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On an almost free damping vibration equation using N-fractional calculus

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Abstract

In this paper, an almost free damping vibration equation is discussed by means of N-fractional calculus. Let $\varphi \in \wp^{\circ} = \{\varphi, 0 \neq |\varphi_v| < \infty, v \in \mathbb{R}\}$. We focus on the following type of equation:

$$\varphi_{m+\varepsilon} + \varphi_{1+(\varepsilon/m)} \cdot a + \varphi \cdot b = 0,$$

where m is an integer and $ab \neq 0$, $a^2 < 4b$, $a > 0$, $\varphi = \varphi(t)$, $|\varepsilon| < 1$, $\varepsilon, t \in \mathbb{R}$. In the case of $m = 2$ and $|\varepsilon| \ll 1$, we call this equation as an almost free damping vibration equation. So the solutions are investigated to be given using N-fractional calculus in the case of $m = 2$. Furthermore, we illustrate the shapes of the solution according to the vibration of ε.

Keywords: Differential equation; Almost damping vibration; Fractional calculus

1. Definition of N-fractional calculus

Nishimoto defines the fractional calculus as follows [1]:

Let $C = \{C_-, C_+\}$, $D = \{D_-, D_+\}$, where C_- is a curve along the cut joining two points z and $-\infty + \mathrm{i}\,\mathrm{Im}(z)$, C_+ is a curve along the cut joining two points z and $\infty + \mathrm{i}\,\mathrm{Im}(z)$ and D_- is a domain surrounded by C_-, and D_+ is a domain surrounded by C_+. Let us define the fractional calculus operator (N-fractional operator) N^v as follows:

$$N^v = \frac{\Gamma(v+1)}{2\pi \mathrm{i}} \int_C \frac{(\cdot)\,\mathrm{d}\zeta}{(\zeta - z)^{v+1}} \quad (v \notin \mathbb{Z}^-) \tag{1}$$

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with

$$N^{-m} = \lim_{v \to -m} N^v \quad (m \in \mathbb{Z}^+). \tag{2}$$

And let $f = f(z)$ be a regular function in D, then the fractional differintegration of an arbitrary order v for $f(z)$ is defined as follows:

$$f_v(z) = \frac{\Gamma(v+1)}{2\pi \mathrm{i}} \int_C \frac{f(\zeta)}{(\zeta - z)^{v+1}} \, \mathrm{d}\zeta, \tag{3}$$

$$(f)_{-m} = \lim_{v \to -m} (f)_v \quad (m \in \mathbb{Z}^+), \tag{4}$$

where

$$-\pi \leqslant \arg(\zeta - z) \leqslant \pi \quad \text{for } C_-,$$

$$0 \leqslant \arg(\zeta - z) \leqslant 2\pi \quad \text{for } C_+,$$

$$\zeta \neq z, \quad z \in \mathbb{C}, \quad v \in \mathbb{R},$$

$$\Gamma, \quad \text{Gamma function.}$$

$(f)_v$ is called the fractional derivatives of order v with respect to z when $v > 0$ or the fractional integrals of order $-v$ when $v < 0$ if $|(f)_v| < \infty$.

The binary operation $\circ$ is defined as

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\beta (N^\alpha f) \quad (\alpha, \beta \in \mathbb{R}) \tag{5}$$

and it was proved that

$$N^\beta (N^\alpha f) = N^{\beta+\alpha} f \quad (\alpha, \beta \in \mathbb{R}). \tag{6}$$

Then the set

$$\{N^v\} = \{N^v | v \in \mathbb{R}\} \tag{7}$$

is an Abelian product group which has the inverse transform operator $(N^v)^{-1} = N^{-v}$ to the fractional operator N^v, for the function f such that $f \in \mathbf{F} = \{f; 0 \neq |f_v| < \infty, v \in \mathbb{R}\}$.

The following results (principal value) for the exponential functions [1,2] are useful in the latter.

(1) $(\mathrm{e}^{ax})_v = a^v \mathrm{e}^{ax}$,

(2) $(\mathrm{e}^{-ax})_v = \mathrm{e}^{-\mathrm{i}\pi v} a^v \mathrm{e}^{-ax}$ for $a \neq 0$.

