigma(p+1)/2} \qquad (2.28)$$

with $\sigma=\frac{n(p-1)}{2(p+1)}$. Since

$$
\begin{aligned}
e^{\frac{4}{p+1}\psi}u^2 &= (e^{2\psi}a(x)b(t)u^2)a(x)^{-1}b(t)^{-1}e^{\left(\frac{4}{p+1}-2\right)\psi}\\
&\le C(e^{2\psi}a(x)b(t)u^2)\left[\left(\frac{\langle x\rangle^{2-\alpha}}{(1+t)^{1+\beta}}\right)^{\frac{\alpha}{2-\alpha}}e^{\left(\frac{4}{p+1}-2\right)\psi}\right]\times(1+t)^{\beta+(1+\beta)\alpha/(2-\alpha)}\\
&\le C(1+t)^{\beta+(1+\beta)\alpha/(2-\alpha)}e^{2\psi}a(x)b(t)u^2
\end{aligned}
$$

and

$$
\begin{aligned}
e^{\frac{4}{(p+1)}\psi}|\nabla\psi|^2u^2 &\le C\frac{\langle x\rangle^{2-2\alpha}}{(1+t)^{2+2\beta}}e^{\frac{1}{2}\left(\frac{4}{p+1}-2\right)\psi}e^{\frac{1}{2}\left(\frac{4}{p+1}-2\right)\psi}e^{2\psi}u^2\\
&\le Ce^{\frac{1}{2}\left(\frac{4}{p+1}-2\right)\psi}e^{2\psi}\left[\left(\frac{\langle x\rangle^{2-\alpha}}{(1+t)^{1+\beta}}\right)^{\frac{2-2\alpha}{2-\alpha}}e^{\frac{1}{2}\left(\frac{4}{p+1}-2\right)\psi}\right]\times(1+t)^{-2(1+\beta)+(1+\beta)(2-2\alpha)/(2-\alpha)}u^2\\
&\le C(1+t)^{-2(1+\beta)/(2-\alpha)}e^{\frac{1}{2}\left(\frac{4}{p+1}-2\right)\psi}e^{2\psi}u^2\\
&\le C(1+t)^{-2(1+\beta)/(2-\alpha)}(1+t)^{\beta+(1+\beta)\alpha/(2-\alpha)}e^{2\psi}a(x)b(t)u^2,
\end{aligned}
$$

we can estimate (2.28) as

$$
\begin{aligned}
J(t;|u|^{p+1}) \le\ & C(1+t)^{[\beta+(1+\beta)\alpha/(2-\alpha)](1-\sigma)(p+1)/2}J(t;a(x)b(t)u^2)^{(1-\sigma)(p+1)/2}\\
&\times[(1+t)^{-1}J(t;a(x)b(t)u^2)+E(t)]^{\sigma(p+1)/2}
\end{aligned}
$$

and hence

$$(t_0+t)^{B+1-\varepsilon}J(t;|u|^{p+1})\le C\left((t_0+t)^{\gamma_1}M(t)^{(p+1)/2}+(t_0+t)^{\gamma_2}M(t)^{(p+1)/2}\right),$$

where

$$
\begin{aligned}
\gamma_1 &= B+1-\varepsilon+\left[\beta+(1+\beta)\frac{\alpha}{2-\alpha}\right]\frac{1-\sigma}{2}(p+1)-\frac{\sigma}{2}(p+1)-(B-\varepsilon)\frac{p+1}{2},\\
\gamma_2 &= B+1-\varepsilon+\left[\beta+(1+\beta)\frac{\alpha}{2-\alpha}\right]\frac{1-\sigma}{2}(p+1)-(B-\varepsilon)\frac{1-\sigma}{2}(p+1)-(B+1-\varepsilon)\frac{\sigma}{2}(p+1).
\end{aligned}
$$

