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Comonotonic approximations for a generalized provisioning problem with application to optimal portfolio selection

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ABSTRACT

In this paper we discuss multiperiod portfolio selection problems related to a specific provisioning problem. Our results are an extension of Dhaene et al. (2005) [14], where optimal constant mix investment strategies are obtained in a provisioning and savings context, using an analytical approach based on the concept of comonotonicity. We derive convex bounds that can be used to estimate the provision to be set up at a specified time in future, to ensure that, after having paid all liabilities up to that moment, all liabilities from that moment on can be fulfilled, with a high probability.

1. Introduction

In this paper we discuss multiperiod portfolio selection problems related to a specific provisioning problem. Our results are an extension of [1], where optimal constant mix investment strategies are obtained in a provisioning and savings context, using an analytical approach based on the concept of comonotonicity. In this analytical framework, we derive convex bounds that can be used to estimate the provision to be set up at a specified time t in future, to ensure that, after having paid all liabilities up to time t, all liabilities from t on can be fulfilled, with a high probability.

We explain how this additional provision can be used to estimate the influence of a temporary change in market parameters. We see how an insurer can get an idea how much a temporary 'crisis' will cost him, and how this will influence his optimal investment portfolio. Also, if an insurer's investment portfolio is not optimal, the results of this paper can be used to check whether postponing rebalancing is acceptable, and if so, for how many years.

We apply our results to optimal portfolio selection problems, and illustrate with numerical examples.

In the following sections a brief introduction is given to respectively risk measures, the theory of comonotonicity, convex bounds of random variables and the framework of optimal portfolio selection in a lognormal setting. Next the general provisioning problem is discussed, and applied to optimal portfolio selection.

1.1. Risk measures and comonotonicity

A *risk measure* is defined as a mapping from a set of random variables, representing the risks at hand, to the real numbers. In other words, a risk measure summarizes the distribution function of a random variable in one single real number. The common notation for a risk measure associated with a random variable X is $\rho[X]$. A risk measure ρ quantifies the riskiness of X: the larger the $\rho[X]$, the more 'dangerous' the risk X.

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Throughout this paper we assume that we are working with (conditioning) random variables such that all (conditional) expectations that are used are well defined and finite.

In this paper the main focus will be on the quantile risk measure, or Value-at-Risk (VaR). The VaR at level p will be denoted by $Q_p(X)$ or $\mathrm{VaR}_p(X)$, and is defined as:

$$Q_p(X) = \mathrm{VaR}_p(X) = F_X^{-1}(p) = \inf\{x \in \mathbb{R} \mid F_X(x) \geq p\}, \quad p \in (0, 1), \tag{1}$$

with $F_X(x) = \Pr\ (X \leq x)$. By convention, we take $\inf \varnothing = +\infty$.

Value-at-Risk measures the worst expected loss under normal market conditions over a specific time interval. It can be used to determine how much can be lost with a given probability over a predetermined time horizon.

Other well-known risk measures are, for example, Tail Value-at-Risk (TVaR), Conditional Tail Expectation (CTE) and Expected Shortfall (ESF). More information on risk measures can be found e.g. in [2] or [1].

A random vector $\underline{X} = (X_1, X_2, \ldots, X_n)$ is said to be *comonotonic* if the individual variables X_i are non-decreasing functions (or all are non-increasing functions) of the same random variable:

$$\underline{X} \stackrel{d}{=} (g_1(Z), g_2(Z), \ldots, g_n(Z)) \tag{2}$$

for some common random variable Z and non-decreasing (or non-increasing) functions g_i. Intuitively, comonotonicity corresponds to an extreme form of positive dependency between the individual variables: increasing the outcome of Z will lead to a simultaneous increase in the different outcomes of $g_i(Z)$.

Comonotonicity of $\underline{X}$ can also be characterized by

$$\underline{X} \stackrel{d}{=} (F_{X_1}^{-1}(U), F_{X_2}^{-1}(U), \ldots, F_{X_n}^{-1}(U)), \tag{3}$$

with U uniformly distributed on the unit interval.

A sum of random variables is called a *comonotonic sum* if its components form a comonotonic vector.

For more characterizations and an overview of the theory of comonotonicity and its many applications in actuarial science and finance we refer to [3–5].

The following result of the comonotonic dependency structure will be crucial in our setting.

Theorem 1 (*Additivity of Quantile Risk Measure for Sums of Comonotonic Risks*)**.** *If the random vector* $(X_1, X_2, \ldots, X_n)$ *is comonotonic, we have that*

$$Q_p\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Q_p(X_i), \tag{4}$$

for all $p \in (0, 1)$.

This additivity property holds in general for all distortion risk measures, such as Tail Value-at-Risk and Expected Shortfall. In case the variables X_i are continuous, the same property holds for the Conditional Tail Expectation. A proof of this theorem and more information about the relationship between risk measures and comonotonicity can be found in [6].

1.2. Convex order bounds for sums of random variables

An extensive introduction to ordering of (distributions of) random variables, including actuarial applications, can be found in [1]. We recall the definition of *stop-loss order* and *convex order*.

Definition 1 (*Stop-Loss Order*)**.** A random variable X is said to precede a random variable Y in *stop-loss order* if X has lower stop-loss premiums than Y:

$$E[(X - d)_+] \leq E[(Y - d)_+], \tag{5}$$

for all $d \in (-\infty, +\infty)$. We denote this as $X \leq_{sl} Y$.

Definition 2 (*Convex Order*)**.** A random variable X is said to precede a random variable Y in *convex order* if $X \leq_{sl} Y$ and $E[X] = E[Y]$. We denote this as $X \leq_{cx} Y$.

Recall that convex order does not imply first order stochastic dominance, which means that $X \leq_{cx} Y$ does not necessarily imply that the distribution functions of X and Y, or, equivalently, their quantiles, are ordered. However, it can easily be seen that convex order does imply that the quantiles are ordered from a certain probability level on: if $X \leq_{cx} Y$, there exists a $p \in (0, 1)$ such that $Q_q[X] \leq Q_q[Y]$ for all $p \leq q \leq 1$. Since in the context of optimal portfolio selection problems we will typically be interested in quantiles in the tail, we can say that the convex upper and lower bounds as defined below respectively overestimate and underestimate the risk. The high accuracy of the convex lower bounds (9), (25) and (27) implies, however, that this underestimation is negligible in our case.

