Equivalence Theorems and Hopf–Galois Extensions

Claudia Menini* and Monica Zuccoli

Dipartimento di Matematica, Università di Ferrara, Via Machiavelli 35,
44100 Ferrara, Italy

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1. INTRODUCTION

In this article we apply equivalence theorems for categories of modules to the category $\mathcal{M}(H)_A^D$ of right $(D - A)$-Hopf modules to get a characterization of Hopf–Galois extensions.

Indeed let $P, Q \in \text{Mod}_R$ and $T = \text{End}(P_R)$, and assume that $P$ belongs to the category $\sigma(Q_R)$ of modules subgenerated by $Q_R$. In Theorem 2.3, by modifying a theorem formulated by Dal Pio and Orsatti [5, Theorem 2.6], we study the situation where, for every $M$ belonging to the category $\sigma(Q_R)$, the map

$$
\Psi_M : \text{Hom}_R(P, M) \otimes T P \rightarrow M
$$

$$
\xi \otimes x \mapsto \xi(x)
$$

is an isomorphism in $\text{Mod}_R$. In particular we prove that this holds iff $\Psi_M$ is an isomorphism for every $M \in \text{Gen}(Q_R)$ and $T P$ is flat iff $\sigma(Q_R) = \sigma(P_R)$ and $P_R$ generates all submodules of $P_R^n$ for every $n \in \mathbb{N}, n > 0$. Let now $H$ be a Hopf algebra over a field $k$, let $A$ be a right $H$-comodule algebra, let $D$ be a right $H$-module coalgebra, and consider the category $\mathcal{M}(H)_A^D$ of right $(D - A)$-Hopf modules, i.e., of those $D$-comodules which are equipped with a suitable right $A$-module structure in such a way that

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these two structures are compatible (see Section 3.1). The starting point for applying the previous result in this setting is that (see Lemma 3.9) $\mathbb{M}(H)^D_A = \sigma(\mathbb{A}_D^* \otimes A)$ as remarked by Caenepeel and Raianu in [3].

Moreover, using a grouplike element $x \in D$, $A$ can be endowed with a suitable $D$-comodule structure in such a way that $A \in \mathbb{M}(H)^D_A$ (see Section 3.13). Since $\text{End}_D(\mathbb{A}_D^* A) \cong A$, (see Theorem 3.15), we may apply the quoted result to the situation where $R = A \#^H D^*$, which is essentially the smash product introduced by Doi in [6], $P_R = A \#^H D^* A$, $Q_R = A \#^D A \otimes D$, and $T = A$. Now if $A, A$ is flat, $\Psi_M$ is an isomorphism for every $M \in \text{Gen}(A_A \otimes D)$ iff $\Psi_{A_D^* A}$ is an isomorphism (see Lemma 3.22) and $\Psi_{A_D^* A}$ is an isomorphism iff

$$\beta_1 : \ A \otimes A \to A \otimes D$$

$$a \otimes b \mapsto \sum a b_0 \otimes x \leftarrow b_1$$

is an isomorphism, i.e., iff $A \subset A$ is a right Hopf-Galois extension.

In this way we get Theorem 3.27, which characterizes right Hopf-Galois extensions $A \subset A$ such that $A_A$ is flat, using the structure of $A$ as a left $A \# D^*$-module. In particular it is shown that these extensions are exactly those for which the weak structure theorem in the sense of [7] holds. Also Theorem 3.27 enables us to apply the well-known Fuller theorem on equivalences of modules (see Theorem 2.5) in our setting. Thus we get Theorem 3.29, which characterizes right Hopf-Galois extensions $A \subset A$ such that $A_A$ is faithfully flat. In particular we prove that this holds iff $A_A \# D^*$ is a quasiprogenerator and $\sigma(\mathbb{A}_D^* A) = \sigma(\mathbb{A}_D^* A \otimes D)$ iff $(-)_A : \mathbb{M}(H)^D_A = \sigma(\mathbb{A}_D^* A \otimes D) \to \text{Mod-}A$ is an equivalence, i.e., the strong structure theorem in the sense of [7] holds. In this case when $D = H$ we thus get part of the famous Schneider’s theorem on Hopf-Galois extensions. Moreover, using the adjointness between the induction functor $F$ and the functor $G = - \Box H$ as described by Caenepeel and Raianu [3], we easily get as a corollary (Corollary 3.32) Doi’s theorem on Hopf-Galois extensions [6, Theorem 2.3].

When $\dim_D(D) < \infty$, then $\sigma(\mathbb{A}_D^* A \otimes D) = A \# D^* \text{Mod}$ as remarked by Doi [6]. Therefore in this case $A$ is a generator of $\sigma(\mathbb{A}_D^* A \otimes D)$ if and only if $A$ is a generator in $A \# D^* \text{Mod}$. Hence this is equivalent, in
the finite case, to $\mathcal{A} \subset \mathcal{A}$ being a right Hopf–Galois extension and $\mathcal{A}$ being flat. Theorem 4.2 characterizes this situation. Also $\mathcal{A}^{\#D^*} \mathcal{A}$ is a quasiprogenerator and $\sigma(\mathcal{A}^{\#D^*} \mathcal{A}) = \sigma(\mathcal{A}^{\#D^*} \mathcal{A} \otimes D)$ in the finite case if and only if $\mathcal{A}^{\#D^*} \mathcal{A}$ is a progenerator. Theorem 4.3 characterizes this situation.

When $D = H$ and $\dim_k(H) < \infty$, $B = A^{\text{GH}} \subset \mathcal{A}$ is a right Hopf–Galois extension if and only if $\mathcal{A}^{\#H^*} \mathcal{A}$ is a generator for the category $\mathcal{A}^{\#H^*} \text{-}\text{Mod}$. Theorem 4.7 characterizes this situation. Here we prove that, in this case, $\mathcal{A}$ is a Frobenius extension of $B$. Theorem 4.7 contains essentially the famous result formulated by Cohen, Fischman, and Montgomery for Hopf–Galois extensions [4, Theorem 1.2 and Theorem 1.2']. Finally, using Theorem 4.7, we can prove that the assumption of faithfully flatness can be weakened in the finite case whenever $D = H$. In fact if $B \subset A$ is a right Hopf–Galois extension, then $\mathcal{A}$ is faithfully flat if and only if $\mathcal{A}$ is a weak generator. Theorem 4.8 characterizes such extensions. Part of this result appears in [4, Theorem 2.2].

We have made some efforts to get a paper that even someone not acquainted with Hopf algebra theory could read. Toward this end we have inserted some details that a Hopf-algebra expert would skip.

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2. PRELIMINARIES

2.1

Throughout this article, all rings have a nonzero identity and all modules are unital. Let $\mathcal{R}$ be a ring, $\mathcal{P}_\mathcal{R}$ a right $\mathcal{R}$-module. We denote by $\text{Gen}(\mathcal{P}_\mathcal{R})$ the full subcategory of $\text{Mod-}\mathcal{R}$ generated by $\mathcal{P}_\mathcal{R}$. The objects of $\text{Gen}(\mathcal{P}_\mathcal{R})$ are the right $\mathcal{R}$-modules $\mathcal{M}_\mathcal{R}$ which are generated by $\mathcal{P}_\mathcal{R}$, i.e., such that there exists a surjective morphism $\mathcal{P}_\mathcal{R}^{(X)} \rightarrow \mathcal{M} \rightarrow 0$ for a suitable set $X$. $\text{Gen}(\mathcal{P}_\mathcal{R})$ is closed under taking epimorphic images and direct sums. Partially following [17], we denote by $\sigma(\mathcal{P}_\mathcal{R})$ the full subcategory of $\text{Mod-}\mathcal{R}$ whose objects are the right $\mathcal{R}$-modules subgenerated by $\mathcal{P}_\mathcal{R}$, i.e., such that there exists an injective morphism $\mathcal{M} \hookrightarrow \mathcal{L}$ for a suitable $\mathcal{L} \in \text{Gen}(\mathcal{P}_\mathcal{R})$. $\sigma(\mathcal{P}_\mathcal{R})$ is closed under taking submodules, epimorphic images and direct sums. Clearly $\text{Gen}(\mathcal{P}_\mathcal{R}) = \sigma(\mathcal{P}_\mathcal{R})$ if and only if $\text{Gen}(\mathcal{P}_\mathcal{R})$ is closed under submodules.
2.2. Theorem. Let \( P_R \in \text{Mod-}R, \ T = \text{End}(P_R) \). Then the following are equivalent.

(a) For every \( n \in \mathbb{N}, \ n > 0, \ P_R \) generates all submodules of \( P_R^n \).
(b) \( \text{Gen}(P_R) = \sigma(P_R) \).
(c) For every \( M \in \sigma(P_R) \) the map

\[
\Psi_M: \ \text{Hom}_R(P, M) \otimes_T P \to M
\]

\[
\xi \otimes x \mapsto \xi(x)
\]

is an isomorphism in \( \text{Mod-}R \).
(d) For every \( M \in \sigma(P_R), \ \Psi_M \) is a surjective morphism in \( \text{Mod-}R \).
(e) \( \tau P \) is flat and the functor

\[
\text{Hom}_R(\tau P, -): \text{Gen}(P_R) \to \text{Mod-}T
\]

is full and faithful.

