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A condensed Cramer's rule for the minimum-norm least-squares solution of linear equations

Jun Ji

Department of Mathematics and Statistics, Kennesaw State University, Kennesaw, GA 30144, United States

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ABSTRACT

In this paper, we will derive the condensed Cramer's rule of Werner for minimal-norm least-squares solution of linear equations $Ax = b$ from the Cramer's rule of Ben-Israel and Verghese. In addition, a new condensed Cramer's rule will be obtained in this paper.

1. Introduction

Let $C_r^{m\times n}$ and C^n denote the class of $m \times n$ complex matrices of rank r and the n-dimensional complex vector space, respectively. For a nonsingular matrix $A \in C_n^{n\times n}$ and a vector $b \in C^n$, the solution $x = (x_1, x_2, \ldots, x_n)^T$ to the system of linear equations

$$Ax = b \tag{1}$$

can be expressed in terms of determinants

$$x_i = \det(A(i \to b))/\det(A), \quad i = 1, 2, \ldots, n, \tag{2}$$

where $X(i \to v)$ denote the matrix obtained from X by replacing the ith column of X with a vector v. This is the well-known Cramer's rule. Among many proofs of Cramer's rule in the literature, the one

E-mail address: jji@kennesaw.edu

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given by Robinson [10] seems simplest. Robinson's proof is based on the fact that the system of linear equations (1) can be rewritten as

$$A\,I(i \to x) = A(i \to b) \tag{3}$$

where I is the identity matrix of order n. By taking determinant on both sides of (3), together with the fact that $\det(I(i \to x)) = x_i$, we have

$$\det(A)\;x_i = \det(A)\det(I(i \to x)) = \det(A\,I(i \to x)) = \det(A(i \to b))$$

which immediately leads to the Cramer's rule in (2).

A Cramer's rule for the minimum-norm least-squares solution of the linear system (1) with a general rectangular matrix $A \in C_r^{m\times n}$ was obtained in 1982 by Ben-Israel and Verghese [1,12]. This Cramer's rule is based on the results of Blattner [3] that the bordered matrix

$$\begin{bmatrix} A & \hat{V} \\ V^* & 0 \end{bmatrix}$$

is nonsingular for $V \in C_{n-r}^{n\times(n-r)}$, $\hat{V} \in C_{m-r}^{m\times(m-r)}$ such that $R(V) = N(A)$, $R(\hat{V}) = N(A^*)$ and its inverse is

$$\begin{bmatrix} A^\dagger & V^{*\dagger} \\ \hat{V}^\dagger & 0 \end{bmatrix}$$

where $X^\dagger$ is the Moore–Penrose inverse of X. Ben-Israel and Verghese [1,12] then applied the classic Cramer's rule in (2) to the following system of linear equations

$$\begin{bmatrix} A & \hat{V} \\ V^* & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix} \tag{4}$$

and proposed the first Cramer's rule for the minimum-norm least-squares solution $x = (x_1, x_2, \ldots, x_n)^T$ of the linear system (1)

$$x_i = \frac{\det\begin{bmatrix} A(i \to b) & \hat{V} \\ V^*(i \to 0) & 0 \end{bmatrix}}{\det\begin{bmatrix} A & \hat{V} \\ V^* & 0 \end{bmatrix}}, \quad i = 1, 2, \ldots, n. \tag{5}$$

Since 1982, the research on Cramer's rule has been very active and is mainly focused either on the extension to various other linear systems or on more condensed form of the rule for the linear system (1) [16,7,13,11,14,6,9,15,8,5]. In particular, Werner in [16] derived different extensions of the Cramer's rule, one of which is a Cramer's rule for the weighted minimum-norm least-squares solution. Just in the case when both weights are being chosen as identity matrices of proper sizes, Werner's extension particularly reduces to the form of

$$x_i = \frac{\det\left((A^*A + VV^*)(i \to A^*b)\right)}{\det(A^*A + VV^*)}, \quad i = 1, 2, 3, \ldots, n, \tag{6}$$

for the minimum-norm least-squares solution. This formula can also be obtained through an explicit expression of Moore–Penrose inverse of matrix A [8]. Through an alternative explicit expression, another condensed Cramer's rule was also presented in [8]:

$$x_i = \sum_{l=1}^{m} \frac{\bar{a}_{li} \det\left((AA^* + \hat{V}\hat{V}^*)(l \to b)\right)}{\det(AA^* + \hat{V}\hat{V}^*)}, \quad i = 1, 2, 3, \ldots, n, \tag{7}$$

where $\bar{a}_{li}$ is the conjugate of a_{li}.