By a simple calculation it follows that if

$$p>1+\frac{2}{n-\alpha},$$

then by taking ε sufficiently small (i.e. δ sufficiently small) both γ_1 and γ_2 are negative. We note that

$$J_\psi(t;|u|^{p+1}) = \int_{\mathbf{R}^n} e^{2\psi}(-\psi_t)|u|^{p+1}dx \le \frac{C}{1+t}\int_{\mathbf{R}^n} e^{(2+\rho)\psi}|u|^{p+1}dx,$$

where ρ is a sufficiently small positive number. Therefore, we can estimate the terms

$$\int_0^t (t_0+\tau)^{B+1-\varepsilon}J_\psi(\tau;|u|^{p+1})d\tau \quad \text{and} \quad \int_0^t (t_0+\tau)^{B-\varepsilon}J(\tau;|u|^{p+1})d\tau$$

in the same manner as before. Noting that

$$P_1+P_2 = \frac{d}{dt}\left[(t_0+t)^{\alpha+\beta}\int_\Omega e^{2\psi}F(u)dx + \int_{\Omega^c} e^{2\psi}\langle x\rangle_K^{\alpha+\beta}F(u)dx\right] + C\int_\Omega e^{2\psi}(1+(t_0+t)^{\alpha+\beta}(-\psi_t))|u|^{p+1}dx + C\int_{\Omega^c} e^{2\psi}(1+\langle x\rangle_K^{\alpha+\beta}(-\psi_t))|u|^{p+1}dx,$$

we have

$$\begin{aligned} P &= \int_0^t (t_0+\tau)^{B-\varepsilon}(P_1+P_2)d\tau \\ &\le CI_0^2 + C(t_0+t)^{B-\varepsilon}\int_\Omega e^{2\psi}(t_0+t)^{\alpha+\beta}F(u)dx + C(t_0+t)^{B-\varepsilon}\int_{\Omega^c} e^{2\psi}\langle x\rangle_K^{\alpha+\beta}F(u)dx \\ &\quad + C\int_0^t (t_0+\tau)^{B-1-\varepsilon}\int_\Omega e^{2\psi}(t_0+\tau)^{\alpha+\beta}F(u)dxd\tau + C\int_0^t (t_0+\tau)^{B-1-\varepsilon}\int_{\Omega^c} e^{2\psi}\langle x\rangle_K^{\alpha+\beta}F(u)dxd\tau \\ &\quad + C\int_0^t (t_0+\tau)^{B-\varepsilon}\int_\Omega e^{2\psi}(1+(t_0+\tau)^{\alpha+\beta}(-\psi_t))|u|^{p+1}dxd\tau \\ &\quad + C\int_0^t (t_0+\tau)^{B-\varepsilon}\int_{\Omega^c} e^{2\psi}(1+\langle x\rangle_K^{\alpha+\beta}(-\psi_t))|u|^{p+1}dxd\tau. \end{aligned}$$

We calculate

$$\begin{aligned} e^{2\psi}\langle x\rangle_K^{\alpha+\beta} &= e^{2A\frac{\langle x\rangle^{2-\alpha}}{(1+t)^{1+\beta}}}\langle x\rangle_K^{\alpha+\beta} \\ &\le Ce^{2A\frac{\langle x\rangle^{2-\alpha}}{(1+t)^{1+\beta}}}\left(\frac{\langle x\rangle^{2-\alpha}}{(1+t)^{1+\beta}}\right)^{\frac{\alpha+\beta}{2-\alpha}}(1+t)^{\frac{(\alpha+\beta)(1+\beta)}{2-\alpha}} \\ &\le Ce^{(2+\rho)\psi}(1+t)^{\frac{(\alpha+\beta)(1+\beta)}{2-\alpha}} \end{aligned}$$

for small $\rho > 0$. Noting that $\frac{(\alpha+\beta)(1+\beta)}{2-\alpha} < 1$ and taking ρ sufficiently small, we can estimate the terms P in the same manner as estimating $(t_0+t)^{B+1-\varepsilon}J(t;|u|^{p+1})$. Consequently, we have a priori estimate for $M(t)$:

$$M(t) \le CI_0^2 + CM(t)^{(p+1)/2}.$$

This shows that the local solution of (1.1) can be extended globally. We note that

$$e^{2\psi}a(x)b(t) \ge c(1+t)^{-(1+\beta)\frac{\alpha}{2-\alpha}-\beta}$$

with some constant $c > 0$. Then we have

$$\int_{\mathbf{R}^n} e^{2\psi}a(x)b(t)u^2dx \ge c(1+t)^{-(1+\beta)\frac{\alpha}{2-\alpha}-\beta}\int_{\mathbf{R}^n} u^2dx. \tag{2.29}$$

This implies the decay estimate of global solution (1.10) and completes the proof of Theorem 1.1.