Moreover if these conditions are fulfilled \( H = \text{Hom}_R(\tau P_R, -) \) induces an equivalence between \( \text{Gen}(P_R) = \sigma(P_R) \) and \( \operatorname{Im}(H) \).

Proof. (a) \( \Rightarrow \) (b) follows from (1) \( \Rightarrow \) (2) of Lemma 1.4 in [18].
(b) \( \Rightarrow \) (c) follows from (1) of Lemma 1.3 in [18].
(c) \( \Rightarrow \) (b), (c) \( \Rightarrow \) (d), and (d) \( \Rightarrow \) (b) are trivial.
(b) \( \Rightarrow \) (e) follows from (2) \( \Rightarrow \) (3) of Lemma 1.4 in [18].
(e) \( \Rightarrow \) (a) follows from (3) \( \Rightarrow \) (1) of Lemma 1.4 in [18].

Note that the equivalences (a) \( \Leftrightarrow \) (b) \( \Leftrightarrow \) (e) and the last assertion appear in [5, Theorem 2.6].

2.3. Theorem. Let \( P, Q \in \text{Mod-}R, \ let \ T = \text{End}(P_R), \ and \ assume \ that \ P \in \sigma(Q_R) \). Then the following are equivalent.

(a) For every \( M \in \text{Gen}(Q_R) \)

\[
\Psi_M: \ \text{Hom}_R(P, M) \otimes_T P \to M
\]

\[
\xi \otimes x \mapsto \xi(x)
\]

is an isomorphism in \( \text{Mod-}R \) and \( \tau P \) is flat.
(b) For every \( M \in \sigma(Q_R), \ \Psi_M \) is an isomorphism in \( \text{Mod-}R \).
(c) $\sigma(Q_R) = \text{Gen}(P_R)$.
(d) $\tau P$ is flat, $\sigma(Q_R) = \sigma(P_R)$ and the functor

$$\text{Hom}_R(P, -) : \text{Gen}(P_R) \to \text{Mod}_T$$

is full and faithful.

(e) $\sigma(Q_R) = \sigma(P_R)$ and $P_R$ generates all submodules of $P_R^n$, for every $n \in \mathbb{N}, n > 0$.

Proof. (a) $\Rightarrow$ (b) Let $M \in \sigma(Q_R)$. Then there exist a right $R$-module $N \in \text{Gen}(Q_R)$ and an injective morphism $i : M \to N$. By applying the functor $H = \text{Hom}_R(P, -)$ to the exact sequence

$$0 \longrightarrow M \overset{i}{\longrightarrow} N \overset{\pi}{\longrightarrow} N/M \longrightarrow 0,$$

where $\pi$ denotes the canonical projection, we get the following exact sequence:

$$0 \longrightarrow H(M) \overset{H(i)}{\longrightarrow} H(N) \overset{H(\pi)}{\longrightarrow} H(N/M).$$

By the flatness of $\tau P$, we obtain the following exact sequence:

$$0 \longrightarrow H(M) \otimes P \overset{f}{\longrightarrow} H(N) \otimes P \overset{h}{\longrightarrow} H(N/M) \otimes P,$$

where $f = H(i) \otimes \text{id}_P$ and $h = H(\pi) \otimes \text{id}_P$. Consider the following commutative diagram:

$$\begin{array}{ccc}
0 & \longrightarrow & H(M) \\
\Psi_M & \downarrow & \\
M & \overset{i}{\longrightarrow} & N \\
\Psi_N/\Psi_M & \downarrow & \\
N/M & \longrightarrow & 0,
\end{array}$$

where $\Psi_N$ and $\Psi_{N/M}$ are isomorphisms as both $N$ and $N/M \in \text{Gen}(Q_R)$. It follows that $\Psi_M$ is an isomorphism too.

(b) $\Rightarrow$ (c) Since $P \in \sigma(Q_R)$, $\text{Gen}(P_R) \subseteq \sigma(Q_R)$. As $\Psi_M$ is an isomorphism for every $M \in \sigma(Q_R)$, $\sigma(Q_R) \subseteq \text{Gen}(P_R)$.

(c) $\Rightarrow$ (d) Since $P \in \sigma(Q_R)$, we get $\text{Gen}(P_R) \subseteq \sigma(P_R) \subseteq \sigma(Q_R) = \text{Gen}(P_R) \subseteq \sigma(P_R)$. Hence $\text{Gen}(P_R) = \sigma(P_R)$ and we can apply (b) $\Rightarrow$ (e) of Theorem 2.2.

(d) $\Leftrightarrow$ (e) follows from (e) $\Leftrightarrow$ (a) in Theorem 2.2.

(d) $\Rightarrow$ (a) By (e) $\Leftrightarrow$ (c) in Theorem 2.2, $\Psi_M$ is an isomorphism for every $M \in \sigma(P_R) = \sigma(Q_R) \supseteq \text{Gen}(Q_R)$.
2.4. Definition. Let \( R \) be a ring, \( M \) a left \( R \)-module. \( R \)-\( M \) is called a weak generator if \( Y \otimes M = 0 \) for every right \( R \)-module \( Y \) implies \( Y = 0 \).

Recall that a module \( P_R \) is called a quasiprogenerator if \( P_R \) is finitely generated, is quasiprojective, and generates each of its submodules.

2.5. Theorem [2, 8]. Let \( \tau P_R \) be a bimodule. Then the following are equivalent.

(a) \( \text{Hom}_R(P, -): \sigma(P_R) \to \text{Mod}-T \) and \( - \otimes_P: \text{Mod}-T \to \sigma(P_R) \)
are inverse category equivalences.

(b) \( \text{Hom}_R(P, -): \sigma(P_R) \to \text{Mod}-T \) is a category equivalence.

(c) \( - \otimes_P: \text{Mod}-T \to \sigma(P_R) \) is a category equivalence.

(d) The map
\[
\Psi_M: \text{Hom}_R(P, M) \otimes_P \to M
\]
\[
\xi \otimes x \mapsto \xi(x)
\]
is an isomorphism for every \( M \in \sigma(P_R) \) and
\[
\Psi'_N: N \to \text{Hom}_R\left(P, N \otimes_P \right)
\]
\[
n \mapsto \left( \Psi'_N(n): P \to N \otimes_P \right)
\]
is an isomorphism for every \( N \in \text{Mod}-T \).

(e) \( \Psi_M \) is an isomorphism for every \( M \in \sigma(P_R) \) and \( \tau P \) is faithfully flat.

(f) \( \Psi_M \) and \( \Psi'_{T/I} \) are isomorphisms for every \( M \leq P_R \) and every \( I \leq T_T \).

(g) \( P_R \) is quasiprojective and generates each of its submodules, \( \tau P \) is a weak generator, and \( T \equiv \text{End}(P_R) \) canonically.

(h) \( P_R \) is a quasiprogenerator and \( T \equiv \text{End}(P_R) \) canonically.

(i) \( \tau P \) is a weak generator, \( \Psi_M \) is an isomorphism for every \( M \in \text{Gen}(P_R) \), and \( T \equiv \text{End}(P_R) \) canonically.

Proof. (a) \( \iff \) (d) \( \iff \) (f) are in Theorem 2.6 in [8].

(a) \( \iff \) (b) \( \iff \) (c) follow by adjoint properties of \( \text{Hom}_R(P, -) \) and \( - \otimes_P \).
(h) ⇒ (i) Since we already know that (h) ⇒ (d), we get that $\text{Gen}(P_R) = \sigma(P_R)$ so that (4) ⇒ (3) of Theorem 10 in [2] applies.

(i) ⇒ (h) By (3) ⇒ (4) of Theorem 10 in [2].

(f) ⇒ (g) By (c) ⇒ (d) of Theorem 2.6 in [8].

(g) ⇒ (e) and (g) ⇒ (f) By Lemma 2.2 in [8] we get $\sigma(P_R) = \text{Gen}(P_R)$ so that, by Lemma 2.1 in [8], $\tau P$ is flat and hence faithfully flat, being a weak generator. Now apply (d) ⇒ (c) and (d) ⇒ (b) of Theorem 2.6 in [8].

(e) ⇒ (i) Since $\Psi_P$ is an isomorphism and $\tau P$ is faithfully flat it is easy and straightforward to prove that the canonical morphism $T \rightarrow \text{End}(P_R)$ is bijective.

We recall now the celebrated Morita theorem.

2.6. Theorem [12]. Let $P_R \in \text{Mod}-R, T = \text{End}(P_R)$. Then the following are equivalent.

(a) $P_R$ is a faithful quasiprogenerator; $\tau P$ is finitely generated.
(b) $P_R$ is a progenerator.
(c) $\tau P$ is a progenerator and $\tau P_R$ is faithfully balanced.
(d) The functor map

\[ \text{Hom}_R(P, -) : \text{Mod}-R \rightarrow \text{Mod}-T \]

\[ M \mapsto \text{Hom}_R(P, M) \]

is an equivalence.

Proof. See [12, Theorems 3.2 and 3.4].