In this paper we will encounter random variables of the form

$$S = \sum_{i=1}^{n} \alpha_i e^{Z_i} \tag{6}$$

where α_i are deterministic constants, and Z_i are linear combinations of the components of a multivariate normal random vector $(Y_1, Y_2, \ldots, Y_n)$: suppose $Z_i = \sum_{j=1}^{n} \lambda_{ij} Y_j$ for $i = 1, \ldots, n$.

The random variable S in (6) is a sum of dependent lognormal random variables. As it is impossible to determine the distribution function of such a sum analytically, we use approximations. Several approximation techniques have been proposed throughout the literature; see e.g. [7–10]. In this paper we will use convex upper and lower bounds based on comonotonicity; see e.g. [11,3,4]. See also [12] or [13] for a comparison of some of the approximation techniques.

The approximations of [11] are based on the following result.

Theorem 2 (*Convex Bounds for Sums of Random Variables*). *For any random vector $(X_1, X_1, \ldots, X_n)$ and any random variable Λ, we have that*

$$S^l = \sum_{i=1}^{n} E[X_i|\Lambda] \leq_{cx} S = \sum_{i=1}^{n} X_i \leq_{cx} \sum_{i=1}^{n} F_{X_i}^{-1}(U) = S^c, \tag{7}$$

with U a uniformly distributed random variable on the unit interval.

A proof of this theorem can be found in [11]. As can be seen from (3), the sum S^c is comonotonic. The special case (6) where S is a sum of dependent lognormal random variables is discussed in detail in [3,4]. Expressions for S^c and S^l are derived in case the cash-flows α_i are positive. The *comonotonic upper bound* S^c is given by

$$S^c = \sum_{i=1}^{n} \alpha_i e^{E[Z_i]+\sigma_{Z_i}\Phi^{-1}(U)}. \tag{8}$$

For the lower bound approximation, the conditioning variable Λ is typically chosen as a linear combination of the variables Y_i. Assume that $\Lambda = \sum_{j=1}^{n} \beta_j Y_j$. In this case the lower bound S^l can be written as:

$$S^l = \sum_{i=1}^{n} \alpha_i e^{E[Z_i]+\frac{1}{2}(1-r_i^2)\sigma_{Z_i}^2 + r_i\sigma_{Z_i}\Phi^{-1}(U)}, \tag{9}$$

where r_i is the correlation between Z_i and Λ. If all coefficients r_i are positive, each of the terms in the sum in (9) is an increasing function of U, which means that S^l is a comonotonic sum. In this case, we call S^l the *comonotonic lower bound*.

If all Y_i are i.i.d., the correlation coefficients are given by

$$r_i = \frac{\sum_{j=1}^{n} \lambda_{ij}\beta_j}{\sqrt{\sum_{j=1}^{n} \lambda_{ij}^2}\sqrt{\sum_{j=1}^{n} \beta_j^2}}, \quad i = 1, \ldots, n. \tag{10}$$

Maximizing the variance of S^l leads, as explained in [14], to the optimal Λ, which is given by

$$\Lambda = \sum_{i=1}^{n} \alpha_i e^{E[Z_i]+\frac{1}{2}\sigma_{Z_i}^2} Z_i. \tag{11}$$

In [15] conditions are derived for the lower bound to be comonotonic in case the cash-flows α_i have changing signs.

As can be seen from Theorem 1, a crucial advantage of the comonotonic bounds is the additivity property, which makes it straightforward to apply risk measures such as quantiles (VaR), TVaR and CTE to S^c and S^l, and hence to determine their distribution function. In [14] expressions are given for the most commonly used risk measures associated with (8) and (9).

1.3. Optimal portfolio selection in a lognormal framework

Throughout this paper we assume the classical continuous-time framework of [16], also known as the Black and Scholes [17] setting. See e.g. [18] for more details on this Black and Scholes setting. We use the same notations and terminology as in [14].

Assume that there are t risky assets or asset classes available in which we can invest. In our examples we assume that there is no risk-free asset class available. The return of the risky assets is modeled by a multivariate geometric Brownian motion: investing an amount of 1 at time $k-1$ in risky asset s grows to $e^{Y_k^s}$ at time k. For a fixed asset s, the random variables

Y_k^s are assumed i.i.d., normally distributed with mean $\mu_s - \frac{1}{2}\sigma_s^2$ and variance σ_s^2. This means that the return of an asset is not influenced by its return in the past. However, within any year, the returns of the different assets are correlated. We have that:

$$Cov[Y_k^{s_1}, Y_l^{s_2}] = \begin{cases} 0 & k \neq l \\ \sigma_{s_1 s_2} & k = l. \end{cases} \tag{12}$$

The drift vector and the variance–covariance matrix of the risky assets are denoted as $\underline{\mu}^T = (\mu_1, \ldots, \mu_t)$ and $\underline{\Sigma}$, respectively.

We restrict to *constant mix strategies*: the fractions invested in the different assets remain constant over time, due to continuous rebalancing. A justification of the use of constant mix investment strategies as an approximation for the more realistic class of strategies where periodic rebalancing is performed can be found in [19]. We also refer to the aforementioned paper for some references on the optimality of constant mix strategies.

A vector describing the portfolio process is denoted as $\underline{\pi}^T = (\pi_1, \ldots, \pi_t)$, where π_i is the proportion invested in risky asset i, with $\sum_{i=1}^t \pi_i = 1$. Although our results also hold in the general case, we assume that short-selling is not allowed, which means that $0 \leq \pi_i \leq 1$ for all i. The drift and volatility corresponding to an investment portfolio $\underline{\pi}$ are written as $\mu(\underline{\pi})$ and $\sigma^2(\underline{\pi})$, and are given by:

$$\mu(\underline{\pi}) = \underline{\pi}^T \underline{\mu} \quad \text{and} \quad \sigma^2(\underline{\pi}) = \underline{\pi}^T \cdot \underline{\Sigma} \cdot \underline{\pi}. \tag{13}$$

The normality assumption for modeling (log) returns on investment has been questioned in the financial literature for the short-term setting (e.g. daily returns—see [20]). However, as both the time period and the investment horizon that we consider throughout this thesis are typically long, the use of a Gaussian model for the stochastic returns can be justified by Central Limit Theorem arguments. Empirical evidence justifying the normality assumption is provided in e.g. [21]. In this paper, four well-known stock market indices in US dollars are investigated, covering all major stock markets in industrial as well as emerging countries. Studying the period 1997–1999, the authors conclude that weekly (and longer period) returns can be considered as normal and independent. Other references where empirical evidence can be found are [22,23].