2. An almost free damping vibration equation

From the point of view of fractional calculus, we have an interest in the vibration of the oscillation differential equation having a damping term. The natural generation of equation is done by replacing the ordinary derivatives with fractional ones which include a slight vibration of ε.

Let us consider the following equation:

$$\varphi_{2+\varepsilon} + \varphi_{1+(\varepsilon/2)} \cdot a + \varphi \cdot b = 0, \tag{8}$$

where $\varphi \in \{\varphi, 0 \neq |\varphi_v| < \infty, v \in \mathbb{R}\}$. Furthermore, if we set

$$ab \neq 0, \quad a^2 < 4b, \ \ a > 0,$$

$$\varphi = \varphi(t), \quad |\varepsilon| < 2, \ \ \varepsilon, t \in \mathbb{R},$$

then the above equation becomes similar to the oscillation model's equation in the case with the damping term $\varphi_1 \cdot a$, with no forcing term. So we call this equation an almost free damping vibration equation when $0 \neq \varepsilon \ll 1$ [4].

In order to consider the particular solution, we put

$$\varphi = \mathrm{e}^{\lambda t}. \tag{9}$$

Operating $N^{2+\varepsilon}$ on both sides of (9), we have

$$\varphi_{2+\varepsilon} = \lambda^{2+\varepsilon} \mathrm{e}^{\lambda t} \tag{10}$$

and operating $N^{1+(\varepsilon/2)}$ on both sides of (9), we have

$$\varphi_{1+(\varepsilon/2)} = \lambda^{1+(\varepsilon/2)} \mathrm{e}^{\lambda t}. \tag{11}$$

We substitute relations (9) and (10) into Eq. (8), and obtain

$$\lambda^{2+\varepsilon} + \lambda^{1+(\varepsilon/2)} a + b = 0. \tag{12}$$

Furthermore, putting

$$\lambda^{1+(\varepsilon/2)} = \delta, \tag{13}$$

we have the following quadratic equation

$$\delta^2 + \delta a + b = 0. \tag{14}$$

By solving this equation, we get the following results from (14).

$$\delta = -(a/2) + \mathrm{i}\omega = r\mathrm{e}^{\mathrm{i}\theta}, \tag{15}$$

$$\delta = -(a/2) - \mathrm{i}\omega = r\mathrm{e}^{-\mathrm{i}\theta}, \tag{16}$$

where

$$r\cos\theta = -a/2, \quad r\sin\theta = \omega, \tag{17}$$

$$(4b - a^2)/4 = \omega^2. \tag{18}$$

By choosing the principal value for the calculations we find the value of λ for $|\varepsilon| < 2$ as follows:

$$\lambda = \delta^{2/(2+\varepsilon)} = (r\mathrm{e}^{\mathrm{i}\theta})^{\sum_{k=0}^{\infty}(-\varepsilon/2)^k}, \tag{19}$$

$$\lambda = \delta^{2/(2+\varepsilon)} = (r\mathrm{e}^{-\mathrm{i}\theta})^{\sum_{k=0}^{\infty}(-\varepsilon/2)^k}. \tag{20}$$

Then, we have the solution:

$$\varphi = \mathrm{e}^{(r\mathrm{e}^{\mathrm{i}\theta})^{\sum_{k=0}^{\infty}(-\varepsilon/2)^k} t}, \tag{21}$$

$$\varphi = \mathrm{e}^{(r\mathrm{e}^{-\mathrm{i}\theta})^{\sum_{k=0}^{\infty}(-\varepsilon/2)^k} t}. \tag{22}$$

We can easily rewrite the solutions as

$$\varphi = \mathrm{e}^{Gt}[\cos Ht + \mathrm{i}\sin Ht] \equiv \varphi^{(\varepsilon)}_{(1)}, \tag{23}$$

$$\varphi = \mathrm{e}^{Gt}[\cos Ht - \mathrm{i}\sin Ht] \equiv \varphi^{(\varepsilon)}_{(2)}, \tag{24}$$

where

$$G = G(r,\theta,\varepsilon) = r^{S(\varepsilon)} \cos\theta S(\varepsilon), \tag{25}$$

$$H = H(r,\theta,\varepsilon) = r^{S(\varepsilon)} \sin\theta S(\varepsilon), \tag{26}$$