3. MAIN RESULTS

3.1

Let $H$ be a Hopf algebra over the field $k$ with comultiplication $\Delta$, counit $\varepsilon$, and antipode $S$. Let $A$ be a right $H$-comodule algebra with structure map $\rho$, and let $B = A^{\text{coH}} = \{ a \in A | \rho(a) = a \otimes 1_A \}$ (see [11] for an explanation and definition). Recall that a right $H$-module coalgebra is a coalgebra $D$ together with a right $H$-module structure $\mu_D : D \otimes H \rightarrow D; d \otimes h \mapsto d \leftarrow h$ such that $\mu_D$ is a coalgebra map, that is,

\[ \Delta_D(d \leftarrow h) = \sum d_1 \leftarrow h_1 \otimes d_2 \leftarrow h_2, \]
\[ \varepsilon_D(d \leftarrow h) = \varepsilon_D(d) \varepsilon(h). \]

Following Doi we introduce the category of right $(D - A)$-Hopf modules denoted by $\mathcal{M}(H)^D_A$ (or simply $\mathcal{M}_A^D$) as follows. An object in $\mathcal{M}(H)^D_A$ is a...
right $D$-comodule with structure map $\rho_M$, endowed with a right $A$-module structure such that, for every $m \in M$ and $a \in A$, $\rho_M(ma) = \sum m_0 a_0 \otimes m_1 \rightarrow a_1$. Clearly a morphism in $\mathcal{M}(H)^D_A$ is defined to be both a morphism of right $D$-comodules and of right $A$-modules.

The definition of left $H$-module coalgebra is given in a similar way. Let $D$ be a left $H$-module coalgebra, $A$ a right $H$-comodule algebra. The category of right–left $(D - A)$-Hopf modules denoted by $\mathcal{M}(H)^D_A$ (or simply $\mathcal{M}(H)^D$) is defined as follows. An object in $\mathcal{M}(H)^D_A$ is a right $D$-comodule endowed with a left $A$-module structure such that, for every $m \in M$ and $a \in A$, $\rho_M(ma) = \sum m_0 a_0 \otimes m_1 \rightarrow a_1$.

3.2. Example. For every $M \in \text{Mod-}A$, $M \otimes D \in \mathcal{M}(H)^D_A$ via $\rho_M \circ \Delta = \text{id}_M \otimes \Delta_D$ and $m \otimes d \cdot a = \sum ma_0 \otimes d \rightarrow a_1$, $m \in M$, $d \in D$, $a \in A$.

Moreover if $M \in \mathcal{M}(H)^D_A$ it is easy to prove that $\rho_M: M \rightarrow M \otimes D$ is a morphism in $\mathcal{M}(H)^D_A$.

3.3. Definition. Let $A$ be a right $H$-comodule algebra, $D$ a right $H$-module coalgebra. Let $A^* = A \otimes D^*$ as a $k$-vector space and let $a \# \gamma = a \otimes \gamma$ for every $a \in A$, $\gamma \in D^*$. Given $a, b \in A$, $\gamma, \chi \in D^*$, set

$$a \# \gamma \cdot b \# \chi = \sum b_0 a_0 \# (b_1 \rightarrow \gamma) \chi,$$

where $\langle h \rightarrow \gamma, d \rangle = \langle \gamma, d \leftarrow h \rangle$ for every $h \in H$, $d \in D$, $\gamma \in D^*$. By linearity this defines a multiplication in $A^* \circ D^*$ which becomes a ring with identity $1 \# \gamma_0$. The ring $A^* \circ D^*$ will be called the smash product of $A$ and $D^*$ (over $H$).

3.4. Definition. Let $A$ be a right $H$-comodule algebra, $D$ a left $H$-module coalgebra. Similarly $A^* \circ D^*$ is defined by setting

$$a \# \gamma \cdot b \# \chi = \sum ab_0 \# (\gamma \leftarrow b_1) \chi,$$

where $\langle \gamma \leftarrow b_1, d \rangle = \langle \gamma, b_1 \rightarrow d \rangle$ for every $d \in D$.

Note that this is exactly the smash product introduced in [6].

3.5. Remark. Assume that the antipode of $H$ is bijective. In this case the opposite ring $H^{op}$ of $H$ is a Hopf algebra with comultiplication $\Delta$, counit $\varepsilon$, and antipode $S$, the composition inverse of $S$. In this case, given a left $H$-module coalgebra $D$, $D$ can be regarded as a right $H^{op}$-module coalgebra and $A^{op}$ as a right $H^{op}$-comodule algebra. Then it is easy to check that $A^{op} \circ D^* = A^* \circ D^*$. 


3.6. Proposition. Assume that $S$ is bijective. Then the map

$$\Lambda: \left( A \# H^* \right)^{\text{op}}_{H^*} \to A^{\text{op}} \# H^*$$

$$\alpha \# \gamma \mapsto \sum \alpha_0 \# (S(\alpha_1) \to \gamma) \cdot S$$

is a ring isomorphism.

Proof. Using the following equalities

$$S(h_1 g) \to ((h_2 \to \chi) \gamma) = (S(g_2) \to \chi)(S(hg_1) \to \gamma),$$

$$[(S(h) \to \gamma) \cdot S] \leftarrow g = [S(hg) \to \gamma] \cdot S,$$

$$(\chi \gamma) \cdot S = (\gamma \cdot S)(\chi \cdot S),$$

which hold for every $h, g \in H$ and $\gamma, \chi \in H^*$, as it is easily checked, it is straightforward to prove that $\Lambda$ is a ring morphism. Now it is easy to prove that

$$h \mapsto (\chi \cdot S) = (\chi \leftarrow S(h)) \cdot \bar{S},$$

$$(\chi \gamma) \cdot \bar{S} = (\gamma \cdot \bar{S})(\chi \cdot \bar{S}).$$

Using these equalities it can be proved that

$$\Lambda^{-1}: A^{\text{op}} \# H^* \to \left( A \# H^* \right)^{\text{op}}_{H^*}$$

$$\alpha \# \gamma \mapsto \sum \alpha_0 \# (\gamma \leftarrow S(\alpha_1)) \cdot \bar{S}. \tag{\textbf{3.6.1}}$$

3.7. Lemma. Let $A$ a right $H$-comodule algebra, $D$ a right $H$-module coalgebra. The maps

$$f: A^{\text{op}} \to A \# D^*$$

$$a \mapsto a \# e_D$$

and

$$g: D^* \to A \# D^*$$

$$\gamma \mapsto 1_A \# \gamma$$

are injective ring morphisms. In particular every left $A \# D^*$-module is a right $A$-module via $f$. 
3.8

Let $M \in \mathcal{M}(H)_A^D$. $M$ becomes a left $A \# D^*$-module by setting

$$(\alpha \# \gamma) \cdot m = \sum m_0 \alpha \langle \gamma, m_1 \rangle.$$ 

We denote this module by $G(M)$. Moreover, given $M, N \in \mathcal{M}(H)_A^D$ and an abelian group morphism $f: M \to N$, $f$ is a morphism in $\mathcal{M}(H)_A^D$ if and only if $f$ is a morphism of left $A \# D^*$-modules. In this case we denote by $G(f): G(M) \to G(N)$ the map $f$ regarded as a morphism in $\mathcal{M}(H)_A^D$. The assignments $M \mapsto G(M)$, $f \mapsto G(f)$ define a covariant functor

$$G: \mathcal{M}(H)_A^D \to A \# D^*\text{-Mod}.$$ 

In particular, by Example 3.2, for every $M \in \text{Mod-}A$, $M \otimes D$ is a left $A \# D^*$-module via

$$(a \# \gamma) \cdot (m \otimes d) = \sum ma_0 \otimes d_1 \leftarrow a_3 \langle \gamma, d_2 \rangle.$$ 

3.9. Lemma [3, 13].

$$\text{Im}(G) = \sigma\left(A \# D^*, A \otimes D\right).$$

Therefore $G$ induces an equivalence

$$\mathcal{M}(H)_A^D \to \sigma\left(A \# D^*, A \otimes D\right).$$

Proof. Let $M \in \mathcal{M}(H)_A^D$. Then $\rho_M: M \to M \otimes D$ is an injective morphism in $\mathcal{M}(H)_A^D$. Let

$$p_M: A^{(M)} \to M$$

$$\left(a_m\right)_{m \in M} \mapsto \sum_{m \in M} ma_m.$$
Then $p_M$ is a surjective morphism in $\text{Mod-}A$ and it is easy to prove that $p_M \otimes \text{id}_D: A(M) \otimes D \to M \otimes D$ is a surjective morphism in $\mathcal{M}(H)^D_A$. As $A(M) \otimes D \cong (A \otimes D)^M$ in $\mathcal{M}(H)^D_A$, we get that $G(M) \in \sigma(A \# D \otimes A \otimes D)$.

Conversely let $M \in \sigma(A \# D \otimes A \otimes D)$. Then there exist a set $X$, a left $A \# D^*$-module $L$ and a surjective morphism of left $A \# D^*$-modules $\pi$ such that

$$
(A \otimes D)^\leftarrow_X M \xrightarrow{\pi} L
$$

As remarked in Lemma 3.7, $M$ is a right $A$-module via $m \cdot a = (a \# e_0)m$. Let us prove that $M$ is a right $D$-comodule. In fact $(A \otimes D)^\leftarrow_X$ is a right $D$-comodule and hence a rational left $D^*$-module. By Lemma 3.7, $\pi$ is a surjective morphism of left $D^*$-modules so that $L$ is a rational $D^*$-module too. Therefore $L$ is a right $D$-comodule so that both $\pi$ and $i$ are morphisms of right $D$-comodules. Now, by Lemma 3.7, both $\pi$ and $i$ are right $A$-module morphisms. As $\pi$ is surjective and $(A \otimes D)^\leftarrow_X \in \mathcal{M}(H)^D_A$, we get that $L$ itself satisfies the requirement of being a right $(D - A)$-Hopf module, i.e., $L \in \mathcal{M}(H)^D_A$. Since $i$ is injective we get that also $M \in \mathcal{M}(H)^D_A$.