Observe that formulas in (5), (6), and (7) involve determinants of matrices of order $m+n-r$, n, and m, respectively. We also observe that in order to determine a single component x_i of the solution vector x, formula (7) needs m determinants of different square matrices of order m for its numerator and one determinant of different square matrix of order m for its denominator. But these $m+1$ determinants are also used for the computation of other components of the solution x. Thus, only $m+1$ determinants of matrices of order m are needed for the solution vector x altogether. In contrast, $n+1$ determinants of matrices of order n are needed for the same solution vector x with (6). Therefore, Werner's formula becomes attractive when $m > n$. But the Cramer's rule in (7) has better computational advantage over other two formulas when $m < n$ though nm additional multiplications are required for the computation of x through (7) from the m ratios of determinants.

In this paper, we will give an easy and direct proof of the condensed Cramer's rule of Werner (6) from the one of Ben-Israel and Verghese (5). With a similar argument, we will develop a new Cramer's rule for the minimum-norm least-squares solution to $Ax = b$. Just like the one in (7), our new formula will also involve determinants of matrices of order m. The new formula requires three determinants of different matrices of order m for each component of x with a total of $2n+1$ determinants of matrices of order m for the solution vector x altogether. In contrast to (7), the new one has a simpler format but it may involve more determinants of matrices of order m.

2. A new condensed Cramer's rule

It is seen from $R(\hat{V}) = N(A^*)$ that $A^*\hat{V} = 0$. Hence, we have

$$\begin{bmatrix} A^* & V \\ \hat{V}^* & 0 \end{bmatrix} \begin{bmatrix} A & \hat{V} \\ V^* & 0 \end{bmatrix} = \begin{bmatrix} A^*A + VV^* & 0 \\ 0 & \hat{V}^*\hat{V} \end{bmatrix} \tag{8}$$

and

$$\begin{bmatrix} A^* & V \\ \hat{V}^* & 0 \end{bmatrix} \begin{bmatrix} A(i \to b) & \hat{V} \\ V^*(i \to 0) & 0 \end{bmatrix} = \begin{bmatrix} A^*(A(i \to b)) + V(V^*(i \to 0)) & 0 \\ \hat{V}^*(A(i \to b)) & \hat{V}^*\hat{V} \end{bmatrix}. \tag{9}$$

One can easily show that

$$\begin{aligned} A^*(A(i \to b)) + V(V^*(i \to 0)) &= (A^*A)(i \to A^*b) + (VV^*)(i \to 0) \\ &= (A^*A + VV^*)(i \to A^*b) \end{aligned}$$

which, together with (9), implies

$$\begin{bmatrix} A^* & V \\ \hat{V}^* & 0 \end{bmatrix} \begin{bmatrix} A(i \to b) & \hat{V} \\ V^*(i \to 0) & 0 \end{bmatrix} = \begin{bmatrix} (A^*A + VV^*)(i \to A^*b) & 0 \\ \hat{V}^*(A(i \to b)) & \hat{V}^*\hat{V} \end{bmatrix}. \tag{10}$$

In view of (8) and (10), together with the fact that $\hat{V}^*\hat{V}$ is nonsingular, the Cramer's rule in (5) can be expressed as

$$x_i = \frac{\det\begin{bmatrix} A^* & V \\ \hat{V}^* & 0 \end{bmatrix} \det\begin{bmatrix} A(i \to b) & \hat{V} \\ V^*(i \to 0) & 0 \end{bmatrix}}{\det\begin{bmatrix} A^* & V \\ \hat{V}^* & 0 \end{bmatrix} \det\begin{bmatrix} A & \hat{V} \\ V^* & 0 \end{bmatrix}} = \frac{\det((A^*A + VV^*)(i \to A^*b))}{\det(A^*A + VV^*)}, \tag{11}$$

for $i = 1, 2, \ldots, n$. Thus, we have derived the condensed Cramer's rule of Werner directly from the one of Ben-Israel and Verghese [1,12].

The Cramer's rule in (11) only depends on A, V, and b. Next, we will develop a formula which depends only on A, $\hat{V}$, and b.

Let e_i denote the ith column of identity matrix I for each i, $1 \leqslant i \leqslant n$. Then the ith column of V^* is V^*e_i and the ith column of the bordered matrix

$$\begin{bmatrix} A(i \to b) & \hat{V} \\ V^*(i \to 0) & 0 \end{bmatrix}$$

can be written as

$$\begin{bmatrix} b \\ V^*e_i \end{bmatrix} - \begin{bmatrix} 0 \\ V^*e_i \end{bmatrix}.$$

From the properties of determinants, we have

$$\det\begin{bmatrix} A(i \to b) & \hat{V} \\ V^*(i \to 0) & 0 \end{bmatrix} = \det\begin{bmatrix} A(i \to b) & \hat{V} \\ V^* & 0 \end{bmatrix} - \det\begin{bmatrix} A(i \to 0) & \hat{V} \\ V^* & 0 \end{bmatrix}. \tag{12}$$

