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Quasinormal resonances of near-extremal Kerr–Newman black holes

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ABSTRACT

We study *analytically* the fundamental resonances of near-extremal, slowly rotating Kerr–Newman black holes. We find a simple analytic expression for these black-hole quasinormal frequencies in terms of the black-hole physical parameters: $\omega = m\Omega - 2i\pi T_{\rm BH}(l+1+n)$, where $T_{\rm BH}$ and Ω are the temperature and angular velocity of the black hole. The mode parameters l and m are the spheroidal harmonic index and the azimuthal harmonic index of a co-rotating mode, respectively. This analytical formula is valid in the regime $\Im\omega \ll \Re\omega \ll M^{-1}$, where M is the black-hole mass.

The statement that black holes have no hair was introduced by Wheeler [1] in the early 1970's. The various no-hair theorems state that the external field of a dynamically formed black hole (or a perturbed black hole) relaxes to a Kerr–Newman spacetime, characterized solely by three parameters: the black-hole mass, charge, and angular momentum. This implies that perturbation fields left outside the black hole would either be radiated away to infinity, or be swallowed by the black hole.

This relaxation phase in the dynamics of perturbed black holes is characterized by 'quasinormal ringing', damped oscillations with a discrete spectrum (see, e.g., [2] for a detailed review). At late times, all perturbations are radiated away in a manner reminiscent of the last pure dying tones of a ringing bell [3–6]. Being the characteristic 'sound' of the black hole itself, these free oscillations are of great importance from the astrophysical point of view. They allow a direct way of identifying the spacetime parameters (the mass, charge, and angular momentum of the black hole). This fact has motivated a flurry of research during the last four decades aiming to compute the quasinormal mode (QNM) spectrum of various types of black-hole spacetimes [2].

The dynamics of black-hole perturbations is governed by the Regge–Wheeler equation [7] in the case of a spherically symmetric Schwarzschild black hole, and by the Teukolsky equation [8] for rotating Kerr–Newman spacetimes. The black hole QNMs correspond to solutions of the wave equations with the physical boundary conditions of purely outgoing waves at spatial infinity and purely ingoing waves crossing the event horizon [9]. Such boundary conditions single out a discrete set of black-hole resonances $\{\omega_n\}$ (assuming a time dependence of the form $e^{-i\omega t}$). (In analogy with standard scattering theory, the QNMs can be regarded as the scattering resonances of the black-hole spacetime. They thus correspond to poles of the transmission and reflection amplitudes of a standard scattering problem in a black-hole spacetime.)

Since the perturbation field can fall into the black hole or radiate to infinity, the perturbation decays and the corresponding QNM frequencies are *complex*. It turns out that there exist an infinite number of quasinormal modes, characterizing oscillations with decreasing relaxation times (increasing imaginary part) [10]. The mode with the smallest imaginary part (known as the fundamental mode) determines the characteristic dynamical timescale τ for generic perturbations to decay.

In this work we determine analytically the *fundamental* (least-damped) resonant frequencies of rotating Kerr–Newman black holes. (For a recent progress in the study of the *highly*-damped resonances, see [11,12].) The spectrum of quasinormal resonances can be studied analytically in the slow rotation, near-extremal limit $(M^2-Q^2-a^2)^{1/2} \ll a \ll M$, where M, Q, and a and the mass, charge, and angular momentum per unit mass of the black hole, respectively. In order to determine the black-hole resonances we shall analyze the scattering of massless scalar and neutrino waves in the Kerr–Newman spacetime [13].[1] The dynamics of a perturbation field Ψ in the Kerr–Newman spacetime is governed by the Teukolsky equation [8]. One may decompose the field as (we use natural units in which $G=c=\hbar=1$)

$$\Psi_{slm}(t,r,\theta,\phi) = e^{im\phi} S_{slm}(\theta;a\omega)\psi_{slm}(r;a\omega)e^{-i\omega t}, \tag{1}$$

where (t,r,θ,ϕ) are the Boyer–Lindquist coordinates, ω is the (conserved) frequency of the mode, l is the spheroidal harmonic

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[1] It is worth mentioning that all attempts to decouple the gravitational and electromagnetic perturbations of the Kerr–Newman spacetime have failed so far, see, e.g., [13].

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index, and m is the azimuthal harmonic index with $-l \leqslant m \leqslant l$. The parameter s is called the spin weight of the field, and is given by $s = \pm\frac{1}{2}$ for massless neutrino perturbations, and $s = 0$ for scalar perturbations. (We shall henceforth omit the indices s, l, m for brevity.) With the decomposition (1), ψ and S obey radial and angular equations, both of confluent Heun type [14,15], coupled by a separation constant $A(a\omega)$.