By means of the foregoing theorem, in the following we will often identify $\mathcal{M}(H)^D_A$ with $\sigma(A \# D \otimes A \otimes D)$.

3.10 [6]

Let $D$ be a right $H$-module coalgebra and $x \in D$ a grouplike element. Let

$$
\pi_x: H \to D \quad h \mapsto x - h.
$$

3.11. Proposition. $\pi_x$ is a morphism of right $H$-module coalgebras.

3.12. Remark. Regarding $H$ as a right $H$-module coalgebra we may consider $\mathcal{M}(H)^D_A$, which will be simply denoted by $\mathcal{M}_A^H$. Note that $A \in \mathcal{M}_A^H$. In fact for every $a, b \in A$ we have $\rho(ab) = \sum a_0 b_0 \otimes a_1 b_1$.

3.13

Let $M \in \mathcal{M}_A^H$ and set

$$
\tilde{\rho}_M = (\text{id}_M \otimes \pi_x) \circ \rho_M: M \to M \otimes D
$$

$$
m \mapsto \sum m_0 \otimes x - m_1.
$$
$M$ can be regarded as an object of $\mathcal{M}(H)^D_A$ via $\tilde{\rho}_M$. We will denote this object by $M_{\pi}$. Given a morphism $f: M \to N$ in $\mathcal{M}_H$, it is easy to prove that $f$ is a morphism $M_{\pi} \to N_{\pi}$ in $\mathcal{M}(H)^D_A$. In this way we get a covariant functor

$$\mathcal{M}_H \to \mathcal{M}(H)^D_A,$$

$$M \mapsto M_{\pi}.$$ 

In the following for a given $M \in \mathcal{M}_H$ we will simply write $M$ instead of $M_{\pi}$, whenever no confusion will arise.

3.14. Definition. For every $M \in \mathcal{M}(H)^D_A$ set

$$Ax = \{ a \in A | \sum a_0 \otimes x \leftarrow a_1 = a \otimes x \}.$$ 

In particular $A_x = \{ a \in A | a_0 \otimes x \leftarrow a_1 = a \otimes x \}$. 

In the particular case where $D = H$ and $x = 1_H$, $M_x$ will be denoted by $M_{\text{coh}}$.

3.15. Theorem. Let $A$ a right $H$-comodule algebra, $D$ a right $H$-module coalgebra, and $x \in D$ a grouplike element. Then

1. For every $M \in \mathcal{M}(H)^D_A$,

$$\nu_M: \text{Hom}_{\mathcal{M}_H}(A, M) \to M_x$$

$$f \mapsto f(1_A)$$

is an isomorphism and its inverse is

$$\Omega_M: M_x \to \text{Hom}_{\mathcal{M}_H}(A, M)$$

$$m \mapsto \left( \Omega_M(m): A \to M \right.$$

$$a \mapsto (a \# e_D) \cdot m \left. \right)$$

2. The map

$$\nu_A: \text{End}_{\mathcal{M}_H}(A) \to A_x$$

is a ring isomorphism. Therefore $A$ is an $A \# D^* - A^*_H$-bimodule.

3. For every $M \in \mathcal{M}(H)^D_A$, $\text{Hom}_{\mathcal{M}_H}(A \# D^*, A^{\# D^*}_A, A^{\# D^*}_A, M)$ is a right $A_x$-module and $\nu_M$ is an isomorphism of right $A_x$-modules.
Proof. (1) Let \( f \in \text{Hom}_{A\# D^*}(A, M) \). Then

\[
\rho_M(f(1_A)) = ((f \otimes \text{id}_D) \circ \tilde{\rho}_A)(1_A) \\
= (f \otimes \text{id}_D)(1_A \otimes x) \\
= f(1_A) \otimes x.
\]

Note that the first equality holds as \( f : A \to M \) is a morphism of left \( D^* \)-modules and hence a right \( D \)-comodule morphism. Therefore we get \( f(1_A) \in M_1 \). Let \( m \in M_1 \). We have

\[
(v_M \circ \Omega_M)(m) = (\Omega_M(m))(1_A) = (1_A \# e_D) \cdot m \\
= me_D(x) = m,
\]

\[
((\Omega_M \circ v_M)(f))(a) = (\Omega_M(f(1_A)))(a) = (a \# e_D) \cdot f(1_A) \\
= f((a \# e_D)1_A) = f(a).
\]

(2) We prove that \( \Omega_A \) is a ring morphism. Let \( b, b' \in A_x \), \( a \in A \). We have

\[
(\Omega_A(bb'))(a) = (a \# e_D) \cdot bb' = (bb')a, \\
(\Omega_A(b) \cdot \Omega_A(b'))(a) = \Omega_A(b)((a \# e_D) \cdot b') = \Omega_A(b)(b'a) \\
= (b'a \# e_D) \cdot b = b(b'a).
\]

(3) Let \( b \in A_x \). We will prove that \( v_M(fb) = v_M(f)b \). We have

\[
v_M(fb) = f(b),
\]

by Section 3.8 and

\[
v_M(f)b = f(1_A)b = (b \# e_D)f(1_A) = f((b \# e_D)1_A) = f(b).
\]

3.16

Let \( V \in \text{Mod}-A_x \). Since \( A \) is an \( A_x = (A \# D^*)_{\text{op}} \)-bimodule, \( V \otimes A \) is a left \( A \# D^* \)-module, where for a given \( \alpha \# \gamma \in A \# D^* \) we have

\[
(\alpha \# \gamma) \cdot (v \otimes a) = v \otimes ((\alpha \# \gamma) \cdot a) \\
= \sum v \otimes a_0 \alpha\langle \gamma, x \leftarrow a_1 \rangle
\]
for every \( v \in V \) and \( a \in A \). Clearly \( V \otimes A \in \text{Gen}(A \# D^* \otimes D) \). It is easy to prove that \( \rho_{V \otimes A} = \text{id}_V \otimes \tilde{\rho} \) and \( (v \otimes a) \cdot b = v \otimes ab \) for every \( v, a, b \in A \). Let \( M \in \mathcal{M}(H)_A^D = \sigma(A \# D^* \otimes D) \).

We set
\[
\Phi_M : M_x \otimes A_x \to M_{x} \\
m \otimes a \mapsto ma.
\]
\( \Phi_M \) is a morphism in \( A \# D^* \)-Mod.

Moreover let
\[
\Psi_M : \text{Hom}_{A \# D^*}(A, M) \otimes A \to M \\
\xi \otimes a \mapsto \xi(a).
\]
Note that this is exactly the map introduced in Theorem 2.2 for \( R = A \# D^* \), \( T = A_x \), and \( P = A \).

3.17. PROPOSITION. For every \( M \in \mathcal{M}(H)_A^D \) the diagram
\[
\begin{array}{c}
\text{Hom}_{A \# D^*}(A, M) \otimes A \\
\downarrow \phi_M \\
M_x \otimes A
\end{array}
\]

is commutative. In particular \( \Psi_M \) is an isomorphism if and only if \( \Phi_M \) is an isomorphism.

Proof. Let \( a \in A \) and \( f \in \text{Hom}_{A \# D^*}(A, M) \). We have
\[
(\Phi_M \circ (\nu_M \otimes \text{id}_A))(f \otimes a) = \Phi_M(f(1_A) \otimes a) = f(1_A)a \\
= f(a) = \Psi_M(f \otimes a),
\]
as \( f \) is a morphism of right \( A \)-modules by Section 3.8.

3.18. LEMMA. For every \( M \in \text{Mod-}A \) the map
\[
\Lambda_M : M \to (M \otimes D)_x \\
m \mapsto m \otimes x
\]
is an isomorphism of right \( A_x \)-modules and its inverse is the map
\[
\Gamma_M : (M \otimes D)_x \to M \\
\sum_{i=1}^n m_i \otimes d_i \mapsto \sum_{i=1}^n m_i e_D(d_i).
\]
Therefore
\[ \Theta_M = \Gamma_M \circ \nu_{M \otimes D} : \text{Hom}_{\mathcal{A} \otimes D^*}(A, M \otimes D) \to M \]
\[ f \mapsto \sum_{i=1}^{n} y_i e_D(d_i), \]
where \( f(1_A) = \sum_{i=1}^{n} y_i \otimes d_i \) is an isomorphism.

3.19
Let \( M \in \text{Mod-}A \) and let \( \sigma_M = \Theta_M \otimes \text{id}_A \). Then
\[ \sigma_M : \text{Hom}_{\mathcal{A} \otimes D^*}(A, M \otimes D) \otimes A \to M \otimes A \]
\[ \quad f \otimes a \mapsto \sum_{i=1}^{n} y_i e_D(d_i) \otimes a, \]
where \( f(1_A) = \sum_{i=1}^{n} y_i \otimes d_i \) is an isomorphism.