The yearly returns $Y_i(\underline{\pi})$ of an investment portfolio $\underline{\pi}$ are independent and normally distributed random variables, with expected value $\mathrm{E}[Y_i(\underline{\pi})] = \mu(\underline{\pi}) - \frac{1}{2}\sigma^2(\underline{\pi})$ and variance $\mathrm{Var}[Y_i(\underline{\pi})] = \sigma^2(\underline{\pi})$.

When no confusion is possible, we omit the dependence on the investment portfolio $\underline{\pi}$ in the notations. Hence, the yearly returns are modeled by the i.i.d., normally distributed random variables Y_i, with mean $\mu - \frac{1}{2}\sigma^2$ and standard deviation σ.

2. Generalized provisioning problem

In this section we discuss the main topic of this paper. We want to determine (an estimate of) the provision to be set up at certain time in future, to ensure that, after having paid the first liabilities, all liabilities from then on can be fulfilled with a high probability. First a general description of the problem is given, followed by the derivation of a solution based on convex order comonotonic bounds. Next the problem is applied to optimal portfolio selection, and illustrated with numerical examples. In the final part of this section some practical interpretations of this provision are described and illustrated.

2.1. Problem description

Consider a series of deterministic liabilities α_i due at time i, for $i = 1, \ldots, n$, with $\alpha_i \geq 0$ for all i. Suppose we have an initial capital $K_0 > 0$ available at time 0. Assume that during the first m years, with $0 < m < n$, an investment strategy $\underline{\pi}_1$ is followed where the return in year i is described by the random variable $Y_i(\underline{\pi}_1)$, with $\mathrm{E}[Y_i(\underline{\pi}_1)] = \mu(\underline{\pi}_1) - \frac{1}{2}\sigma^2(\underline{\pi}_1)$ and $\mathrm{Var}[Y_i(\underline{\pi}_1)] = \sigma^2(\underline{\pi}_1)$. The random variables $Y_i(\underline{\pi}_1)$ are iid and normally distributed, for $i = 1, \ldots, m$. After m years, a different investment strategy is followed, with return in year j equal to $Y_j(\underline{\pi}_2)$. The random variables $Y_j(\underline{\pi}_2)$ are iid and normally distributed, with $\mathrm{E}[Y_j(\underline{\pi}_2)] = \mu(\underline{\pi}_2) - \frac{1}{2}\sigma^2(\underline{\pi}_2)$ and $\mathrm{Var}[Y_j(\underline{\pi}_2)] = \sigma^2(\underline{\pi}_2)$. We assume that the random variables $Y_i(\underline{\pi}_1)$ and $Y_j(\underline{\pi}_2)$ are independent for all i and j.

We want to determine (an estimate of) the provision to be set up at time m, with $0 < m < n$, to ensure that, after having paid the first m liabilities, all future liabilities can be fulfilled, incorporating a certain ruin probability ϵ. We denote this additional reserve at time m by K_m. Formally, we want to determine K_m such that:

$$\Pr\left[K_0 e^{\sum_{i=1}^m Y_i(\underline{\pi}_1)} - \sum_{i=1}^m \alpha_i e^{\sum_{j=i+1}^m Y_j(\underline{\pi}_1)} + K_m \geq \sum_{i=1}^{n-m} \alpha_{m+i} e^{-\sum_{j=1}^i Y_j(\underline{\pi}_2)}\right] \geq (1-\epsilon), \tag{14}$$

for some small ϵ. In other words, the reserve K_m is equal to the following quantile:

$$K_m = Q_{1-\epsilon}\left[\sum_{i=1}^{n-m} \alpha_{t+i} e^{-\sum_{j=1}^i Y_j(\underline{\pi}_2)} - K_0 e^{\sum_{i=1}^m Y_i(\underline{\pi}_1)} + \sum_{i=1}^m \alpha_i e^{\sum_{j=i+1}^m Y_j(\underline{\pi}_1)}\right]. \tag{15}$$

Note that K_m is not necessarily positive. A negative K_m means that the investor can withdraw an amount $-K_m$ from the account at time m, and still fulfill future liabilities, incorporating a ruin probability ϵ. If $K_m = 0$, no additional reserve is needed at time m, but at the same time nothing can be withdrawn from the account.

Note also that (14) is a long-term survival probability, over the whole investment period of n years. For example, a survival probability of 85% over a period of 30 years corresponds to a yearly survival probability of approximately 99.46%, since $0.85 \approx (0.9946)^{30}$.

In the following section expressions are derived for respectively the convex upper bound and lower bound approximation.

2.2. Derivation of convex bounds

Within the quantile function in (15) we have sums of dependent lognormal random variables. As explained in Section 1.2, it is impossible to determine the distribution function of these sums exactly. Therefore we derive analytical approximations, based on the concept of comonotonicity, which are easy to compute. The results in this section are a generalization of [14].

The bounds derived in [14] cannot be applied directly to compute (15), as the terms within the quantile have different signs. Also, Theorem 1 from [15] cannot be applied here, since the conditions of the theorem are not necessarily satisfied. Therefore we have to use a different approach to determine a value for the reserve K_m.

Denote $Z = \sum_{i=1}^{m} Y_i(\underline{\pi}_1)$. Applying the law of total probability, conditioning on Z, the left-hand side of inequality (14) becomes:

$$\int_{-\infty}^{\infty} \Pr\left[\sum_{i=1}^{m} \alpha_i e^{\sum_{j=i+1}^{m} Y_j(\underline{\pi}_1)|Z=z} + \sum_{i=1}^{n-m} \alpha_{m+i} e^{-\sum_{j=1}^{i} Y_j(\underline{\pi}_2)} \le K_0 e^z + K_m\right] \frac{1}{\sigma_Z}\phi\left(\frac{z-\mu_Z}{\sigma_Z}\right) dz, \tag{16}$$

with $\mu_Z = E[Z] = m\mu(\underline{\pi}_1)$ and $\sigma_Z = \sqrt{m}\sigma(\underline{\pi}_1)$.

