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A geometric version of the Robinson–Schensted correspondence for skew oscillating tableaux

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Abstract

We consider an analogue of the Robinson–Schensted correspondence for skew oscillating tableaux and we propose a geometric version of this correspondence, extending similar constructions for standard (Combinatoire et représentation du groupe symétrique, Lecture Notes in Mathematics, Vol. 579, Springer, Berlin, 1977, pp. 29–58) and oscillating tableaux (Formal Power Series and Algebraic Combinatorics, FPSAC'99, Univ. Politecnicà de Catalunya, 1999, pp. 141–152). We deduce from this geometric construction new proofs of some combinatorial properties of this correspondence.

1. Introduction

The Robinson–Schensted correspondence is a classical combinatorial construction relating permutations and pairs of standard tableaux having the same shape, defined independently by Robinson [10] and Schensted [16]. The paper of Schensted was followed by numerous works dealing with the combinatorial properties of this correspondence. We can cite, for example, papers of Schützenberger [17,18], Knuth [9], Viennot [20] and Beissinger [1]. In Sagan's book [14, Chapter 3], the interested reader can find a complete exposition of the Robinson–Schensted correspondence, its combinatorial properties and the connection between tableaux, the symmetric group and the theory of symmetric functions. More recently, people have been interested in the extension of this correspondence to various generalizations, in the Young poset, of standard tableaux, like skew (and generalized skew) tableaux [15], oscillating tableaux (related to the representations of the symplectic group) [19,4,12], skew oscillating tableaux [5,12,11], shifted tableaux [21,13,1] and generalized tableaux [9].

In this article we consider of the extension of this correspondence to the family of skew oscillating tableaux, defined independently first by Roby [12,11], and later by Dulucq and Sagan [5], and we define an analogue of the geometric construction due to Viennot [20] which allows us to give geometric (and intuitive) proofs of some

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combinatorial properties of this correspondence. In particular, we prove a new result about the number of odd height columns in the final shape of a skew oscillating tableau, which extends similar results for standard tableaux [18,1] and skew tableaux [15].

In the following two sections, we give basic definitions about biwords, permutations and tableaux, and we briefly recall the correspondence for skew oscillating tableaux. Next, we describe our extension of the geometric construction of Viennot and its implications.

2. Definitions and notations

We use the notation $\lambda=(\lambda_1,\ldots,\lambda_k)$ for both a *partition* and the corresponding *Ferrers diagram.*[1] If λ and μ are two Ferrers diagrams such that $\mu\subseteq\lambda$, the corresponding *skew shape* λ/μ is the set of *cells* belonging to λ but not to μ. If $|\lambda/\mu|=n$, then we write $\lambda/\mu\vdash n$.

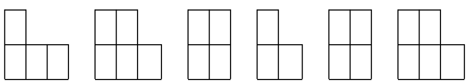
A *partial tableau* T of shape λ/μ is a labeling of the cells of the skew shape λ/μ such that the columns and rows are strictly increasing (from bottom to top and left to right) and the labels are distinct. The tableau is called *standard* if the labels are 1 through $n=|\lambda/\mu|$. We denote by $T(i,j)$ the label of the cell in the ith row (from the bottom) and jth column (from the left), so that $c\in T$ means $c=T(i,j)$ for some i,j. The sets of partial and standard tableaux of shape λ/μ will be denoted by $\mathrm{PT}(\lambda/\mu)$ and $\mathrm{ST}(\lambda/\mu)$, respectively. The set of labelings of the cells of the skew shape λ/μ with columns and rows strictly decreasing (from bottom to top and left to right) will be denoted by $\overline{\mathrm{PT}}(\lambda/\mu)$. For example, if $\lambda=(5,3)$ and $\mu=(2,1)$, the following three tableaux belong to $\mathrm{PT}(\lambda/\mu)$, $\mathrm{ST}(\lambda/\mu)$ and $\overline{\mathrm{PT}}(\lambda/\mu)$, respectively.

	2	9		
		4	6	8

	2	5		
		1	3	4

	4	1		
		5	3	2

A *skew oscillating tableau* T of length n, initial shape α, and final shape β is a sequence of Ferrers diagrams $(\alpha=\lambda^0,\lambda^1,\ldots,\lambda^n=\beta)$, where λ^k is obtained from λ^{k-1} by adding or removing a cell. If $\alpha=\emptyset$, T is called an *oscillating* tableau. We denote by $\mathrm{SO}_n(\alpha,\beta)$ the set of skew oscillating tableaux of length n, initial shape α and final shape β. For example, if $\alpha=(3,1)$ and $\beta=(3,2)$, the following tableau belongs to $\mathrm{SO}_5(\alpha,\beta)$.