Proof of Corollary 1.4. In a similar way to derive (2.29), we have

$$\int_{\mathbf{R}^n} e^{2\psi}a(x)b(t)u^2dx \ge c(1+t)^{-(1+\beta)\frac{\alpha}{2-\alpha}-\beta}\int_{\mathbf{R}^n} e^{(2A-\mu)\frac{\langle x\rangle^{2-\alpha}}{(1+t)^{\beta}}}u^2dx.$$

By noting that

$$\frac{\langle x\rangle^{2-\alpha}}{(1+t)^{1+\beta}} \ge (1+t)^\rho$$

on $\Omega_\rho(t)$ and Theorem 1.1, it follows that

$$\begin{aligned}
&(1+t)^{-(1+\beta)\frac{\alpha}{2-\alpha}-\beta}\int_{\Omega_\rho(t)} e^{(2A-\mu)(1+t)^\rho}(u_t^2+|\nabla u|^2+u^2)dx \\
&\le C(1+t)^{-(1+\beta)\frac{\alpha}{2-\alpha}-\beta}\int_{\Omega_\rho(t)} e^{(2A-\mu)\frac{\langle x\rangle^{2-\alpha}}{(1+t)^\beta}}(u_t^2+|\nabla u|^2+u^2)dx \\
&\le C\int_{\mathbf{R}^n} e^{2\psi}(u_t^2+|\nabla u|^2+a(x)b(t)u^2)dx \\
&\le C(1+t)^{-B+\varepsilon}.
\end{aligned}$$

Thus, we obtain

$$\int_{\Omega_\rho(t)}(u_t^2+|\nabla u|^2+u^2)dx \le C(1+t)^{-\frac{(1+\beta)(n-2\alpha)}{2-\alpha}+\varepsilon}e^{-(2A-\mu)(1+t)^\rho}.$$

This proves Corollary 1.4. □

Acknowledgments

The author is deeply grateful to Professors Ryo Ikehata, Kenji Nishihara, Tatsuo Nishitani, Akitaka Matsumura and Michael Reissig. They gave me constructive comments and warm encouragement again and again.