Denoting $S(z) = S_1(z) + S_2$, with $S_1(z) = \sum_{i=1}^{m} \alpha_i e^{\sum_{j=i+1}^{m} Y_j(\underline{\pi}_1)|Z=z}$, and $S_2 = \sum_{i=1}^{n-m} \alpha_{m+i} e^{-\sum_{j=1}^{i} Y_j(\underline{\pi}_2)}$, we can rewrite (16) as

$$\int_{-\infty}^{\infty} \Pr[S(z) \le K_0 e^z] \frac{1}{\sigma_Z}\phi\left(\frac{z-\mu_Z}{\sigma_Z}\right) dz. \tag{17}$$

To approximate the distribution function of $S(z)$, we can use its comonotonic upper bound $S^c(z)$ or lower bound $S^l(z)$, as defined by (8) and (9). Important is that here it is possible to apply the results from Dhaene et al. [14], because all terms in $S(z)$ are of the same sign.

2.2.1. Upper bound approximation

To compute the probability within integral (16), we can approximate $S(z)$ by its comonotonic upper bound $S^c(z)$ as follows:

$$S(z) \le_{cx} S^c(z) = S_1^c(z) + S_2^c = \sum_{i=1}^{m} \alpha_i F^{-1}_{e^{\sum_{j=i+1}^{m} Y_j(\underline{\pi}_1)|Z=z}}(U) + \sum_{i=1}^{n-m} \alpha_{m+i} F^{-1}_{e^{-\sum_{j=1}^{i} Y_j(\underline{\pi}_2)}}(U), \tag{18}$$

with U uniformly distributed on the unit interval.

As shown in [4], the random variables $\sum_{j=i+1}^{m} Y_j(\underline{\pi}_1)|Z = z$ are normally distributed for any z. It can easily be seen that its expected value and variance are given by:

$$E\left[\sum_{j=i+1}^{m} Y_j(\underline{\pi}_1)|Z=z\right] = \frac{m-i}{m} z \quad \text{and} \quad \mathrm{Var}\left[\sum_{j=i+1}^{m} Y_j(\underline{\pi}_1)|Z=z\right] = \frac{i(m-i)}{m}\sigma^2(\underline{\pi}_1). \tag{19}$$

Using (8) and (19), $S_1^c(z)$ can be rewritten as:

$$S_1^c(z) = \sum_{i=1}^{m} \alpha_i \exp\left(\frac{m-i}{m} z + \sqrt{\frac{i(m-i)}{m}}\sigma(\underline{\pi}_1)\Phi^{-1}(U)\right). \tag{20}$$

We also have an expression for S_2^c:

$$S_2^c = \sum_{i=1}^{n-m} \alpha_{m+i} \exp\left(-i\left(\mu(\underline{\pi}_2) - \frac{1}{2}\sigma^2(\underline{\pi}_2)\right) + \sqrt{i}\sigma(\underline{\pi}_2)\Phi^{-1}(U)\right). \tag{21}$$

Hence, using the additivity property (see Theorem 1), we can compute the quantiles of $S^c(z)$ as:

$$\begin{aligned} Q_{1-p}[S^c(z)] = &\sum_{i=1}^{m} \alpha_i \exp\left(\frac{m-i}{m}z - \sqrt{\frac{\mathrm{i}(m-i)}{m}}\sigma(\underline{\pi}_1)\Phi^{-1}(p)\right) \\ &+ \sum_{i=1}^{n-m} \alpha_{m+i} \exp\left(-\mathrm{i}\left(\mu(\underline{\pi}_2) - \frac{1}{2}\sigma^2(\underline{\pi}_2)\right) - \sqrt{\mathrm{i}}\sigma(\underline{\pi}_2)\Phi^{-1}(p)\right). \end{aligned} \tag{22}$$

This result can be used to determine the distribution function of $S^c(z)$, which can then be used to approximate integral (16).

2.2.2. Lower bound approximation

We can also approximate $S(z)$ using convex lower bounds. We have that

$$S_1(z) \geq_{cx} S_1^l(z) = E[S_1(z)|\Lambda_1(z)]. \tag{23}$$

Using (11) and (19) we get:

$$\Lambda_1(z) = \sum_{i=1}^{m} \alpha_i e^{\frac{m-i}{m}z + \frac{1}{2}\frac{\mathrm{i}(m-i)}{m}\sigma^2(\underline{\pi}_1)} \left(\sum_{j=i+1}^{m} Y_j(\underline{\pi}_1)|Z = z\right). \tag{24}$$

Using (9) and (19), we can write the lower bound S_1^l as:

$$S_1^l(z) = \sum_{i=1}^{m} \alpha_i \exp\left(\frac{m-i}{m}z + \frac{1}{2}(1-r_i^2)\frac{\mathrm{i}(m-i)}{m}\sigma^2(\underline{\pi}_1) + r_i\sqrt{\frac{\mathrm{i}(m-i)}{m}}\sigma(\underline{\pi}_1)\Phi^{-1}(U_1)\right), \tag{25}$$

with U_1 uniformly distributed on the unit interval. The correlation coefficients r_i can be determined using (10). Using the additivity property explained in Theorem 1, the quantiles of $S_1^l(z)$ can be determined as:

$$Q_{1-p}[S_1^l(z)] = \sum_{i=1}^{m} \alpha_i \exp\left(\frac{m-i}{m}z + \frac{1}{2}(1-r_i^2)\frac{\mathrm{i}(m-i)}{m}\sigma^2(\underline{\pi}_1) - r_i\sqrt{\frac{\mathrm{i}(m-i)}{m}}\sigma(\underline{\pi}_1)\Phi^{-1}(p)\right). \tag{26}$$