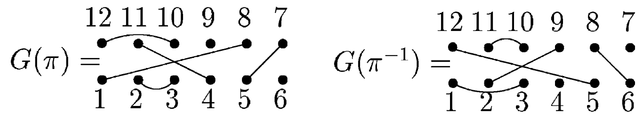
[1] We display Ferrers diagram in "French" notation (the smallest part λ_k in the top row).

The family of skew oscillating tableaux generalizes the families of standard and oscillating tableaux. Indeed, a skew oscillating tableau of $\mathrm{SO}_n(\mu,\lambda)$ in which, for $1 \leqslant k \leqslant n$, λ^k is obtained from λ^{k-1} by adding a cell in a standard tableau of $\mathrm{ST}(\lambda/\mu)$, the label of a cell being given by the step of creation of this cell.

By convention, we denote by S_n the set of the permutations of $[n]=\{1,2,\dots,n\}$ and by I_n the set of the involutions of $[n]$, and, for a permutation σ, we denote its inverse by σ^{-1}.

A *biword* π on $[n]$ is a set of vertical pairs of positive integers of $[n]$ which are pairwise disjoint, $\pi = \begin{pmatrix} i_1 & i_2 & \dots & i_k \\ j_1 & j_2 & \dots & j_k \end{pmatrix}$. Here, we consider biwords such that $i_l > j_l$, for $1 \leqslant l \leqslant k$. We write such biwords with the convention $i_1 > i_2 > \cdots > i_k$ and we define $\hat{\pi} = \{i_1,\dots,i_k\}$ and $\hat{\pi} = \{j_1,\dots,j_k\}$. We denote the set of these biwords on $[n]$ by BW_n. The inverse π^{-1} of a biword π is obtained by changing every pair (i_l, j_l) to $(n+1-j_l, n+1-i_l)$. Moreover, we can represent a biword π on $[2n]$ with a graph denoted $G(\pi)$ formed with two parallel lines of n vertices such that the n vertices of the bottom (resp. top) line are labeled by $1,\dots,n$ (resp. $2n,\dots,n+1$) and the edges are given by the pairs of π. It follows that $G(\pi^{-1})$ is obtained from $G(\pi)$ by a horizontal symmetry.

$$\pi = \begin{pmatrix} 12 & 11 & 8 & 7 & 3 \\ 10 & 4 & 1 & 5 & 2 \end{pmatrix}$$



Remark 2.1. If a biword π of BW_{2n} has n pairs of integers (i_l, j_l) such that $n+1 \leqslant i_l \leqslant 2n$ and $1 \leqslant j_l \leqslant n$, it is equivalent to the permutation $\sigma = j_n\, j_{n-1} \dots j_1$ of S_n, and the biword π^{-1} is equivalent to the permutation σ^{-1}.

3. The correspondence for skew oscillating tableaux

In [16], Schensted describes an algorithm which associates to a permutation σ of S_n a pair of standard tableaux (P,Q) having the same shape λ ($\mu=\emptyset$) such that $\lambda \vdash n$. In [12,11], Roby introduces a similar result for pairs of skew oscillating tableaux having the same length, initial shape and final shape (see Theorem 3.6 below). This result also appears in a paper by Dulucq and Sagan [5]. In the rest of this section, we present the algorithm of Dulucq and Sagan, which relies on three notions of insertion in a partial tableau (see Example 3.1 below). Let P be a partial tableau of shape λ/μ.

(1) *External insertion* inserts an integer x in P as defined by Schensted (see [16] or [14]). We denote the new tableau obtained by this process by $\mathrm{ExtIns}(P,x)$.

(2) *Internal insertion* was defined by Sagan and Stanley in [15]. Let $P(u,v)$ be a cell of P such that the cells $P(u-1,v)$ and $P(u,v-1)$, if they exist, belong to μ. The internal insertion of the cell $P(u,v)$ removes from its label x from this cell and inserts this integer x into the row $(u+1)$ of P using the external insertion algorithm. We denote the new tableau obtained by this process by $\mathrm{IntIns}(P,u,v)$.
(3) *Empty insertion* adds an empty cell $P(u,v)$, where the cells $P(u-1,v)$ and $P(u,v-1)$, if they exist, belong to μ. We denote the new tableau obtained by this process by $\mathrm{EmptyIns}(P,u,v)$.