From $R(V) = N(A)$, we have $AV = 0$ and thus,

$$\begin{bmatrix} A & \hat{V} \\ V^* & 0 \end{bmatrix} \begin{bmatrix} A^* & V \\ \hat{V}^* & 0 \end{bmatrix} = \begin{bmatrix} AA^* + \hat{V}\hat{V}^* & 0 \\ 0 & V^*V \end{bmatrix}, \tag{13}$$

$$\begin{bmatrix} A(i \to b) & \hat{V} \\ V^* & 0 \end{bmatrix} \begin{bmatrix} A^* & V \\ \hat{V}^* & 0 \end{bmatrix} = \begin{bmatrix} (A(i \to b))A^* + \hat{V}\hat{V}^* & (A(i \to b))V \\ 0 & V^*V \end{bmatrix}, \tag{14}$$

and

$$\begin{bmatrix} A(i \to 0) & \hat{V} \\ V^* & 0 \end{bmatrix} \begin{bmatrix} A^* & V \\ \hat{V}^* & 0 \end{bmatrix} = \begin{bmatrix} (A(i \to 0))A^* + \hat{V}\hat{V}^* & (A(i \to 0))V \\ 0 & V^*V \end{bmatrix}. \tag{15}$$

In view of (12)–(15), together with the fact that V^*V is nonsingular, we can rewrite the Cramer's rule in (5) as follows

$$x_i = \frac{\det\begin{bmatrix} A(i \to b) & \hat{V} \\ V^*(i \to 0) & 0 \end{bmatrix} \det\begin{bmatrix} A^* & V \\ \hat{V}^* & 0 \end{bmatrix}}{\det\begin{bmatrix} A & \hat{V} \\ V^* & 0 \end{bmatrix} \det\begin{bmatrix} A^* & V \\ \hat{V}^* & 0 \end{bmatrix}}$$

$$= \frac{\left(\det\begin{bmatrix} A(i \to b) & \hat{V} \\ V^* & 0 \end{bmatrix} - \det\begin{bmatrix} A(i \to 0) & \hat{V} \\ V^* & 0 \end{bmatrix}\right) \det\begin{bmatrix} A^* & V \\ \hat{V}^* & 0 \end{bmatrix}}{\det\left(\begin{bmatrix} A & \hat{V} \\ V^* & 0 \end{bmatrix}\begin{bmatrix} A^* & V \\ \hat{V}^* & 0 \end{bmatrix}\right)}$$

$$= \frac{\det\left(\begin{bmatrix} A(i \to b) & \hat{V} \\ V^* & 0 \end{bmatrix}\begin{bmatrix} A^* & V \\ \hat{V}^* & 0 \end{bmatrix}\right) - \det\left(\begin{bmatrix} A(i \to 0) & \hat{V} \\ V^* & 0 \end{bmatrix}\begin{bmatrix} A^* & V \\ \hat{V}^* & 0 \end{bmatrix}\right)}{\det(AA^* + \hat{V}\hat{V}^*)\det(V^*V)}$$

$$= \frac{\det((A(i \to b))A^* + \hat{V}\hat{V}^*) - \det((A(i \to 0))A^* + \hat{V}\hat{V}^*)}{\det(AA^* + \hat{V}\hat{V}^*)}, \quad i = 1, 2, \ldots, n.$$

In summary, we have derived the following result.

Theorem 1. *Let $A \in C_r^{m\times n}$ and let $\hat{V} \in C_{m-r}^{m\times(m-r)}$ be a matrix such that $R(\hat{V}) = N(A^*)$. Then the minimum-norm least-squares solution $x = (x_1, x_2, \ldots, x_n)^T$ of $Ax = b$ is given by*

$$x_i = \frac{\det((A(i \to b))A^* + \hat{V}\hat{V}^*) - \det((A(i \to 0))A^* + \hat{V}\hat{V}^*)}{\det(AA^* + \hat{V}\hat{V}^*)}, \quad i = 1, 2, \ldots, n. \tag{16}$$

When A is of full row rank, i.e., $r = m$, the matrix $\hat{V}$ in our derivation will disappear. In such a case, the result in (16) is reduced to the one in [9]:

$$x_i = \frac{\det((A(i \to b))A^*) - \det((A(i \to 0))A^*)}{\det(AA^*)}, \quad i = 1, 2, \ldots, n. \tag{17}$$

We comment that $Ax = b$ is always consistent and may have multiple solutions when $r = m$. It is only shown by Lakshminarayanan et al. [9] that their Cramer's rule in (17) gives a solution to the linear system $Ax = b$. However, it is indeed the minimum-norm solution according to Theorem 1. Furthermore, if $m = n = r$, then

$$\det((A(i \to 0))A^*) = \det(A(i \to 0))\det(A^*) = 0$$

since $\det(A(i \to 0)) = 0$. Due to the facts that $\det(AA^*) = \det(A)\det(A^*)$ and

$$\det((A(i \to b))A^*) = \det(A(i \to b))\det(A^*),$$

the formula in (17) immediately becomes the Cramer's rule in (2).

Though a special case has been in the literature for a few years, the general result as stated in Theorem 1 seems new to us.

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