Let \( \beta^M = (\Phi_{M \otimes D})^\#(\Lambda_M \otimes \text{id}_A) \), we have
\[ \beta^M : M \otimes A \to M \otimes D \]
\[ m \otimes a \mapsto (m \otimes x) a = \sum ma_0 \otimes x \leftarrow a_1. \]

3.20. **Proposition.** For every \( M \in \text{Mod-}A \) the diagram
\[ \text{Hom}_{\mathcal{A} \otimes D^*}(A, M \otimes D) \otimes A \xrightarrow{\sigma_M} M \otimes D \]
\[ \quad \Phi_{M \otimes D} \quad \beta^M \]
\[ M \otimes A \]
\[ A_\lambda \]
\[ A_\lambda \]
is commutative. In particular \( \Psi_{M \otimes D} \) is injective (resp. surjective) if and only if \( \beta^M \) is injective (resp. surjective).

In the following \( \beta^A \) will be simply denoted by \( \beta \) so that
\[ \beta : A \otimes A \to A \otimes D \]
\[ a \otimes b \mapsto \sum ab_0 \otimes x \leftarrow b_1. \]

3.21. **Lemma.** The map
\[ \mu : A \# D^* \to A \]
\[ a \# y \mapsto a \# y \cdot 1_A \]
is a surjective morphism of left $A \# D^*_{H}$-modules. In particular $A$ is a cyclic left $A \# D^*_{H}$-module.

Proof. $a \# e_D \cdot 1_A = a e_D(x) = a$. 

3.22. Lemma. Assume that $A$ is flat and that $A \# D^*_{H}$ is an isomorphism. Then for every $M \in \sigma(A \# D^*_{H})$, $\Psi_M$ is an injective morphism. In particular $\Psi_M$ is an isomorphism for every $M \in \text{Gen}(A \# D^*_{H})$.

Proof. Note that, as $\Psi_{A \# D}$ is an isomorphism, $A \# D$ is generated by $A$ in $A \# D^*_{H}$-Mod so that

$$\text{Gen}(A \# D^*_{H}) \subseteq \text{Gen}(A \# D^*_{H})$$

Let $M \in \sigma(A \# D^*_{H}) = \mathcal{M}(H)^D_{A}$. As $M \in \text{Mod-}A$, there exists an exact sequence

$$A^{(Y)} \rightarrow A^{(X)} \rightarrow M \rightarrow 0,$$

where $X$ and $Y$ are suitable sets. By applying the functor $- \otimes D$ we get that the sequence

$$A^{(Y)} \otimes D \rightarrow A^{(X)} \otimes D \rightarrow M \otimes D \rightarrow 0$$

is exact in $A \# D^*_{H}$-Mod. Recalling now that, by Lemma 3.21, $A \# D^*_{H}$ is cyclic, we get the following isomorphisms of right $A$-modules:

$$\text{Hom}_{A \# D^*_{H}}(A, A \# D) \equiv \text{Hom}_{A \# D^*_{H}}(A, (A \# D)^{(X)})$$

$$\equiv \left(\text{Hom}_{A \# D^*_{H}}(A, A \# D)^{(X)}\right).$$

Now it is easy to prove that, given a nonempty set $X$, $\Psi_{A \# D}^{(X) \otimes D}$ is an isomorphism, $\Psi_{A \# D}$ being an isomorphism. Let $H = \text{Hom}_{A \# D^*_{H}}(A, -)$. 

From the exact sequence (\( \ast \)) we get the following diagram:

\[
\begin{array}{ccc}
H(A, A^{(Y)} \otimes D) & \xrightarrow{f} & H(A, A^{(X)} \otimes D) \\
\theta_{x^1} & \downarrow & \theta_{x^1} \\
A^{(Y)} & \xrightarrow{\psi_M} & A^{(X)} \\
\theta_{x^1} & \downarrow & \\
A & \xrightarrow{\psi_M D} & M \\
\end{array}
\]

It is easy to prove that this diagram is commutative; moreover its vertical arrows are isomorphisms by Lemma 3.18. It follows that the sequence

\[
H(A, A^{(Y)} \otimes D) \rightarrow H(A, A^{(X)} \otimes D) \rightarrow H(A, M \otimes D) \rightarrow 0 \quad (\ast \ast)
\]

is exact. Let us prove that \( \psi_M \otimes D \) is an isomorphism. By applying the functor \( - \otimes A \) to the exact sequence (\( \ast \ast \)) we get the diagram

\[
\begin{array}{ccc}
H(A, A^{(Y)} \otimes D) \otimes A & \xrightarrow{g} & H(A, A^{(X)} \otimes D) \otimes A \\
\psi_{A^{(Y)} \otimes D} & \downarrow & \psi_{A^{(X)} \otimes D} \\
A^{(Y)} \otimes D & \xrightarrow{\psi_M \otimes D} & A^{(X)} \otimes D \\
\theta_{x^1} & \downarrow & \theta_{x^2} \\
M \otimes D & \rightarrow & M \otimes D \\
\end{array}
\]

whose rows are exact. It is straightforward to prove that this diagram is commutative. Since \( \psi_{A^{(Y)} \otimes D} \) and \( \psi_{A^{(X)} \otimes D} \) are isomorphisms we get that \( \psi_M \otimes D \) is an isomorphism. By Example 3.2 and Section 3.8, \( \rho_M : M \rightarrow M \otimes D \) is an injective morphism in \( \mathfrak{M}(H)^D_A = \sigma(\mathfrak{A}^D \otimes A \otimes D) \). Therefore, since \( A \) is flat, we get the exact sequence

\[
0 \rightarrow H(A, M) \otimes A \rightarrow H(A, M \otimes D) \otimes A
\]

so that from the commutative diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{\psi_M} & H(A, M) \otimes A \\
\downarrow & & \downarrow \psi_M \otimes D \\
M & \rightarrow & M \otimes D
\end{array}
\]

we get that \( \psi_M \) is injective.

3.23. \textbf{Definition.} Let \( A \) a right \( H \)-comodule algebra, \( D \) a right \( H \)-module coalgebra, \( x \in D \) a grouplike element. We say that \( A \subset A \) is a \textit{right Hopf–Galois extension} if the map

\[
\beta_x : \ A \otimes A \rightarrow A \otimes D
\]

\[
a \otimes b \mapsto \sum ab \otimes x \leftarrow b_1
\]

is bijective.
3.24. Definition. Let $A$ a right $H$-comodule algebra, $D$ a left $H$-module coalgebra, $x \in D$ a grouplike element. We say that $A_x \subseteq A$ is a left Hopf–Galois extension if the map

$$\beta'_x: A \otimes A \to A \otimes D$$

$$a \otimes b \mapsto \sum a_0 b \otimes a_1 \to x$$

is bijective.

3.25. Remark. Taking $D = H$, we have that $H$ can be considered both as a right and as a left $H$-module coalgebra with respect to the natural $H$-module structure of $H$. Moreover $x = 1_H$ is a grouplike element of $H$. In this case we simply set $\beta = \beta_1$, and $\beta' = \beta'_1$. Whenever the antipode $S$ of $H$ is bijective with composition inverse $\overline{S}$, the map

$$\theta: A \otimes H \to A \otimes H$$

$$a \otimes h \mapsto \sum a_0 \otimes a_1 S(h)$$

is bijective and its inverse is the map

$$\theta': A \otimes H \to A \otimes H$$

$$a \otimes h \mapsto \sum a_0 \otimes \overline{S}(h) a_1.$$

Moreover it is straightforward to prove that $\beta' = \theta \circ \beta$ (see [10, Proposition 1.2]).

3.26. Remark. If $A = D = H$, then $H$ is a right (left) Hopf–Galois extension. In fact it is easy to prove that $H^{\text{coH}} = k \cdot 1_H$ and that the map

$$\alpha: H \otimes H \to H \otimes H$$

$$h \otimes g \mapsto \sum h S(g_1) \otimes g_2$$

is the two-sided inverse of the map

$$\beta: H \otimes H \to H \otimes H$$

$$h \otimes g \mapsto \sum h g_1 \otimes g_2.$$

3.27. Theorem. Let $k$ be a field, $H$ a Hopf algebra, $A$ a right $H$-comodule algebra, $D$ a right $H$-module coalgebra, and $x \in D$ a grouplike element.
Then the following are equivalent.

(a) $A_x \subset A$ is a right Hopf–Galois extension and $A_x A$ is flat.

(b) For every $M \in \mathcal{M}(H)_A = \sigma(\mathcal{A}_D, A \otimes D)$,

$$\Phi_M : \frac{M_x \otimes A}{A_x} \to M$$

$$m \otimes a \mapsto ma$$

is an isomorphism of left $A \# D^\ast$-modules, i.e., the weak structure theorem in the sense of [7] holds.

(c) The map

$$\Psi_{A \# D^\ast} : \text{Hom}_{A \# D^\ast}((A, A \otimes D) \otimes A \to A \otimes D$$

$$\xi \otimes a \mapsto \xi(a)$$

is an isomorphism of left $A \# D^\ast$-modules and $A_x A$ is flat.

(d) For every $M \in \text{Gen}(\mathcal{A}_D, A \otimes D)$,

$$\Psi_M : \text{Hom}_{A \# D^\ast}((A, M) \otimes A \to M$$

$$\xi \otimes a \mapsto \xi(a)$$

is an isomorphism of left $A \# D^\ast$-modules and $A_x A$ is flat.