S_2 can be approximated by a convex lower bound S_2^l in a similar way:

$$S_2 \geq_{cx} S_2^l = E[S_2|\Lambda_2]. \tag{27}$$

The conditioning variable Λ_2 is given by

$$\Lambda_2 = \sum_{i=1}^{n-m} \alpha_{m+i} e^{-\mathrm{i}\mu(\underline{\pi}_2) + \frac{1}{2}\mathrm{i}\sigma^2(\underline{\pi}_2)} \left(-\sum_{j=1}^{i} Y_j(\underline{\pi}_2)\right). \tag{28}$$

Using (9) we get the following expression for S_2^l:

$$S_2^l = \sum_{i=1}^{n-m} \alpha_{m+i} \exp\left(-\mathrm{i}\mu(\underline{\pi}_2) + \left(1 - \frac{1}{2}(r_i')^2\right)\mathrm{i}\sigma^2(\underline{\pi}_2) + r_i'\sqrt{\mathrm{i}}\sigma(\underline{\pi}_2)\Phi^{-1}(U_2)\right), \tag{29}$$

with U_2 uniformly distributed on the unit interval. The correlation coefficients r_i' can be determined using (10). Using the additivity property, the quantiles of S_2^l can be determined using:

$$Q_{1-p}[S_2^l] = \sum_{i=1}^{n-m} \alpha_{m+i} \exp\left(-\mathrm{i}\mu(\underline{\pi}_2) + \left(1 - \frac{1}{2}(r_i')^2\right)\mathrm{i}\sigma^2(\underline{\pi}_2) - r_i'\sqrt{\mathrm{i}}\sigma(\underline{\pi}_2)\Phi^{-1}(p)\right). \tag{30}$$

We approximate $S(z) = S_1(z) + S_2$ by the sum $S^l(z) = S_1^l(z) + S_2^l$. The approximation $S^l(z)$ is a convex lower bound for $S(z)$, since convex order is closed under convolution for independent risks (see e.g. [1]). The quantiles of $S^l(z)$ can be computed by adding (26) and (30). This allows us to compute the distribution function of $S^l(z)$, and hence to approximate integral (16).

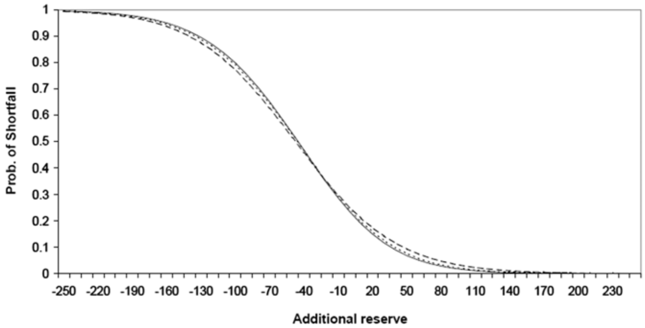

Fig. 1. Comparison of upper bound (dashed line) and lower bound (dotted line) to simulated results (solid line), $\mu(\underline{\pi}_1) = \mu(\underline{\pi}_2) = 0.05$ and $\sigma(\underline{\pi}_1) = \sigma(\underline{\pi}_2) = 0.1$.

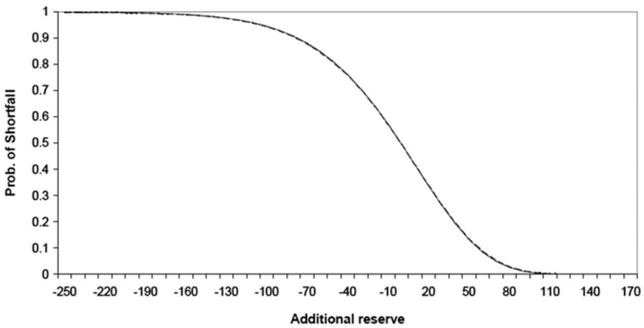

Fig. 2. Comparison of upper bound (dashed line) and lower bound (dotted line) to simulated results (solid line), $\mu(\underline{\pi}_1) = 0.05, \sigma(\underline{\pi}_1) = 0.1, \mu(\underline{\pi}_2) = 0.02$ and $\sigma(\underline{\pi}_2) = 0.01$.

2.3. Numerical illustration

Assume that $n = 30$, $\alpha_i = 10$ for $i = 1, \ldots, 30$, $K_0 = 200$ and $m = 5$. Furthermore, assume as a first example $\mu_X = \mu_Y = 0.05$ and $\sigma_X = \sigma_Y = 0.1$. Using this setting we can compute the probability of shortfall

$$\Pr\left[K_0 e^{\sum_{i=1}^{m} Y_i(\underline{\pi}_1)} - \sum_{i=1}^{m} \alpha_i e^{\sum_{j=i+1}^{m} Y_j(\underline{\pi}_1)} + K_m \le \sum_{i=1}^{n-m} \alpha_{m+i} e^{-\sum_{j=1}^{i} Y_j(\underline{\pi}_2)}\right] \tag{31}$$

for a range of reserves K_m. In Fig. 1 our lower and upper bound approximations are compared to simulated results. We observe that both approximations perform very well, especially the lower bound. The figure also illustrates the intuitive fact that increasing the additional reserve K_m decreases the probability of shortfall. As a second example, suppose a more conservative strategy is followed after 5 years. More precisely, assume that $\mu(\underline{\pi}_1) = 0.05, \sigma(\underline{\pi}_1) = 0.1, \mu(\underline{\pi}_2) = 0.02$ and $\sigma(\underline{\pi}_2) = 0.01$. Computing the probability of shortfall (31) for different reserves K_m leads to Fig. 2. In this second example we see that our approximations are even closer to the simulated results, as it is almost not possible to distinguish the lines. Detailed numerical results of these examples can be found in Tables 5 and 6 in the Appendix.