$$S(\varepsilon) = \sum_{k=0}^{\infty} \left(-\frac{\varepsilon}{2}\right)^k.$$

When $|\varepsilon| \ll 1$, we can write

$$S(\varepsilon) \approx 1 - \frac{\varepsilon}{2}.$$

Also, we can further rewrite the solutions as follows:

$$\varphi \simeq \mathrm{e}^{Pt}[\cos Qt + \mathrm{i}\sin Qt] \equiv \varphi_{(1)}, \tag{27}$$

$$\varphi \simeq \mathrm{e}^{Pt}[\cos Qt - \mathrm{i}\sin Qt] \equiv \varphi_{(2)}, \tag{28}$$

where

$$P = P(r,\theta,\varepsilon) = r^{1-(\varepsilon/2)} \cos\theta\{1 - (\varepsilon/2)\}, \tag{29}$$

$$Q = Q(r,\theta,\varepsilon) = r^{1-(\varepsilon/2)} \sin\theta\{1 - (\varepsilon/2)\}. \tag{30}$$

3. The general solutions of almost free damping vibration equations

Similar to the case of the particular solution, setting

$$\varphi = \mathrm{e}^{\lambda t},$$

we have

$$\lambda = \delta^{2/(2+\varepsilon)} = \begin{cases} (r\mathrm{e}^{\mathrm{i}\theta})^{2/(2+\varepsilon)}, \\ (r\mathrm{e}^{-\mathrm{i}\theta})^{2/(2+\varepsilon)}. \end{cases}$$

Here, $\varepsilon \in \mathbb{R}$, $\lambda^{1+(\varepsilon/2)} = \delta$, and r and θ are the same as (17) and (18).

Now this λ can be written by

$$(r\mathrm{e}^{\mathrm{i}\theta})^{2/(2+\varepsilon)} = r^{2/(2+\varepsilon)}(\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{\mathrm{i}2n\pi})^{2/(2+\varepsilon)} \quad (n \in \mathbb{Z}_0^+)$$
$$= r^{2/(2+\varepsilon)}\{\cos P(n) + \mathrm{i}\sin P(n)\}, \tag{31}$$

where

$$P(n) = P(n, \theta, \varepsilon) = \frac{2(\theta + 2n\pi)}{2+\varepsilon}. \tag{32}$$

Therefore, as one solution, we have

$$\varphi = \mathrm{e}^{(r\mathrm{e}^{\mathrm{i}\theta})^{2/(2+\varepsilon)}t}$$
$$= \exp\{r^{2/(2+\varepsilon)}(\cos P(n) + \mathrm{i}\sin P(n))t\} \equiv \varphi_{(1)}|_{(n)}. \tag{33}$$

In the same way, for another solution, we have

$$(r\mathrm{e}^{-\mathrm{i}\theta})^{2/(2+\varepsilon)} = r^{2/(2+\varepsilon)}\{\cos Q(n) + \mathrm{i}\sin Q(n)\}, \tag{34}$$

$$Q(n) = Q(n, \theta, \varepsilon) = \frac{2(-\theta + 2n\pi)}{2+\varepsilon}. \tag{35}$$

Therefore, we can get

$$\varphi = \mathrm{e}^{(r\mathrm{e}^{-\mathrm{i}\theta})^{2/(2+\varepsilon)}t}$$
$$= \exp\{r^{2/(2+\varepsilon)}(\cos Q(n) + \mathrm{i}\sin Q(n))t\} \equiv \varphi_{(2)}|_{(n)}. \tag{36}$$

Then putting

$$\varphi_{(1)} = \sum_{n=0}^{s} a_n \cdot \varphi_{(1)}|_{(n)}, \tag{37}$$

$$\varphi_{(2)} = \sum_{n=0}^{s} b_n \cdot \varphi_{(2)}|_{(n)}, \tag{38}$$

we can get the general solution for $\varepsilon \in \mathbb{R}$ as follows:

$$\varphi = \varphi_{(1)} + \varphi_{(2)} = \sum_{n=0}^{s} \{a_n \cdot \varphi_{(1)}|_{(n)} + b_n \cdot \varphi_{(2)}|_{(n)}\}, \tag{39}$$

where a_n, b_n is an arbitrary constant and s is finite [infinite] when ε is the rational [irrational] number.