(e) For every $M \in \sigma(\mathcal{A}_D, A \otimes D)$,

$$\Psi_M : \text{Hom}_{A \# D^\ast}((A, M) \otimes A \to M$$

$$\xi \otimes a \mapsto \xi(a)$$

is an isomorphism of left $A \# D^\ast$-modules.

(f) $\sigma(\mathcal{A}_D, A \otimes D) = \text{Gen}(\mathcal{A}_D, A)$.

(g) $A_x A$ is flat, $\sigma(\mathcal{A}_D, A) = \sigma(\mathcal{A}_D, A \otimes D)$, and

$$\text{Hom}_{A \# D^\ast}(A, -) : \text{Gen}(\mathcal{A}_D, A) \to \text{Mod}_{A_x}$$

$$M \mapsto \text{Hom}_{A \# D^\ast}(A, M)$$

is full and faithful.

(h) $\sigma(\mathcal{A}_D, A) = \sigma(\mathcal{A}_D, A \otimes D)$ and $\mathcal{A}_D, A$ generates all submodules of $A^n$, for every $n \in \mathbb{N}$, $n > 0$. 

Proof. (a) \(\iff\) (c) follows from Proposition 3.20.
(b) \(\iff\) (e) follows from Proposition 3.17.
(c) \(\Rightarrow\) (d) follows from Lemma 3.22.
(d) \(\Rightarrow\) (c) is trivial.
(d) \(\iff\) (e) \(\iff\) (f) \(\iff\) (g) \(\iff\) (h) follow from Theorem 2.3 applied to \(R = A \# D^*,\) \(T = A,\) \(P = A,\) and \(Q = A \otimes D,\) in view of Theorem 3.15, Remark 3.12, and Section 3.13. 

3.28. Remark. Clearly analogous results hold in the category \(\text{M}(H)^D,\) where \(A \#^* D^*\) and \(\beta'\) play the role of \(\beta\) and \(A \#^* D^*,\) respectively.

3.29. Theorem. Let \(k\) be a field, \(H\) a Hopf algebra, \(A\) a right \(H\)-comodule algebra, \(D\) a right \(H\)-module coalgebra, and \(x \in D\) a grouplike element. Then the following are equivalent.

(a) \(A,\langle x\rangle A\) is a right Hopf–Galois extension and \(\langle x\rangle A\) is faithfully flat.
(b) The map

\[
(-)_x: \text{M}(H)^D_A = \alpha(\langle A \# D^* \rangle A \otimes D) \to \text{Mod}A_x
\]

\[
M \mapsto M_x
\]

is an equivalence, i.e., the strong structure theorem in the sense of \([7]\) holds.
(c) The map

\[
\Psi_{A \otimes D}: \text{Hom}_{A \# D^*}(A, A \otimes D) \otimes A \to A \otimes D
\]

\[
\xi \otimes a \mapsto \xi(a)
\]

is an isomorphism and \(\langle x\rangle A\) is faithfully flat.
(d) The map

\[
\Psi_M: \text{Hom}_{A \# D^*}(A, M) \otimes A \to M
\]

\[
\xi \otimes a \mapsto \xi(a)
\]
is an isomorphism for every $M \in \left( A_{\#D^*} A \otimes D \right)$ and

\[ \Psi'_N: N \to \text{Hom}_{A_{\#D^*}} \left( A, N \otimes A \right) \]

\[ n \mapsto \begin{pmatrix} A \to N \otimes A \\ a \to n \otimes a \end{pmatrix} \]

is an isomorphism for every $N \in \text{Mod}-A_x$.

(e) The map

\[ \text{Hom}_{A_{\#D^*}} \left( A, - \right): \sigma \left( A_{\#D^*} A \otimes D \right) \to \text{Mod}-A_x \]

\[ M \mapsto \text{Hom}_{A_{\#D^*}} \left( A, M \right) \]

is an equivalence.

(f) The map

\[ - \otimes A: \text{Mod}-A_x \to \sigma \left( A_{\#D^*} A \otimes D \right) \]

\[ N \mapsto N \otimes A \]

is an equivalence.

(g) $A_{\#D^*} A$ is quasiprojective and generates each of its submodules, $A_x A$ is a weak generator, and $\sigma \left( A_{\#D^*} A \otimes D \right) = \sigma \left( A_{\#D^*} A \right)$.

(h) $A_{\#D^*} A$ is a quasiprogenerator and $\sigma \left( A_{\#D^*} A \otimes D \right) = \sigma \left( A_{\#D^*} A \right)$.

(i) $A_x A$ is a weak generator, $\Psi_M$ is an isomorphism for every $M \in \text{Gen}(A_{\#D^*} A)$, and $\sigma \left( A_{\#D^*} A \otimes D \right) = \sigma \left( A_{\#D^*} A \right)$.

**Proof.** Recall now, by Theorem 3.15, that $\text{End}_{A_{\#D^*}} \left( A \right) \cong A_x$.

(a) $\Rightarrow$ (c) follows from Proposition 3.20.

(b) $\Rightarrow$ (e) follows from Theorem 3.15.

(c) $\Rightarrow$ (d) By (c) $\Rightarrow$ (e) in Theorem 3.27, $\Psi_M$ is an isomorphism for every $M \in \sigma \left( A_{\#D^*} A \otimes D \right)$. Moreover, by (c) $\Rightarrow$ (h) in the same theorem, $\sigma \left( A_{\#D^*} A \otimes D \right) = \sigma \left( A_{\#D^*} A \right)$.

(d) $\Rightarrow$ (c) and (d) $\Rightarrow$ (e). By (e) $\Rightarrow$ (g) in Theorem 3.27 we get that $\sigma \left( A_{\#D^*} A \otimes D \right) = \sigma \left( A_{\#D^*} A \right)$. Now (d) $\Rightarrow$ (c) follows from (d) $\Rightarrow$ (e) in
Theorem 2.5 and (d) \Rightarrow (e) follows from (d) \Rightarrow (b) in the same theorem.

(e) \Rightarrow (f) Note that for every $N \in \text{Mod-}A$, we have $N \otimes_A A \in \text{Gen}(\mathcal{A}D^*,A)$, therefore $\text{Im}(- \otimes_A A) \subseteq \sigma(A\mathcal{D}^*,A)$.

Since $- \otimes_A A$ is a left adjoint of $\text{Hom}_{\mathcal{D}}(A,-)$ we get (f).

(f) \Rightarrow (g) We have

$$\sigma(A\mathcal{D}^*,A) = \text{Im}(- \otimes_A A) \subseteq \text{Gen}(\mathcal{A}D^*,A) \subseteq \sigma(A\mathcal{D}^*,A) \subseteq \sigma(A\mathcal{D}^*,A \otimes D).$$

Now apply (c) \Rightarrow (g) of Theorem 2.5.

(g) \Rightarrow (c) Since $A \otimes D \in \sigma(A\mathcal{D}^*,A \otimes D) = \sigma(A\mathcal{D}^*,A)$, by (g) \Rightarrow (e) in Theorem 2.5, we get that $\Psi_{A\mathcal{D}}$ is an isomorphism and $A\mathcal{A}$ is faithfully flat.

(g) \Rightarrow (h) \iff (i) follow from (g) \Rightarrow (h) \iff (i) in Theorem 2.5.

3.30. Remark. Clearly analogous results hold in the category $\mathcal{M}(H)^D$, where $A \mathcal{D}^*$ and $\mathcal{D}^*$ play the role of $\mathcal{D}$ and $A \mathcal{D}^*$, respectively.

3.31. Remark. The equivalence (a) \iff (f) for $D = H$ is part of the famous Schneider's theorem [14, Theorem 1].

3.32. Corollary [6, Theorem 2.3]. Let $H$ be a Hopf algebra, $A$ a right $H$-comodule algebra, $D$ a right $H$-module coalgebra, and $x \in D$ a grouplike element. Assume that $H$ is a faithfully coflat left $D$-comodule via $\pi_x$, i.e., $h \mapsto \Sigma(\xi \leftarrow h_x) \otimes h_2$. Then $A \mathcal{A}$ is a right Hopf--Galois extension and $A\mathcal{A}$ is flat (resp. faithfully flat) if $B \subseteq A$ is a right Hopf--Galois extension and $\mathcal{D}^*$ is flat (resp. faithfully flat). In this case Theorem 3.27 (resp. Theorem 3.29) applies.

Proof. Consider

$$\pi_x: \quad H \to D \quad \quad h \mapsto x \leftarrow h.$$ 

$\pi_x$ is a coalgebra morphism. Therefore by Theorems 1.1 and 1.3 in [3] the induction functor $F: \mathcal{M}(H)^D \to \mathcal{M}(H)^D$ defined by $F(M) = M$ endowed with the right $D$-comodule structure $\rho_{F(M)}(m) = \Sigma m_0 \otimes \pi_x(m_1)$ has, as a
right adjoint, the functor $G: \mathcal{M}(H)^D_A \to \mathcal{M}(H)^H_A$ defined by $G(M') = M' \boxtimes H$, with structure map $\rho_{GM'}(\sum m'_i \otimes h_i) = \sum m'_i \otimes h_{i,1} \otimes h_{i,2}$ and $(\sum m'_i \otimes h_i) \cdot a = \sum m'_i a_0 \otimes h_i a_1$.