The angular functions $S(\theta; a\omega)$ are the spin-weighted spheroidal harmonics which are solutions of the angular equation [8,15]

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right) + \left[a^2\omega^2\cos^2\theta - 2a\omega s\cos\theta - \frac{(m + s\cos\theta)^2}{\sin^2\theta} + s + A\right]S = 0. \tag{2}$$

The angular functions are required to be regular at the poles $\theta = 0$ and $\theta = \pi$. These boundary conditions pick out a discrete set of eigenvalues labeled by an integer l. In the $a\omega \ll 1$ limit these angular functions become the familiar spin-weighted spherical harmonics with the corresponding angular eigenvalues $A = l(l+1) - s(s+1) + O(a\omega)$.[2]

The radial Teukolsky equation is given by

$$\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{d\psi}{dr}\right) + \left[\frac{K^2 - 2is(r-M)K}{\Delta} - a^2\omega^2 + 2ma\omega - A + 4is\omega r\right]\psi = 0, \tag{3}$$

where $\Delta \equiv r^2 - 2Mr + Q^2 + a^2$ and $K \equiv (r^2 + a^2)\omega - am$. The zeroes of Δ, $r_\pm = M \pm (M^2 - Q^2 - a^2)^{1/2}$, are the black hole (event and inner) horizons.

For the scattering problem one should impose physical boundary conditions of purely ingoing waves at the black-hole horizon and a mixture of both ingoing and outgoing waves at infinity (these correspond to incident and scattered waves, respectively). That is,

$$\psi \sim \begin{cases} e^{-i\omega y} + \mathcal{R}(\omega)e^{i\omega y} & \text{as } r \to \infty \ (y \to \infty), \\ \mathcal{T}(\omega)e^{-i(\omega - m\Omega)y} & \text{as } r \to r_+ \ (y \to -\infty), \end{cases} \tag{4}$$

where the "tortoise" radial coordinate y is defined by $dy = [(r^2 + a^2)/\Delta]\,dr$. Here $\Omega \equiv \frac{a}{r_+^2 + a^2}$ is the angular velocity of the black-hole horizon. The coefficients $\mathcal{T}(\omega)$ and $\mathcal{R}(\omega)$ are the transmission and reflection amplitudes for a wave incident from infinity. The discrete black-hole resonances are the poles of these transmission and reflection amplitudes. (The pole structure reflects the fact that the QNMs correspond to purely outgoing waves at spatial infinity.) These resonances determine the ringdown response of a black hole to outside perturbations.

The transmission and reflection amplitudes satisfy the usual probability conservation equation $|\mathcal{T}(\omega)|^2 + |\mathcal{R}(\omega)|^2 = 1$. The calculation of these scattering amplitudes in the low frequency limit, $\Im\omega \ll \Re\omega \ll M^{-1}$, is a common practice in the physics of black holes, see, e.g., [13,16,17] and references therein. Define

$$x \equiv \frac{r - r_+}{r_+ - r_-}, \qquad \varpi \equiv \frac{\omega - m\Omega}{4\pi T_{\text{BH}}}, \qquad k \equiv \omega(r_+ - r_-), \tag{5}$$

where $T_{\text{BH}} = \frac{(r_+ - r_-)}{4\pi(r_+^2 + a^2)}$ is the Bekenstein–Hawking temperature of the black hole. Then a solution of Eq. (3) obeying the ingoing boundary conditions at the horizon ($r \to r_+$, $kx \ll 1$) is given by [18,19]

$$\psi = x^{-s-i\varpi}(x+1)^{-s+i\varpi}\,{}_2F_1(-l-s, l-s+1; 1-s-2i\varpi; -x), \tag{6}$$

where ${}_2F_1(a, b; c; z)$ is the hypergeometric function. In the asymptotic ($r \gg M$, $x \gg |\varpi| + 1$) limit one finds the solution [18,19]

$$\psi = C_1 e^{-ikx}x^{l-s}\,{}_1F_1(l-s+1; 2l+2; 2ikx) + C_2 e^{-ikx}x^{-l-s-1}\,{}_1F_1(-l-s; -2l; 2ikx), \tag{7}$$

where ${}_1F_1(a; c; z)$ is the confluent hypergeometric function. The coefficients C_1 and C_2 can be determined by matching the two solutions in the overlap region $|\varpi| + 1 \ll x \ll 1/k$. This yields

$$C_1 = \frac{\Gamma(2l+1)\Gamma(1-s-2i\varpi)}{\Gamma(l-s+1)\Gamma(l+1-2i\varpi)}, \tag{8}$$

and

$$C_2 = \frac{\Gamma(-2l-1)\Gamma(1-s-2i\varpi)}{\Gamma(-l-s)\Gamma(-l-2i\varpi)}. \tag{9}$$