2.4. Application to optimal portfolio selection

We can easily use our results in an optimal portfolio selection setting. In this section, we describe two possible optimization problems. For example, suppose we have an initial capital K_0 available at time 0, and suppose we know that we will add an extra capital K_m at time m. Suppose also that the investment strategy followed during the first m years is fixed, and is given by $\underline{\pi}_1$. In this case we can optimize the investment strategy to be followed from year m on. The optimal portfolio is the one leading to a maximal survival probability p^*:

$$p^* = \max_{\underline{\pi}} \Pr\left[K_0 e^{\sum_{i=1}^{m} Y_i(\underline{\pi}_1)} - \sum_{i=1}^{m} \alpha_i e^{\sum_{j=i+1}^{m} Y_j(\underline{\pi}_1)} + K_m \ge \sum_{i=1}^{n-m} \alpha_{m+i} e^{-\sum_{j=1}^{i} Y_j(\underline{\pi})}\right]. \tag{32}$$

As a second and perhaps more relevant optimization, suppose we have given an initial capital K_0 and a desired level of ruin probability ϵ. Suppose again that the investment strategy followed during the first m years is fixed, and is given by $\underline{\pi}_1$. We

Table 1
Minimal reserves K_5^* and optimal strategies for given certainty levels ϵ.

	ϵ			
	0.15	0.10	0.05	0.01
π_1^*	53.54%	60.78%	68.54%	76.74%
π_2^*	46.46%	39.22%	31.46%	23.26%
$\mu(\underline{\pi}^*)$	7.86%	7.57%	7.26%	6.93%
$\sigma(\underline{\pi}^*)$	12.84%	12.09%	11.39%	10.78%
K_5^*	30.74	46.65	70.78	119.94

can then optimize the investment strategy to be followed from year m on, by looking for the portfolio leading to a minimal additional reserve K_m^* at time m:

$$K_m^* = \min_{\underline{\pi}} Q_{1-\epsilon}\left[K_0 e^{\sum_{i=1}^{m} Y_i(\underline{\pi}_1)} - \sum_{i=1}^{m} \alpha_i e^{\sum_{j=i+1}^{m} Y_j(\underline{\pi}_1)} - \sum_{i=1}^{n-m} \alpha_{m+i} e^{-\sum_{j=1}^{i} Y_j(\underline{\pi})}\right]. \tag{33}$$

We refer to Section 2.5.2 for a possible practical application of these optimal portfolio selection problems. In the following paragraph, we illustrate optimization problem (33) numerically.

2.4.1. Numerical illustration

Assume that $n = 20$, and $\alpha_i = 10$ for $i = 1, \ldots, 20$. Suppose we have two risky asset classes available in which we can invest, with drift vector $\underline{\mu}^T = (0.06, 0.10)$, standard deviations $\underline{\sigma}^T = (0.10, 0.20)$ and correlation $\rho_{1,2} = 0.50$. Furthermore, take $K_0 = 150$ and suppose during the first 5 years the investment strategy $\underline{\pi}_1 = (0.75, 0.25)$ is followed, with parameters $\mu(\underline{\pi}_1) = 0.07$ and $\sigma(\underline{\pi}_1) = 0.15$. In this paragraph, we use the lower bound approximation as defined in Section 2.2.2 to determine the optimal investment strategy $\underline{\pi}^*$, leading to a minimal reserve K_m^*, as described by (33). As illustrated in Section 2.3, this lower bound approximation is in general significantly more accurate than the upper bound.

Assuming that $m = 5$, the results of our optimization for different values of ϵ are given in Table 1. These results show that increasing the certainty level leads to a more conservative optimal strategy, and a higher required additional reserve K_5^* at time 5. For example, if from year 5 on the strategy (0.6061, 0.3939) is followed, and if an amount of 46.65 is put on the account at time 5, there is 90% certainty that all liabilities can be paid. Following any other investment strategy, or adding less than 46.65 at time 5, would lead to a survival probability of less than 90%.

2.5. Interpretations of additional reserve

In this section we give interpretations for the reserve K_m, illustrated with numerical examples. Throughout this section, we use the comonotonic lower bound approximations derived in the previous sections to solve the optimization problems.

2.5.1. Effect of a temporary change in market parameters

Estimating the additional reserve K_m can be useful to quantify the effect of temporary changes in the market parameters. Suppose an insurance company has determined its investment portfolio using long-term estimates for the parameters describing the financial market. To estimate the influence of a temporary change in market parameters, assume that during the first m years the market behaves differently, with different parameters μ and σ.

If we assume a temporary improvement of market conditions (asset classes with higher drifts and/or lower variances), the reserve K_m as defined by (15) can be interpreted as the amount of money that will be available on the account at time m due to these favorable short-term market conditions (assuming that we use the same ruin probability ε).

Similarly, if we would assume temporary adverse market conditions, the reserve K_m is an estimate of the amount of money the insurer will have to put on the account at time m in order to recover from this short-term "crisis".

Also, the insurer can see how these temporary (un)favorable parameters change its optimal investment strategy: assuming that the market behaves unusually well (bad) during the first years, how will the optimal investment strategy look like afterward. For a given reserve K_m (e.g. $K_m = 0$), the influence of (un)favorable temporary market conditions on the ruin probability ε can also be investigated.

Example. Take $K_0 = 175$, $n = 30$, and $\alpha_i = 10$ for $i = 1, \ldots, 30$, and assume that we have the 2 asset classes as in Section 2.4.1. Maximizing the survival probability, which is the probability of being able to pay all the liabilities, leads to an optimal investment strategy $\underline{\pi} = (0.5804, 0.4196)$, with $\mu(\underline{\pi}) = 0.0768$, $\sigma(\underline{\pi}) = 0.1236$ and corresponding maximal survival probability 85%. Note that this is a survival probability over the whole investment period of 30 years, corresponding to a yearly survival probability of approximately 99.46%. Assume that the insurer invests according to this optimal strategy.