Since $H$ is a faithfully coflat left $D$-comodule, $G$ is a faithful and exact functor. Assume now that $B \subset A$ is a right Hopf-Galois extension and $B$ is flat. Then, by Theorem 3.27, $A$ is a generator of $\mathcal{M}(H)^H_B$. Let $\varphi: M_1 \to M_2$ be a nonzero morphism in $\mathcal{M}(H)^H_B$. Then $G(\varphi): G(M_1) \to G(M_2)$ is a nonzero morphism in $\mathcal{M}(H)^H_B$ and hence $\text{Hom}_B(A, G(\varphi)): \text{Hom}_B(A, G(M_1)) \to \text{Hom}_B(A, G(M_2))$ is a nonzero morphism, $A$ being a generator of $\mathcal{M}(H)^H_B$. Thus we get that $A = F(A)$ is a generator in $\mathcal{M}(H)^D_A$ so that $\text{Gen}(\mathcal{M}(H)^D_A, A) = \sigma(\mathcal{M}(H)^D_A \otimes \hat{D})$ and $A \subset A$ is right $H$-Galois and $A$ is flat.

Assume now that $B$ is faithfully flat. Then, by Theorem 3.29, $\text{Hom}_B(\mathcal{M}(H)^D_A, -) = (-)^{\text{coH}}: \mathcal{M}(H)^D_A \to \text{Mod-}B$ is an equivalence. It follows that the functor $\text{Hom}_B(\mathcal{M}(H)^D_A, -)$ is exact, so that $A = F(A)$ is a projective object of $\sigma(\mathcal{M}(H)^D_A \otimes \hat{D}) = \text{Gen}(\mathcal{M}(H)^D_A, A)$. In particular $A = F(A)$ is quasiprojective. Since $A$ is a cyclic left $A$-$D^*$-module we conclude that $A = F(A)$ is a quasiprogenerator.

3.33. Remark. We insert the following well-known facts for the generic reader’s sake. By applying Theorem 3.29 for $D = A = H$ (see Remark 3.26) we get “the fundamental theorem of Hopf-modules.”

The functor $(-)^{\text{coH}}: \mathcal{M}_H^H \to \text{k-M od}$

$M \mapsto M^{\text{coH}}$

is an equivalence. Its inverse is the functor

$- \otimes H: \text{k-M od} \to \mathcal{M}_H^H$

$V \mapsto V \otimes H$.

We recall now that $H^\square = H^*_{\text{rat}} \in \mathcal{M}_H^H$ with respect to the right $H$-module structure defined by setting $\langle \gamma \leftarrow h, \chi \rangle = \langle \gamma, \chi S(h) \rangle$ for every $\gamma \in H^*_{\text{rat}}$ and $x \in H$ (see [15, Theorem 5.1.2]) and $(H^\square)^{\text{coH}} = \{ x \in H^* | \chi = \langle \gamma, 1_{H^*} \rangle \chi \}$ for every $x \in H^* = H^\square_{\gamma}$, the space of left integrals in $H^*$ (or left integrals on $H^*$). Therefore we get that

$$\Phi_H: \int_{H^*} H^\square \otimes H \to H^\square_{\gamma}$$

$$\gamma \otimes h \mapsto \gamma \leftarrow h$$

is an isomorphism.
If \( \dim_k(H) < \infty \), \( H \sqcap = H^* \) so that we get that

\[
\Phi_{H^*}: \int_H^i \otimes H \to H^*
\]

\[
\gamma \otimes h \mapsto \gamma h
\]

is an isomorphism. Since \( \dim_k(H) = \dim_k(H^*) \) we get, in particular, \( \dim_k(\int_H^i) = 1 \). Assume now that \( h \in \ker(S) \) and let \( 0 \neq T \in \int_H^i \). Then for every \( x \in H \) we have \( \langle T - h, x \rangle = \langle T, xS(h) \rangle = 0 \). Hence \( \Phi_{H^*}(T \otimes h) = T - h = 0 \), by injectivity of \( \Phi_{H^*} \), we get \( T \otimes h = 0 \) and hence \( h = 0 \). Therefore \( S: H \to H \) is injective and hence also bijective.

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Let now \( T \neq 0 \) be a fixed element of \( \int_H^i \). Then, by the foregoing, the map

\[
\varphi: H \to H^*
\]

\[
h \mapsto T - \overline{S}(h)
\]

is an isomorphism. Note that, for every \( x \in H \), we have

\[
\langle T - \overline{S}(h), x \rangle = \langle T, xS(\overline{S}(h)) \rangle
\]

\[
= \langle T, xh \rangle
\]

\[
= \langle h \mapsto T, x \rangle.
\]

Therefore, for every \( h \in H, T \in H^* \), we have

\[
\varphi(h) = h \mapsto T.
\]

4. THE FINITE CASE

4.1. Lemma. Assume that \( \dim_k(D) < \infty \). Then

\[
\sigma \left( A \#_{H^*} A \otimes D \right) = A \#_{H^*} D \cdot \text{Mod}.
\]

Proof. See [6, page 375].

Using the above result, from Theorems 3.27 and 3.29 we get the following results.

4.2. Theorem. Let \( A \) be a right \( H \)-comodule algebra, \( D \) a finite-dimen-

sional right \( H \)-module coalgebra, and \( x \in D \) a grouplike element. Then the
following are equivalent.

(a) \( A_x \subset A \) is a right Hopf–Galois extension and \( A_x A \) is flat.

(f) \( A_{\# H}^D A \) is a generator in \( A \# D^*-\text{Mod} \).

(i) \( A_{\# H}^D A_{\# H}^{op} \) is faithfully balanced and \( A_{\# H}^{op} A \) is finitely generated and projective.

Moreover each of these conditions is equivalent to each of the conditions (b), (c), (d), (e), (g), and (h) of Theorem 3.27 with \( \sigma(A_{\# H}^D A \otimes D) = A \# D^*-\text{Mod} \).

Proof. (a) \( \iff \) (b) \( \iff \) (c) \( \iff \) (d) \( \iff \) (e) \( \iff \) (f) \( \iff \) (g) \( \iff \) (h). follow from Theorem 3.27 and Lemma 4.1.

(f) \( \iff \) (i) follows from Theorem 17.8 in [1].

4.3. Theorem. Let \( A \) be a right \( H \)-comodule algebra, \( D \) a finite-dimensional right \( H \)-module coalgebra, and \( x \in D \) a grouplike element. Then the following are equivalent.

(a) \( A_x \subset A \) is a right Hopf–Galois extension and \( A_x A \) is faithfully flat.

(f) \( A_{\# H}^D A \) is a faithful quasiprognerator, \( A_x A \) is finitely generated.

(g) \( A_{\# H}^D A \) is a prognerator.

(h) \( A_x A \) is a prognerator and \( A_{\# H}^D A_{\# H}^{op} \) is faithfully balanced.

Moreover each of these conditions is equivalent to each of the conditions (b), (c), (d), and (e) of Theorem 3.29 with \( \sigma(A_{\# H}^D A \otimes D) = A \# D^*-\text{Mod} \).

Proof. Apply Lemma 4.1, Theorem 3.29, and Theorem 2.6.

4.4. Definition (see [9]). Let \( B \) be a subring of a ring \( A \) with identity \( 1_A \) (such that \( 1_A \in B \)). \( A \) is a Frobenius extension of \( B \) if \( A_B \) is finitely generated and projective and moreover

\[
_b A_A \cong {_b \text{H} \text{om}_B(A_B, B_B)}.
\]

4.5. Remark (see [9, Bemerkung 1]). \( A \) is a Frobenius extension of \( B \) if and only if \( _b A_A \) is finitely generated and projective and

\[
_A A_B \cong _A \text{H} \text{om}_B(A_B, B_B).
\]
4.6. **Remark.** Let $H$ be a finite-dimensional Hopf algebra and $A$ a right $H$-comodule algebra. Regarding $H$ as a left $H$-module coalgebra, we have that $A \in A \mathcal{M}(H)^H = A \# H^* \text{-Mod}$. Now, by Remark 3.5, $A \# H^* = A_{H^*}^{\text{op}} \# H^*$ and, by Proposition 3.6,

$$
\Lambda: \left( A \# H^* \right)^{\text{op}}_{H^*} \to A_{H^*}^{\text{op}} \# H^*
$$

$$
a \# \gamma \mapsto \sum \alpha_0 \# (S(\alpha_1) \to \gamma) \circ S
$$

is a ring isomorphism. Therefore $A$ becomes a right $A \# H^*$-module with respect to

$$
a \cdot (\alpha \# \chi) = \Lambda(\alpha \# \chi) \cdot a
$$

$$
= (\chi \circ S) \cdot (\alpha a),
$$

$a \in A$, $\alpha \# \chi \in A \# H^*$. This is the right $A \# H^*$-module structure of $A$ considered in the following theorem.

4.7. **Theorem (4, Theorems 1.2 and 1.2')**. Let $H$ be a finite-dimensional Hopf algebra and $A$ a right $H$-comodule algebra, $B = A_{H^*}^{\text{Grind}}$. Then the following are equivalent.