Conversely, the *deletion* of the cell $P(u,v)$, denoted by $\mathrm{Del}(P,u,v)$, can be an *empty deletion* if the cell is an empty cell, an *internal deletion* if the process (this is the classical process of deletion defined by Schensted [16]) ends in filling a cell of μ, or an *external deletion* if the process ends with the removal of an integer from P.

Example 3.1.

$$P=\begin{array}{|c|c|c|c}\cline{1-3} & 2 & 6 & \\ \hline & & 3 & \multicolumn{1}{c|}{8} \\ \hline\end{array},\quad \mathrm{ExtIns}(P,5)=\begin{array}{|c|c|c|c|}\hline & 2 & 6 & 8 \\ \hline & & 3 & 5 \\ \hline\end{array},\quad \mathrm{Del}(P,2,3)=\begin{array}{|c|c|c|c}\cline{1-2} & 2 & \multicolumn{1}{c}{} & \\ \hline & & 6 & \multicolumn{1}{c|}{8} \\ \hline\end{array},$$

$$P=\begin{array}{|c|c|c|c}\cline{1-1} 4 & \multicolumn{1}{c}{} & & \\ \cline{1-3} & 2 & 6 & \\ \hline & & 3 & \multicolumn{1}{c|}{8} \\ \hline\end{array},\quad \mathrm{ExtIns}(P,1,3)=\begin{array}{|c|c|c|c}\cline{1-2} 4 & 6 & \multicolumn{1}{c}{} & \\ \cline{1-3} & 2 & 3 & \\ \hline & & & \multicolumn{1}{c|}{8} \\ \hline\end{array},\quad \mathrm{Del}(P,3,1)=\begin{array}{|c|c|c|c}\cline{1-3} & 4 & 6 & \\ \hline & 2 & 3 & \multicolumn{1}{c|}{8} \\ \hline\end{array},$$

$$P=\begin{array}{|c|c|c|}\cline{1-1} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline & & 6 \\ \hline & & 3 \\ \hline\end{array},\quad \mathrm{EmptyIns}(P,3,2)=\begin{array}{|c|c|c|}\cline{1-2} & & \multicolumn{1}{c}{} \\ \hline & & 6 \\ \hline & & 3 \\ \hline\end{array},\quad \mathrm{Del}(P,3,1)=\begin{array}{|c|c|c|}\hline & & 6 \\ \hline & & 3 \\ \hline\end{array}.$$

The correspondence for skew oscillating tableaux (Theorem 3.6) relies on the following algorithms Φ and Φ^{-1} relating the triples (π,T,U) of $\mathrm{BW}_n \times \bigcup_{\mu\subseteq\alpha\cap\beta}[\mathrm{PT}(\beta/\mu)\times \mathrm{PT}(\alpha/\mu)]$, such that $\pi\dot{\cup}T\dot{\cup}U=[n]$ (where $\dot{\cup}$ denotes the disjoint union), to the skew oscillating tableaux P of $\mathrm{SO}_n(\alpha,\beta)$.

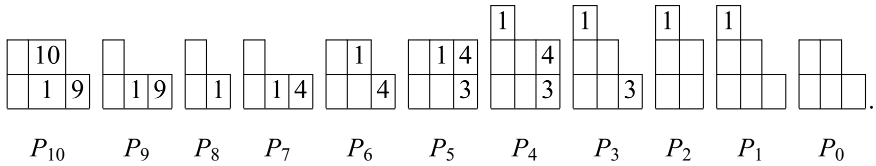
Algorithm 1. $\Phi(\pi,T,U)$

(1) Let $P_n=T$.
(2) For i from n to 1:
  (a) if there is a cell $P_i(u,v)=i$, then remove this cell to obtain P_{i-1},
  (b) else if the pair $(i,x)\in\pi$, then $P_{i-1}=\mathrm{ExtIns}(P_i,x)$,
  (c) else if $U(u,v)=i$ and $P_i(u,v)$ exists, then $P_{i-1}=\mathrm{IntIns}(P_i,u,v)$,
  (d) else $(U(u,v)=i,\ P_i(u,v)$ does not exist), $P_{i-1}=\mathrm{EmptyIns}(P_i,u,v)$.
(3) The tableaux P_i have respective shapes λ^i/μ^i and $P=(\lambda^0,\ldots,\lambda^n)$.