Finally, the asymptotic form of the confluent hypergeometric functions [18,19] can be used to write the solution in the form given by Eq. (4). After some algebra one finds

$$\left|\mathcal{T}(\omega)\right|^2 = \Re\left\{\left[\frac{(l-s)!(l+s)!}{(2l)!(2l+1)!}\right]^2 \frac{\Gamma(l+1-2i\varpi)}{\Gamma(-l-2i\varpi)}(2ik)^{2l+1}\right\}, \tag{10}$$

for the transmission probability.

The quasinormal frequencies are the scattering resonances of the black-hole spacetime. They thus correspond to poles of the transmission and reflection amplitudes. Taking cognizance of Eq. (10) and using the well-known pole structure of the Gamma functions [19], one finds the resonance condition $l + 1 - 2i\varpi = -n$, where $n \geqslant 0$ is a non-negative integer. This yields a simple formula for the black-hole resonances:

$$\omega = m\Omega - 2i\pi T_{\text{BH}}(l + 1 + n), \tag{11}$$

in the near-extremal limit. It is worth emphasizing again that this formula is valid in the $\Im\omega \ll \Re\omega \ll M^{-1}$ regime. This requires $(M^2 - Q^2 - a^2)^{1/2} \ll a \ll M$ and $m > 0$.[3]

In summary, we have studied analytically the quasinormal mode spectrum of near-extremal, slowly rotating Kerr–Newman black holes. It was shown that the fundamental resonances can be expressed in terms of the black-hole physical parameters: the temperature T_{BH}, and the horizon angular velocity Ω.

The fundamental resonances are expected to dominate the relaxation dynamics of a perturbed black-hole spacetime. Taking cognizance of Eq. (11), one realizes that in the near-extremal limit $\Im\omega$ approaches zero linearly with the black-hole temperature T_{BH} for all modes co-rotating with the black hole (i.e., modes having $m > 0$). We therefore conclude that the characteristic relaxation timescale $\tau \sim 1/\Im\omega$ of the black hole is of the order of $O(T_{\text{BH}}^{-1})$.[4]

[2] In the small $a\omega$ limit we shall take $A = l(l+1) - s(s+1)$ in Eq. (3), where l is nearly an integer with a small correction of order $O(a\omega)$.

[3] Taking cognizance of Eq. (10), one finds that the total reflection modes (TRM), which are characterized by the condition $\mathcal{T}(\omega) = 0$, are given by the requirement $1/\Gamma(-l - 2i\varpi) = 0$, that is $-l - 2i\varpi = -n$. We recall that l, as *defined* from the relation $A = l(l+1) - s(s+1)$ [where the separation constants $\{A\}$ themselves are obtained from Eq. (2)], is nearly an integer with a small correction of order $O(a\omega)$. This implies that each quasinormal frequency is separated from a nearby total reflection frequency by a small term of order $O(a\omega)$. Thus, in order to determine the QNMs in numerical calculations (and distinguish them from the TRMs), one would have to use numerical schemes of very high precision.

[4] We note that a spherically symmetric Schwarzschild black hole has only one time/length scale—its horizon radius, r_+ (or equivalently, its mass M). One therefore expects to find $\tau \sim r_+$ (and $\omega \sim r_+^{-1}$) on dimensional grounds. On the other hand, Kerr–Newman black holes have an additional lengthscale—the black-hole inverse temperature T_{BH}^{-1}. Here we have established that the relevant relaxation timescale of the perturbed black hole is determined by its inverse temperature, T_{BH}^{-1}, and not by its horizon radius r_+. We emphasize that T_{BH}^{-1} is much larger than r_+ in

It is worth emphasizing that this result, $\tau \sim T_{\rm BH}^{-1}$, is in accord with the spirit of the recently proposed universal relaxation bound [20,21].

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the extremal limit, $T_{\rm BH} \to 0$. Thus, our result $\Im\omega = O(T_{\rm BH})$ is stronger than a relation of the form $\Im\omega \sim r_+^{-1}$, which one could have anticipated from some naive dimensionality considerations. In particular, our analytical results imply that extremal Kerr–Newman black holes have *infinitely* long relaxation times.