References

[1] F. John, Blow-up of solutions of nonlinear wave equations in three space dimensions, Manuscripta Math. 28 (1979) 235–268.
[2] R. Glassey, Existence in the large $\square u = F(u)$ in two dimensions, Math. Z. 178 (1981) 233–261.
[3] T. Sideris, Nonexistence of global solutions to semilinear wave equations in high dimensions, J. Differential Equations 52 (1984) 378–406.
[4] W. Strauss, Nonlinear scattering theory at low energy, J. Funct. Anal. 41 (1981) 110–133.
[5] W. Strauss, Nonlinear Wave Equations, in: C.B.M.S. Lecture Notes, vol. 73, Amer. Math. Soc., Providence, RI, 1989.
[6] Y. Zhou, Cauchy problem for semilinear wave equations in four space dimensions with small initial data, J. Partial Differential Equations 8 (1995) 135–144.
[7] H. Lindblad, C. Sogge, Long-time existence for small amplitude semilinear wave equations, Amer. J. Math. 118 (1996) 1047–1135.
[8] V. Georgiev, H. Lindblad, C. Sogge, Weighted Strichartz estimates and global existence for semilinear wave equation, Amer. J. Math. 119 (1997) 1291–1319.
[9] M. Rammaha, Nonlinear wave equations in high dimensions, Differ. Equ. Appl. (1988) 322–326. Ohio University.
[10] D. Tataru, Strichartz estimates in the hyperbolic space and global existence for the semilinear wave equation, Trans. Amer. Math. Soc. 353 (2001) 795–807.
[11] A. Matsumura, On the asymptotic behavior of solutions of semi-linear wave equations, Publ. Res. Inst. Math. Sci. 12 (1976) 169–189.
[12] K. Nishihara, $L^p - L^q$ estimates of solutions to the damped wave equation in 3-dimensional space and their application, Math. Z. 244 (2003) 631–649.
[13] K. Mochizuki, Scattering theory for wave equations with dissipative terms, Publ. Res. Inst. Math. Sci. 12 (1976–1977) 383–390.
[14] J. Wirth, Wave equations with time-dependent dissipation. I. Non-effective dissipation, J. Differential Equations 222 (2006) 487–514.
[15] J. Wirth, Wave equations with time-dependent dissipation. II. Effective dissipation, J. Differential Equations 232 (2007) 74–103.
[16] G. Todorova, B. Yordanov, Weighted L^2-estimates of dissipative wave equations with variable coefficients, J. Differential Equations 246 (2009) 4497–4518.
[17] J.S. Kenigson, J.J. Kenigson, Energy decay estimates for the dissipative wave equation with space–time dependent potential, Math. Methods Appl. Sci. 34 (1) (2011) 48–62.
[18] R. Ikehata, G. Todorova, B. Yordanov, Optimal decay rate of the energy for wave equations with critical potential, J. Math. Soc. Japan (in press).
[19] S. Kawashima, M. Nakao, K. Ono, On the decay property of solutions to the Cauchy problem of the semilinear wave equation with a dissipative term, J. Math. Soc. Japan 47 (1995) 617–653.
[20] R. Karch, Selfsimilar profiles in large time asymptotics of solutions to damped wave equations, Studia Math. 143 (2000) 175–197.
[21] N. Hayashi, E.I. Kaikina, P.I. Naumkin, Asymptotics for nonlinear damped wave equations with large initial data, Sib. Élektron. Mat. Izv. 4 (2007) 249–277.
[22] R. Ikehata, K. Nishihara, H. Zhao, Global asymptotics of solutions to the Cauchy problem for the damped wave equation with absorption, J. Differential Equations 226 (2006) 1–29.
[23] K. Nishihara, Global asymptotics for the damped wave equation with absorption in higher dimensional space, J. Math. Soc. Japan 58 (2006) 805–836.
[24] K. Nishihara, H. Zhao, Decay properties of solutions to the Cauchy problem for the damped wave equation with absorption, J. Math. Anal. Appl. 313 (2006) 598–610.
[25] N. Hayashi, E.I. Kaikina, P.I. Naumkin, Damped wave equation in the subcritical case, J. Differential Equations 207 (2004) 161–194.
[26] N. Hayashi, E.I. Kaikina, P.I. Naumkin, Damped wave equation with a critical nonlinearity, Trans. Amer. Math. Soc. 358 (2006) 1165–1185.
[27] N. Hayashi, E.I. Kaikina, P.I. Naumkin, On the critical nonlinear damped wave equation with large initial data, J. Math. Anal. Appl. 334 (2007) 1400–1425.
[28] K. Nishihara, Decay properties for the damped wave equation with space dependent potential and absorbed semilinear term, Comm. Partial Differential Equations 35 (2010) 1402–1418.
[29] K. Nishihara, J. Zhai, Asymptotic behaviors of solutions for time dependent damped wave equations, J. Math. Anal. Appl. 360 (2009) 412–421.
[30] J. Lin, K. Nishihara, J. Zhai, Decay property of solutions for damped wave equations with space–time dependent damping term, J. Math. Anal. Appl. 374 (2011) 602–614.
[31] J. Lin, K. Nishihara, J. Zhai, Critical exponent for the semilinear wave equation with time-dependent damping, Preprint.
[32] G. Todorova, B. Yordanov, Critical exponent for a nonlinear wave equation with damping, C. R. Acad. Sci. Paris I 330 (2000) 557–562.
[33] G. Todorova, B. Yordanov, Critical exponent for a nonlinear wave equation with damping, J. Differential Equations 174 (2001) 464–489.
[34] T.T. Li, Y. Zhou, Breakdown of solutions to $\square u + u_t = |u|^{1+\alpha}$, Discrete Contin. Dyn. Syst. 1 (1995) 503–520.
[35] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. I 13 (1966) 109–124.
[36] Qi S. Zhang, A blow-up result for a nonlinear wave equation with damping: the critical case, C. R. Acad. Sci. Paris I 333 (2001) 109–114.
[37] R. Ikehata, K. Tanizawa, Global existence of solutions for semilinear damped wave equations in $\mathbf{R}^N$ with noncompactly supported initial data, Nonlinear Anal. 61 (2005) 1189–1208.
[38] R. Ikehata, G. Todorova, B. Yordanov, Critical exponent for semilinear wave equations with space-dependent potential, Funkcial. Ekvac. 52 (2009) 411–435.
[39] K. Nishihara, Asymptotic behavior of solutions to the semilinear wave equation with time-dependent damping, Tokyo J. Math. 34 (2011) 327–343.