Example 3.2. Let (π, T, U) be the following triple:

$$\pi = \begin{pmatrix} 8 & 6 \\ 4 & 3 \end{pmatrix}, \quad T = \begin{array}{|c|c|c} \cline{1-2} & 10 & \\ \hline & 1 & \multicolumn{1}{c|}{9} \\ \hline \end{array}, \quad U = \begin{array}{|c|c|c} \cline{1-2} & 5 & \\ \hline & 7 & \multicolumn{1}{c|}{2} \\ \hline \end{array}.$$

The execution of $\Phi(\pi, T, U)$ produces the following sequence of tableaux:



Remark 3.3. It is easy to verify the property of the shapes of the partial tableaux P_i that, for $i \in \{0, \ldots, n\}$, $\mu \subseteq \mu^i$.

Algorithm 2. $\Phi^{-1}(P)$ (Suppose $P = (\alpha = \lambda^0, \ldots, \lambda^n = \beta)$.)

(1) Let $\pi = \emptyset$, $T_0 = \alpha$ and $U_0 = \alpha$.
(2) For i from 1 to n:
  (a) if $\lambda^i = \lambda^{i-1} + (u, v)$, then add in T_{i-1} a cell in position (u, v) with label i to obtain T_i, $U_i = U_{i-1}$,
  (b) else $(\lambda_i = \lambda_{i-1} - (u, v))$ $T_i = \mathrm{Del}(T_{i-1}, u, v)$:
    (i) if this deletion is external (the integer x is evacuated from T_{i-1}), then add the pair (i, x) to π, $U_i = U_{i-1}$.
    (ii) else if it is an internal deletion (the cell $T_{i-1}(u', v')$ becomes labeled), then label the cell $U_{i-1}(u', v')$ with i to obtain U_i,
    (iii) else (it is an empty deletion), label the cell $U_{i-1}(u, v)$ with i to obtain U_i.
(3) Finally, $T = T_n$ and $U = U_n$.

Theorem 3.4 (Dulucq and Sagan [5] and Roby [12]). *Let α and β be two partitions and n an integer. Φ is a bijection from triples (π, T, U) of* $\mathrm{BW}_n \times \bigcup_{\mu \subseteq \alpha \cap \beta} [\mathrm{PT}(\beta/\mu) \times \mathrm{PT}(\alpha/\mu)]$ *such that $\pi \dot{\cup} T \dot{\cup} U = [n]$ to tableaux P of* $\mathrm{SO}_n(\alpha, \beta)$.

Remark 3.5. If the skew oscillating tableau P is a skew tableau of shape β/α, then $\Phi^{-1}(P) = (\emptyset, P, \alpha)$.

Theorem 3.6 (Dulucq and Sagan [5] and Roby [12]). *Let α be a partition and $n = 2m$ an even integer. There is a bijection RS_{SO} from triples (π, T, U) of* $\mathrm{BW}_n \times \bigcup_{\mu \subseteq \alpha} [\mathrm{PT}(\alpha/\mu) \times \mathrm{PT}(\alpha/\mu)]$ *such that $\pi \dot{\cup} T \dot{\cup} U = [n]$ to pairs of tableaux (P, Q) of* $\bigcup_\beta [\mathrm{SO}_m(\alpha, \beta) \times \mathrm{SO}_m(\alpha, \beta)]$.

This result follows immediately from Theorem 3.4 and the fact that $(\lambda^0(=\alpha),\ldots,\lambda^m(=\beta),\ldots,\lambda^n(=\alpha))$ is equivalent to the pair (P,Q) of skew oscillating tableaux with $P=(\lambda^0,\lambda^1,\ldots,\lambda^m)$ and $Q=(\lambda^n,\lambda^{n-1},\ldots,\lambda^m)$.

Remark 3.7. If P and Q are two standard tableaux, the tableaux U and T are empty and the biword π is equivalent to a permutation of S_m (Remark 2.1), which is the permutation obtained with the classical Schensted algorithm.