Table 2
Reserve K_m at time m in the case of (un)favorable short-term market conditions.

m	2	3	4	5	10	15
$2 * \underline{\sigma}$	28.68	39.53	49.17	58.02	96.29	132.04
$\underline{\mu} - 2\%$	7.07	11.00	14.89	18.73	37.75	57.27
$\underline{\mu} - 1\%$	3.66	5.99	8.31	10.61	21.96	33.66
$\underline{\mu} + 1\%$	−3.34	−4.45	−5.60	−6.83	−14.95	−27.30
$\underline{\mu} + 2\%$	−6.94	−9.89	−12.96	−16.20	−36.50	−66.41
$0.5 * \underline{\sigma}$	−5.68	−8.06	−10.38	−12.69	−24.53	−38.58

Table 3
Survival probability in the case of (un)favorable short-term market conditions ($K_m = 0$).

m	2	3	4	5	10	15
$2 * \underline{\sigma}$	74.6%	71.3%	68.8%	66.8%	60.4%	56.7%
$\underline{\mu} - 2\%$	82.1%	80.5%	78.9%	77.4%	71.5%	67.6%
$\underline{\mu} - 1\%$	83.6%	82.7%	81.8%	81.1%	78.2%	76.5%
$\underline{\mu} + 1\%$	86.2%	86.6%	86.9%	87.1%	88.4%	89.5%
$\underline{\mu} + 2\%$	87.4%	88.2%	88.9%	89.6%	92.0%	93.5%
$0.5 * \underline{\sigma}$	88.4%	89.8%	91.0%	92.0%	95.8%	97.7%

Table 4
Minimal reserves K_m^* and optimal strategies for different values of m.

	m				
	2	3	4	5	10
π_1^*	52.26%	50.75%	49.25%	48.24%	45.23%
π_2^*	47.74%	49.25%	50.75%	51.76%	54.77%
$\mu(\underline{\pi}^*)$	7.91%	7.97%	8.03%	8.07%	8.19%
$\sigma(\underline{\pi}^*)$	12.98%	13.14%	13.31%	13.43%	13.78%
K_m^*	3.59	5.38	6.94	8.35	13.90

Suppose the insurer wants to check the influence of unusual short-term market conditions. In Table 2, additional reserves K_m are given for different market assumptions, and different values of m. In all examples, the survival probability is 85%. For example, if every asset class has a drift 2% higher than normal for a period of 5 years, an amount of 16.20 can be withdrawn from the account at time 5. If the standard deviations of the asset classes are double for a period of 10 years, the insurer will have to put 96.29 on the account at time 10 in order to keep the same survival probability of 85%.

In Table 3 the influence of a change in short-term market conditions on the survival probability is illustrated. Suppose the insurer does not want to invest extra money at time m, or $K_m = 0$. We see from the table that the more (un)favorable the market conditions are, and the longer these conditions last, the higher (lower) the survival probability becomes.

2.5.2. Postponing rebalancing of investment portfolio

Suppose an insurance company knows that its current investment portfolio is not optimal. Assume however that the insurer does not want to change to a different investment strategy immediately, but prefers to wait for a period of m years. In this case, the reserve K_m as defined by (15) can serve as an estimate of the cost of postponing the rebalancing (incorporating, of course, a certain ruin probability). Also the influence of postponing rebalancing on the optimal investment strategy can be investigated.

Similarly, suppose the insurer knows how much money he will have available in m years to put on its account (e.g. $K_m = 0$ if he does not want to invest extra money). In that case, the insurer can determine the influence of postponing the rebalancing on the ruin probability and on the optimal investment strategy. This way the insurer can get an idea of the maximum number of years m for which postponing changing its investment strategy is acceptable.

Example. Take $K_0 = 175$, $n = 30$, and $\alpha_i = 10$ for $i = 1, \ldots, 30$, and assume that we have the two asset classes as described in Section 2.4.1. Maximizing the survival probability over the whole investment period leads to an optimal investment strategy $\underline{\pi} = (0.5804, 0.4196)$, with $\mu(\underline{\pi}) = 0.0768$, $\sigma(\underline{\pi}) = 0.1236$ and corresponding maximal survival probability 85%.

Suppose the insurer has currently an investment portfolio given by $(0.25, 0.75)$, with corresponding drift 0.09 and standard deviation 0.1639. In other words, the insurer's current portfolio is more risky than the optimal one. Suppose the insurer does not want to rebalance immediately, but would like to keep its current strategy for m years. Also suppose the insurer would still like to have a survival probability of 85%. Using (33) we can determine the optimal investment strategy $\underline{\pi}^*$, to be followed from time m on, leading to a minimal additional reserve K_m^*. This minimal reserve will always be positive: since the insurer does not invest optimally during the first m years, he can not reach the survival probability of 85% (which is the maximal survival probability) without investing an extra amount of money at time m. For different values of m, the results are given in Table 4. For example, for $m = 5$ we find as a result $\underline{\pi}^* = (0.4824, 0.5176)$, with $\mu(\underline{\pi}^*) = 0.0807$ and

Table 5
Comparison of the upper bound and lower bound to simulated results, with $\mu(\underline{\pi}_1) = \mu(\underline{\pi}_2) = 0.05$ and $\sigma(\underline{\pi}_1) = \sigma(\underline{\pi}_2) = 0.1$.

K_m	Upper bound	Simulation	Lower bound
−250	0.99495	0.99644	0.99589
−240	0.99319	0.99512	0.99443
−230	0.99085	0.99340	0.99248
−220	0.98776	0.99115	0.98989
−210	0.98372	0.98803	0.98647
−200	0.97846	0.98383	0.98199
−190	0.97168	0.97851	0.97615
−180	0.96301	0.97153	0.96861
−170	0.95205	0.96245	0.95897
−160	0.93832	0.95100	0.94676
−150	0.92135	0.93660	0.93147
−140	0.90063	0.91839	0.91256
−130	0.87569	0.89626	0.88947
−120	0.84614	0.86913	0.86168
−110	0.81170	0.83670	0.82879
−100	0.77227	0.79913	0.79051
−90	0.72801	0.75489	0.74680
−80	0.67930	0.70569	0.69788
−70	0.62686	0.65106	0.64431
−60	0.57165	0.59203	0.58699
−50	0.51485	0.53085	0.52714
−40	0.45780	0.46770	0.46621
−30	0.40183	0.40432	0.40581
−20	0.34822	0.34431	0.34751
−10	0.29803	0.28737	0.29276
0	0.25206	0.23562	0.24268
10	0.21083	0.18979	0.19807
20	0.17456	0.15064	0.15929
30	0.14320	0.11729	0.12636
40	0.11652	0.09065	0.09900
50	0.09413	0.06881	0.07670
60	0.07557	0.05187	0.05885
70	0.06035	0.03886	0.04476
80	0.04798	0.02886	0.03380
90	0.03801	0.02123	0.02537
100	0.03002	0.01563	0.01894
110	0.02366	0.01136	0.01409
120	0.01861	0.00830	0.01044
130	0.01462	0.00611	0.00772
140	0.01148	0.00440	0.00570
150	0.00900	0.00322	0.00420
160	0.00706	0.00231	0.00309
170	0.00554	0.00172	0.00228
180	0.00435	0.00126	0.00168
190	0.00342	0.00091	0.00124
200	0.00268	0.00066	0.00091
210	0.00211	0.00047	0.00067
220	0.00166	0.00037	0.00050
230	0.00131	0.00027	0.00037
240	0.00104	0.00020	0.00027
250	0.00082	0.00014	0.00020