When we put

$$P(n) = P(n, \theta, \varepsilon) = (\theta + 2n\pi)S(\varepsilon), \tag{40}$$

$$Q(n) = Q(n, \theta, \varepsilon) = (-\theta + 2n\pi)S(\varepsilon), \tag{41}$$

$$S(\varepsilon) = \frac{2}{2+\varepsilon}, \tag{42}$$

we can get the general solution according to the case of ε by rewriting $s(\varepsilon)$ as follows [4]:

(i) For $\varepsilon \in \mathbb{R}$

$$S(\varepsilon) = \frac{2}{(2+\varepsilon)}.$$

(ii) For $|\varepsilon| < 2$

$$S(\varepsilon) = \sum_{k=0}^{\infty} \left(-\frac{\varepsilon}{2}\right)^k.$$

(iii) For $|\varepsilon| \ll 1$

$$S(\varepsilon) \approx 1 - \frac{\varepsilon}{2},$$

we have the "almost general solution" in this case.

4. Behavior of solutions for ε

(I) For $\varepsilon = 0$ we have the following two particular solutions

$$\begin{aligned} \varphi_{(1)}|_{(0)} &= e^{-(a/2)t}\{\cos \omega t + i \sin \omega t\} \\ &= e^{-(a/2)t}\left\{\cos \frac{\sqrt{4b-a^2}}{2}t + i \sin \frac{\sqrt{4b-a^2}}{2}t\right\}, \end{aligned} \tag{43}$$

$$\begin{aligned} \varphi_{(2)}|_{(0)} &= e^{-(a/2)t}\{\cos \omega t - i \sin \omega t\} \\ &= e^{-(a/2)t}\left\{\cos \frac{\sqrt{4b-a^2}}{2}t - i \sin \frac{\sqrt{4b-a^2}}{2}t\right\}. \end{aligned} \tag{44}$$

The equation for $a = b = 1$, which is

$$\varphi_{2+\varepsilon} + \varphi_{1+(\varepsilon/2)} + \varphi = 0, \tag{45}$$

has particular solutions written by

$$\varphi|_{(1)}^{\varepsilon} = e^{Gt}\{\cos Ht + i \sin Ht\}, \tag{46}$$

$$\varphi|_{(2)}^{\varepsilon} = e^{Gt}\{\cos Ht - i \sin Ht\}, \tag{47}$$

where

$$G = G(r, \theta, \varepsilon) = r^{S(\varepsilon)} \cos \theta\, S(\varepsilon), \tag{48}$$

$$H = H(r, \theta, \varepsilon) = r^{S(\varepsilon)} \sin \theta\, S(\varepsilon), \tag{49}$$

$$S(\varepsilon) = \sum_{k=0}^{\infty} \left(-\frac{\varepsilon}{2}\right)^k, \tag{50}$$

$$r \cos \theta = -\frac{1}{2}, \quad r \sin \theta = \frac{\sqrt{3}}{2}. \tag{51}$$

And then under the same condition, the ordinary differential equation for $\varepsilon = 0$ is written by

$$\varphi'' + \varphi' + \varphi = 0 \tag{52}$$

which has the following solutions:

$$\varphi_{(1)} = \mathrm{e}^{-(1/2)t}\left\{\cos\frac{\sqrt{3}}{2}t + \mathrm{i}\sin\frac{\sqrt{3}}{2}t\right\}, \tag{53}$$

$$\varphi_{(2)} = \mathrm{e}^{-(1/2)t}\left\{\cos\frac{\sqrt{3}}{2}t - \mathrm{i}\sin\frac{\sqrt{3}}{2}t\right\}. \tag{54}$$