4. Analogue of the geometric construction of Viennot

In this section, we consider a triple (π,T,U) and a skew oscillating tableau P of length n as defined in Theorem 3.4, with the condition $\alpha=\beta$ (hence n is even and we write $n=2m$), and we denote by $P_n,\ldots,P_0$ the partial tableaux produced by the algorithm Φ (see Example 3.2). We describe an alternative algorithm allowing us to derive the tableaux $P_n,\ldots,P_0$ from the triple (π,T,U), based on a geometric representation of such a triple in the subset $\{0,\ldots,m\}\times\{0,\ldots,m\}$ of the discrete plane. This construction is an extension (to the correspondence for skew oscillating tableaux) of the work of Viennot [20].

In our algorithm, we associate to the triple (π,T,U) a subset of $\{0,\ldots,m\}\times\{0,\ldots,m\}$, called the *valid domain* of (π,T,U) (Definition 4.1) and a list $(A_1,B_1),\ldots,(A_k,B_k)$ of pairs of sets of points of $\{0,\ldots,m\}\times\{0,\ldots,m\}$ (this list is computed by Algorithm 3). Next, to each pair (A_i,B_i) we associate a set of paths in $\{0,\ldots,m\}\times\{0,\ldots,m\}$, called the *shadow lines* [2] of (A_i,B_i) (Definition 4.2), and we derive the partial tableaux $P_n,\ldots,P_0$ from these sets of shadow lines and from the valid domain (Theorem 4.5). First, we define the notion of valid domain (see Example 4.4 for an illustration).

Definition 4.1. Let X be the decreasing mapping from $\{0,\ldots,m\}$ to $U\dot\cup\hat\pi\dot\cup\{n+1\}$ such that $X(j)$ is the jth greatest element of $U\dot\cup\hat\pi\dot\cup\{n+1\}$, and let Y be the increasing mapping from $\{0,\ldots,m\}$ to $T\dot\cup\check\pi\dot\cup\{0\}$ such that $Y(j)$ is the jth lowest element of $T\dot\cup\check\pi\dot\cup\{0\}$. The *valid domain of* (π,T,U) is the set of points (x,y) (with $0\leqslant x,y\leqslant m$) such that $X(x)>Y(y)$.

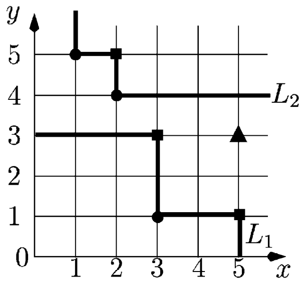
Now, given two sets A and B of points of $\{0,\ldots,m\}\times\{0,\ldots,m\}$, we define the notion of *shadow lines* of the pair (A,B). We call the points of A (resp. B) the A-points (resp. B-points).

Definition 4.2. Let $A=\{(x_1,y_1),\ldots,(x_p,y_p)\}$ and $B=\{(x_{p+1},y_{p+1}),\ldots,(x_{p+q},y_{p+q})\}$ be two sets of points of $\{0,\ldots,m\}\times\{0,\ldots,m\}$ such that for every pair of distinct integers (i,j) $(1\leqslant i,j\leqslant p+q)$, $x_i\neq x_j$ and $y_i\neq y_j$.

[2] We use the terminology of Viennot, although it is not really accurate in our extension of his work.

- The shadow $S(A,B)$ of (A,B) is the set of points (x,y) $(x,y \geqslant 0)$, such that either there is an A-point (x_i, y_i) with $x_i \leqslant x$ and $y_i \leqslant y$, or there is a B-point (x_i, y_i) with $x_i \leqslant x$ or $y_i \leqslant y$.
- The shadow lines of (A,B) are defined recursively. The first shadow line L_1 is the boundary of $S(A,B)$. To construct the shadow line L_{i+1}, remove the A-points belonging to L_i and the B-point having the smallest ordinate (if such a B-point exists) and construct the shadow line of the remaining points. This procedure ends when there is no remaining point.
- The NE-corners of a shadow line are the points (x,y) on the shadow line such that $(x+1,y)$ and $(x,y+1)$ do not belong to this shadow line.
- The SW-corners of a shadow line are the A-points belonging to this line.

Example 4.3. In the following example, where the A-points are represented by circles, the B-points by triangles, and the NE-corners by squares, we have two shadow lines L_1 and L_2.