(a) $B \subset A$ is a right Hopf–Galois extension.

(a') $B \subset A$ is a left Hopf–Galois extension.

(b) The map

$$
\beta: A \otimes_B A \to A \otimes H
$$

$$
a \otimes b \mapsto \sum a b_0 \otimes b_1
$$

is surjective.

(b') The map

$$
\beta': A \otimes_B A \to A \otimes H
$$

$$
a \otimes b \mapsto \sum a b_0 \otimes a_1
$$

is surjective.

(c) The map

$$
\Psi_{A \# H}: \text{Hom}_{A \# H}(A, A \otimes H) \otimes_B A \to A \otimes H
$$

$$
\xi \otimes a \mapsto \xi(a)
$$
is surjective.

(d) \( A \#_{H}^\ast \) is a generator for the category \( A \# H^\ast \text{-mod} \).

(d') \( A \#_{H}^{\ast} \) is a generator for the category \( \text{Mod} \#_{H}^{\ast} \).

(e) \( A \#_{H}^{\ast} A_{B} \) is faithfully balanced and \( B \) is a generator for the category \( A \#_{H}^{\ast} \).

(e') \( B_{\#} A \#_{H}^{\ast} \) is faithfully balanced and \( A_{B} \) is a generator for the category \( B \#_{H}^{\ast} \).

(f) For every \( M \in \text{Mod} \#_{H}^{\ast} \),

\[
\Psi_M: \text{Hom}_{A \#_{H}^{\ast}}(A, M) \otimes_B A \to M
\]

\[\xi \otimes a \mapsto \xi(a)\]

is an isomorphism.

(g) For every \( M \in \text{Mod} \#_{H}^{\ast} \),

\[
\Phi_M: M^\text{coH} \otimes_B A \to M
\]

\[m \otimes a \mapsto ma\]

is an isomorphism.

Moreover if these conditions are satisfied, \( A \) is a Frobenius extension of \( B \).

**Proof.**  (a) \(\Leftrightarrow\) (a') and (b) \(\Leftrightarrow\) (b') follow by Remark 3.25.

(b) \(\Leftrightarrow\) (c) follows by Proposition 3.20.

(d) \(\Rightarrow\) (a) follows by Theorem 4.2(f) \(\Rightarrow\) (a).

(d) \(\Leftrightarrow\) (e) \(\Leftrightarrow\) (f) \(\Leftrightarrow\) (g) follow by Theorem 4.2.

(a) \(\Rightarrow\) (b) is trivial.

(b') \(\Rightarrow\) (d) Let us fix \( 0 \neq T \in j^\ast_{H} \). Let \( v = (id_A \otimes \varphi) \circ \beta ', \) where \( \varphi: H \to H^\ast \) is the isomorphism defined in Section 3.34. Then \( v: A \otimes A \to A \# H^\ast \) is surjective. We prove that \( v \) is a morphism of left \( A \# H^\ast \)-modules, where \( A \otimes A \) is endowed with its \( A \# H^\ast \)-module structure defined as in Section 3.16. Since \( A \otimes A \in \text{Gen}(A \#_{H}^{\ast} A) \) this will prove (d).
First of all note that for every \(a, b \in A\) we have \(v(a \otimes b) = (b \# T)(a \# e)\).

Let \(a, b \in A\), \(\alpha \# \chi \in A \# H^*\). Then
\[
v((\alpha \# \chi) \cdot (a \otimes b)) = v((a \otimes (\alpha \# \chi) \cdot b))
\]
\[
= v(a \otimes \sum b_\alpha \chi, b_1)
\]
\[
= (\sum b_\alpha \chi, b_1) \cdot (a \# e)
\]
\[
= (\sum b_\alpha \chi(b_1 \rightarrow \chi)T) \cdot (a \# e)
\]
\[
= (\alpha \# \chi) \cdot (b \# T) \cdot (a \# e)
\]
\[
= (\alpha \# \chi) \cdot v(a \otimes b).
\]

The equivalences \((a') \leftrightarrow (b') \leftrightarrow (d') \leftrightarrow (e')\) can be proved in an analogous way in view of Remark 4.6.

Assume now that the foregoing equivalent conditions hold and let us prove that \(A\) is a Frobenius extension of \(B\). From \((e')\) we get that \(A_B\) is finitely generated and projective. Let us prove that \(A_B = B \text{Hom}_B(A_B, B_A)\). Let \(R = (A \# H^*)^p\) and let \(i: A \rightarrow R: a' \mapsto a' \# e\).

Then, by Lemma 3.7, \(i\) is an injective ring morphism. Note that \(A\) is a left (right) \(R\)-module, so that \(A\) can be considered as a left (right) \(A\)-module via \(i\). Now it is easy to prove that this left (right) \(A\)-module structure coincides with the usual left (right) \(A\)-module structure on \(A\).

Since \(B \otimes A_R\) is faithfully balanced \(B \otimes B_R \equiv A \otimes A_R\), so that
\[
\text{Hom}_B(RA_B, BA_R) \equiv B \left[ \text{Hom}_B(RA_B, \text{Hom}_R(BA_R, BA_R)) \right]_R
\]
\[
\equiv B \left[ \text{Hom}_R(RA_B \otimes BA_R, BA_R) \right]_R
\]

and, by the foregoing, we get that
\[
\text{Hom}_B(AA_B \otimes BA_R, BA_R) \equiv B \left[ \text{Hom}_R(AA_B \otimes BA_R, BA_R) \right]_A.
\]

Let us fix a \(0 \neq T \in \mathfrak{t}_H^1\) and let \(v: A \otimes B_R \rightarrow A_R\) be the map defined in the proof of \((b') \Rightarrow (d)\), \(v = (\text{id}_{A_B} \otimes \varphi) \circ \beta'\) so that, by \((a')\), \(v\) is bijective. Moreover \(v\) is a morphism of right \(R\)-modules and for every \(a, b \in A\) we know that \(v(a \otimes b) = (b \# T)(a \# e)\). Let us prove that \(v\) is a morphism of left \(A\)-modules. Let \(a' \in A\). Then \(v(a'(a \otimes b)) = v((a'a \otimes b))
\]
\[
= (b \# T)(a'a \# e) = (b \# T)(a \# e)(a' \# e) = v(a \otimes b)(a' \# e) = a'v(a \otimes b).\]
Therefore we get

\[ B \text{Hom}_B(A, B) \cong B \text{Hom}_R(A, R) \cong B A, \]

where the last map is a morphism of right \( A \)-modules by the remarks made above on the structure of \( A \).

4.8. Theorem. Let \( H \) a finite-dimensional Hopf algebra, \( A \) a right \( H \)-comodule algebra, and \( B = A^{\text{coH}} \). Then the following are equivalent.

(a) \( B \subset A \) is a right Hopf–Galois extension and \( B A \) is a weak generator.

(a') \( B \subset A \) is a left Hopf–Galois extension and \( A_B \) is a weak generator.

(b) The functor map

\[ \text{Hom}_{A^H}(A, -) : A^H \text{-Mod} \to \text{Mod-B} \]

\[ M \mapsto \text{Hom}_{A^H}(A, M) \]

is an equivalence.

(c) \( A^H A \) is a faithful quasiprogenerator and \( B A \) is finitely generated.

(c') \( A^H A \) is a faithful quasiprogenerator and \( A_B \) is finitely generated.

(d) \( A^H A \) is a progenerator.

(d') \( A_B \) is a progenerator.

(e) \( B A \) is a progenerator and \( B A^H A \) is faithfully balanced.

(e') \( A_B \) is a progenerator and \( B A \) is faithfully balanced.

Proof. (a) \(\to\) (a') By Theorem 4.7, \( B \subset A \) is a left Hopf–Galois extension. Let \( B L \in B\text{-Mod}, L \neq 0 \). Since \( B A \) is a progenerator and \( A^H A_B \equiv A\text{Hom}_B(A_B, B_B) \), by Theorem 4.7 we have

\[ 0 \neq \text{Hom}_B(B A_B, B) \cong \text{Hom}_B(B A_B, B_B) \cong B \text{Hom}_B(B A_B, B_B) \cong B B L \]

\[ \cong \text{Hom}_B(B A_B, B B L) \cong A B B L. \]

(a') \(\to\) (a) By Theorem 4.7, \( A_B \) is a progenerator and \( B A_A \equiv B \text{Hom}_B(A_B, B_B) \). A proof similar to (a) \(\to\) (a') applies.

(a) \(\to\) (b) By Theorem 4.7, \( B A \) is projective so that \( B A \) is faithfully flat
since $\mathcal{A}$ is a weak generator. (b) follows by (a) $\Rightarrow$ (e) in Theorem 4.3.

(b) $\Rightarrow$ (a) By (e) $\Rightarrow$ (a) in Theorem 4.3.

(c) $\iff$ (d) $\iff$ (e) follow by Theorem 4.3.

Note added in proof. After the submission of this paper, the authors received the preprint "Galois Extensions for Co-Frobenius Hopf Algebras" by M. Beattie, S. Dascalescu, and Ş. Raianu where, using Theorem 3.27 above, it is proved that for every co-Frobenius Hopf algebra $H$, any $H$-Galois $H$-comodule algebra $\mathcal{A}$ is a flat left $\mathcal{A}^{coH}$-module.

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