$\sigma(\underline{\pi}^*) = 0.1343$. The minimal additional reserve at time 5 amounts to $K_5^* = 8.35$. In other words, if the insurer wants to postpone rebalancing for 5 years, and if he wants to keep the same survival probability of 85%, we estimate that he has to invest an additional amount of 8.35 at time 5, and change to the strategy $\underline{\pi}^*$. If he does not invest according to $\underline{\pi}^*$, he will have to put more than 8.35 on the account at time 5 to have a survival probability of 85%.

Note that the optimal strategies $\underline{\pi}^*$ in Table 4 are more risky than the optimal strategy $\underline{\pi}$ above. However, it might not be meaningful to compare these strategies as they are obtained using different optimization criteria.

From the results in Table 4 we can see that increasing m, hence delaying the moment of rebalancing, leads to an increase in the additional reserve K_m^*. Also we see that the optimal strategy to be followed from time m on becomes more risky for increasing m. Our results allow us to solve several other interesting optimization problems. For example, another possible optimization problem would be to fix an investment portfolio for the first m years, and to choose a value of K_m (e.g. $K_m = 0$), and to determine the investment strategy to be followed from time m on such that the survival probability is maximized. For $K_m = 0$, this maximal probability would be less than 85%, unless the insurer invests according to the optimal strategy $\underline{\pi} = (0.5804, 0.4196)$ as explained above.

Table 6
Comparison of the upper bound and lower bound to simulated results, with $\mu(\underline{\pi}_1) = 0.05$, $\sigma(\underline{\pi}_1) = 0.1$, $\mu(\underline{\pi}_2) = 0.02$ and $\sigma(\underline{\pi}_2) = 0.01$.

K_m	Upper bound	Simulation	Lower bound
−250	0.99943	0.99947	0.99944
−240	0.99922	0.99925	0.99923
−230	0.99892	0.99897	0.99894
−220	0.99851	0.99858	0.99854
−210	0.99795	0.99807	0.99798
−200	0.99718	0.99733	0.99723
−190	0.99614	0.99631	0.99620
−180	0.99472	0.99499	0.99480
−170	0.99280	0.99312	0.99291
−160	0.99022	0.99059	0.99037
−150	0.98678	0.98724	0.98696
−140	0.98219	0.98291	0.98243
−130	0.97612	0.97684	0.97643
−120	0.96816	0.96919	0.96854
−110	0.95779	0.95924	0.95827
−100	0.94441	0.94593	0.94499
−90	0.92733	0.92900	0.92802
−80	0.90576	0.90784	0.90657
−70	0.87887	0.88127	0.87979
−60	0.84581	0.84864	0.84684
−50	0.80584	0.80854	0.80693
−40	0.75835	0.76116	0.75946
−30	0.70308	0.70566	0.70414
−20	0.64021	0.64259	0.64112
−10	0.57053	0.57252	0.57120
0	0.49554	0.49640	0.49586
10	0.41748	0.41648	0.41738
20	0.33928	0.33760	0.33870
30	0.26427	0.26126	0.26321
40	0.19581	0.19192	0.19437
50	0.13681	0.13258	0.13513
60	0.08922	0.08476	0.08749
70	0.05366	0.04952	0.05208
80	0.02934	0.02619	0.02806
90	0.01434	0.01205	0.01343
100	0.00614	0.00477	0.00559
110	0.00225	0.00155	0.00196
120	0.00069	0.00041	0.00056
130	0.00017	0.00008	0.00013
140	0.00003	0.00001	0.00002
150	0.00000	0.00000	0.00000
160	0.00000	0.00000	0.00000
170	0.00000	0.00000	0.00000

2.6. Conclusion

In this paper we discussed a general provisioning problem. We derived approximations that can be used to determine an estimate at time 0 of the provision to be set up at a certain time in future, to ensure, after having paid the first liabilities, that all future liabilities can be fulfilled, incorporating a specified (low) ruin probability. We derived a convex lower and an upper bound based on comonotonicity to determine an accurate and easily computable approximation for this reserve. We applied our results in an optimal portfolio selection framework, and illustrated it with numerical examples.

We have seen that the general provisioning problem can be useful in practice. As a first plausible interpretation, the additional reserve can be used to quantify the effect of temporary changes in market conditions. We have seen for example that such changes can significantly influence the long-term survival probability. Secondly, the setting discussed in this paper can be used to see if and how long postponing rebalancing of the investment portfolio can be justified.

This paper gives an interesting extension to comonotonic approximations, and their application to optimal portfolio selection. Future research would for example consist in deriving analytical expressions for other, more general risk measures such as Tail Value-at-risk (TVaR) or Conditional Tail Expectation (CTE). Other future work could consist in trying to combine the results of [24] with the results of this paper, solving more general life-cycle investment problems.

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Appendix. Numerical results

The tables in this appendix contain the numerical results of the examples of Section 2.3, comparing our lower bound and upper bound approximations to results obtained using simulation. Table 5 contains the results of the first example, where $\mu(\underline{\pi}_1) = \mu(\underline{\pi}_2) = 0.05$ and $\sigma(\underline{\pi}_1) = \sigma(\underline{\pi}_2) = 0.1$. Table 6 contains the results of the second example, where $\mu(\underline{\pi}_1) = 0.05$, $\sigma(\underline{\pi}_1) = 0.1$, $\mu(\underline{\pi}_2) = 0.02$ and $\sigma(\underline{\pi}_2) = 0.01$.

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