(II) For $\varepsilon \ll 1$, $S(\varepsilon)$ becomes

$$S(\varepsilon) \approx 1 - \frac{\varepsilon}{2}. \tag{55}$$

Then (46) and (47) can be written as

$$\begin{aligned}\varphi|^{\varepsilon}_{(1)} \approx{}& \exp\left\{r^{(1-\varepsilon/2)}\left\{\cos\theta\left(1-\frac{\varepsilon}{2}\right)\right\}t\right\}\\ &\times\cos\left\{r^{(1-\varepsilon/2)}\sin\theta\left(1-\frac{\varepsilon}{2}\right)\right\}t + \mathrm{i}\sin\left\{r^{(1-\varepsilon/2)}\sin\theta\left(1-\frac{\varepsilon}{2}\right)\right\}t,\end{aligned} \tag{56}$$

$$\begin{aligned}\varphi|^{\varepsilon}_{(2)} \approx{}& \exp\left\{r^{(1-\varepsilon/2)}\left\{\cos\theta\left(1-\frac{\varepsilon}{2}\right)\right\}t\right\}\\ &\times\left\{\cos\left\{r^{(1-\varepsilon/2)}\left\{\sin\theta\left(1-\frac{\varepsilon}{2}\right)\right\}t\right\} - \mathrm{i}\sin\left\{r^{(1-\varepsilon/2)}\left\{\sin\theta\left(1-\frac{\varepsilon}{2}\right)\right\}t\right\}\right\}.\end{aligned} \tag{57}$$

From these equations, we can get the relation

$$\begin{aligned}\mathrm{Re}(\varphi|^{\varepsilon}_{(1)}) &= \mathrm{Re}(\varphi|^{\varepsilon}_{(2)})\\ &= \exp\left\{r^{(1-\varepsilon/2)}\left\{\cos\theta\left(1-\frac{\varepsilon}{2}\right)\right\}t\right\}\cos\left\{r^{(1-\varepsilon/2)}\sin\theta\left(1-\frac{\varepsilon}{2}\right)\right\}t,\end{aligned} \tag{58}$$

where $\mathrm{Re}(C)$ denotes the real part of complex number C.

On the other hand, from (53) and (54) we can observe the relation

$$\mathrm{Re}(\varphi|_{(1)}) = \mathrm{Re}(\varphi|_{(2)}) = \mathrm{e}^{-(1/2)t}\cos\frac{\sqrt{3}}{2}t. \tag{59}$$

According to these relations, we can easily find that

$$\lim_{\varepsilon\to 0}\mathrm{Re}(\varphi|^{\varepsilon}_{(1)}) = \mathrm{Re}(\varphi|_{(1)}). \tag{60}$$

The graph of Eq. (59) illustrates a damping vibration which is a simple harmonic oscillation $\cos\sqrt{3}/2t$ with an amplitude $\mathrm{e}^{-(1/2)t}$. Therefore, the graph of Eq. (58) should be similar to the graph of (59) as $|\varepsilon| \to 0$.

We illustrate the graphs of real part of equations for (59) and the same for (58) with $\varepsilon = 0.1, 0.2, 0.5$. Fig. 1 is for (59) and Fig. 2 is for (58).

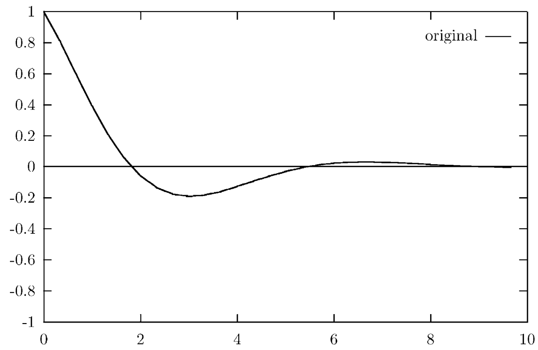

Fig. 1. Real part of $\varphi|_{(1)}$ and $\varphi|_{(2)}$.

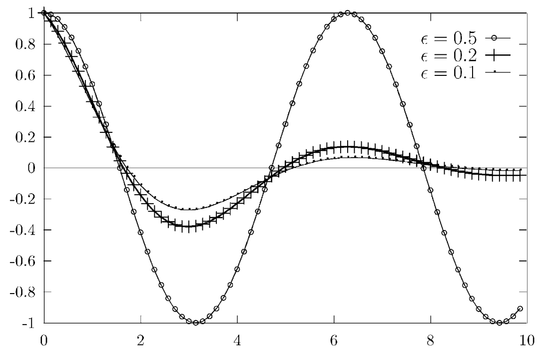

Fig. 2. Real part of $\varphi|_{(1)}^{\varepsilon}$ and $\varphi|_{(2)}^{\varepsilon}$ for $\varepsilon = 0.1, 0.2, 0.5$.

5. Uncited Reference

[3]

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