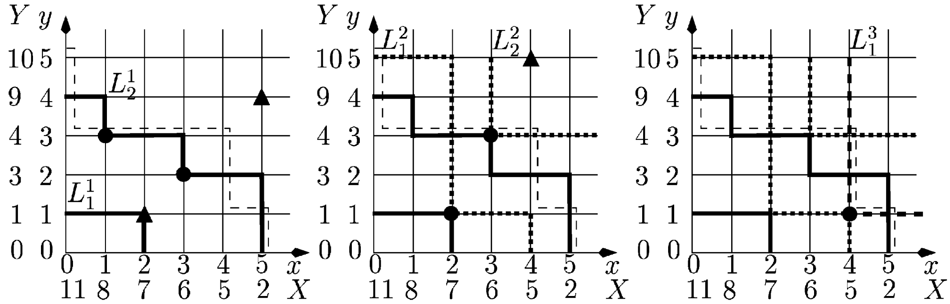


We can now give the algorithm computing the list $(A_1,B_1),\ldots,(A_k,B_k)$ of pairs of sets of points associated to the triple (π,T,U).

Algorithm 3. Let (π,T,U) be a triple as defined in Theorem 3.4, with $\alpha=\beta$.

(1) Let $A_1 = \{(x,y) \mid (X(x),Y(y)) \in \pi\}$.
(2) Let $B_1 = \{(x,y) \mid X(y)$ (resp. $Y(x)$) is the label of the cell $U_1(1,k)$ (resp. $T_1(1,k)$), for all the k such that $U_1(1,k)$ is labeled$\}$.
(3) Denote by L^1 the set of shadow lines $L_1^1,\ldots,L_l^1$ associated to (A_1,B_1).
(4) Let $i=2$.
(5) While $A_{i-1} \neq \emptyset$ and $B_{i-1} \neq \emptyset$:
    (a) Let $A_i = \{(x,y) \mid (x,y)$ is a NE-corner of a shadow line of L^{i-1} and (x,y) belongs to the valid domain of $(\pi,T,U)\}$.
    (b) Let $B_i = \{(x,x) \mid X(y)$ (resp. $Y(x)$) is the label of the cell $U_1(i,k)$ (resp. $T_1(i,k)$), for all the k such that $U_1(i,k)$ is labeled$\}$.
    (c) Denote by L^i the set of shadow lines associated to (A_i,B_i).

Example 4.4. In the following figure, representing the successive steps in the construction of the shadow lines associated to the triple (π,T,U) given in Example 3.2, the valid domain consists of all the points of $\{0,\dots,m\} \times \{0,\dots,m\}$ below the thin dotted line, and the A-points (resp. B-points) are represented by circles (resp. triangles).



In the following theorem, the main result of this section, we show how we can easily derive the tableaux $P_n,\dots,P_0$ from the shadow lines associated to the triple (π,T,U). We recall that for $i \in \{0,\dots,n\}$, if P_i has shape λ^i/μ^i, then $\mu \subseteq \mu^i$ (Remark 3.3). We call the kth step the transformation of the tableau P_k into the tableau P_{k-1}.

Theorem 4.5. *Following, from left to right, the shadow line L^i_j associated to (π,T,U) describes the states of the (μ_i+j)th cell of the ith row of the tableaux $P_n,\dots,P_0$ in the following way*:

(1) *if $(0,y)$ belongs to the line, for some $y \in [m]$, then the cell is present in P_n, with label $Y(y)$,*
(2) *if the line leaves the valid domain through (x,y), the cell is removed during the step $Y(y)$,*
(3) *if the SW-corner (x,y) belongs to the line, during the step $X(x)$ the cell is labeled with $Y(y)$ (resp. created with label $Y(y)$), if the point (x,y') on the line having maximal $y' \leqslant m$ is in (resp. not in) the valid domain,*
(4) *if $(x,0)$ belongs to the line, during the step $X(x)$ the label of the cell is removed (resp. the cell is created with no label), if the point (x,y') on the line having maximal $y' \leqslant m$ is in (resp. not in) the valid domain.*

Example 4.6. Consider the triple (π,T,U) of the Example 3.2 and the associated shadow lines given in Example 4.4. We can illustrate the previous theorem with the shadow line L^1_2, which describes the third cell of the first row of the tableaux $P_n,\dots,P_0$ ($\mu=(1,1)$). The point $(0,4)$ belongs to L^1_2, with $Y(4)=9$, and the cell has label 9 in P_{10}. Immediately, the line leaves the valid domain through $(0,4)$ and during step 9, the cell is removed. Next, the SW-corner $(1,3)$ belongs to the line and the cell is created during the step 8 with label 4 ($X(1)=8$ and $Y(3)=4$). Then the SW-corner $(3,2)$ belongs to this line and we can verify that during the step 6, the label of the cell is changed to 3 ($X(3)=6$ and $Y(2)=3$). Lastly, it leaves the valid domain through

$(4,2)$ and goes through $(5,0)$ and the cell is removed during step 3 ($Y(2)=3$) and added again, as an empty cell, during step 2 ($X(5)=2$).

In order to prove Theorem 4.5 we follow the scheme of the proof of the result of Viennot given by Sagan [14, Section 3.8] and we focus on the first row of the tableaux $P_n,\dots,P_0$ and on the shadow lines of L^1.

Lemma 4.7. *Let $0 \leqslant k \leqslant m$ be an integer. Applying the rules* (1), (2), (3) *and* (4) *of Theorem* 4.5 *to the restriction of the shadow lines of L^1 to the points having an abscissa lower than or equal to k describes the behavior of the cells of the first row of the tableaux $P_n,\dots,P_{X(k)-1}$. Moreover, if the shadow line L^1_j intersects the line $x=k$ in the valid domain and the lowest point of intersection between these two lines is (k,y_j), then the (μ_1+j)th cell of the first row of the tableau $P_{X(k)-1}$ is labeled by $Y(y_j)$* (*resp. unlabeled*) *if $y_j>0$* (*resp. $y_j=0$*).

Proof. We prove this lemma by induction on k. If $k=0$, it follows from Algorithm 3 and the definition of B_1 that the shadow line L^1_j intersects the line $x=0$ on the point $(0,y_j)$ (which lies in the valid domain) if and only if the (μ_1+j)th cell of T is labeled by $Y(y_j)$. The result for $k=0$ follows from this observation, the rule (1), and from the fact that, by definition of Φ, $P_n=T$. Now we assume that the result holds for the line $x=k$ ($0\leqslant k<m$) and we consider the line $x=k+1$.

(a) First, we consider what happens in steps $X(k)-1,\dots,X(k+1)+1$. We can notice that, by definition of the mapping X, none of the integers $X(k)-1,\dots,X(k+1)+1$ appears in $\hat{\pi}$ or as the label of a cell of U. It follows from the definition of Φ that during the steps $X(k)-1,\dots,X(k+1)+1$, the only operations performed by Φ are the suppressions of the cells having these integers for labels. On the other hand, a shadow line L^1_j leaves the valid domain, through a point (k,y_j), if and only if $X(k+1)<Y(y_j)<X(k)$. Hence, it follows from the rule (2) that the result holds for the tableaux $P_n,\dots,P_{X(k+1)}$.

(b) Now, it remains to consider the step $X(k+1)$. First, we can deduce from Algorithm 3 that the line $x=k+1$ intersects at most one shadow line of L^1 on a vertical segment, the other shadow lines of L^1 intersecting $x=k+1$ on a horizontal segment. For every shadow line L^1_j, we denote by (k,z_j) (resp. $(k+1,y_j)$) its lowest intersection point with the line $x=k$ (resp. $x=k+1$), if such a point exists.

(b.1) If $x=k+1$ intersects no shadow line of L^1 on a vertical segment, the rules of Theorem 4.5 imply that no cell of the first row of $P_{X(k+1)}$ is modified. On the other hand, it follows from the definition of the shadow lines that there is no A-point of A_1 or B-point of B_1 on the line $x=k+1$, which implies that $X(k+1)\notin\hat{\pi}$ and that no cell of the first row of U is labeled by $X(k+1)$. We deduce from this fact and from the definition of Φ that during the step $X(k+1)$, no cell of the first row of $P_{X(k+1)}$ is modified, and the lemma holds in this case.

(b.2) Now, we assume that the line $x=k+1$ intersects the shadow line L^1_j on a vertical segment. This implies that $(k+1,y_j)$ lies in the valid domain and that, for

$l \neq j$, if $z_l > 0$, then $y_l = z_l$. First, we can deduce from the induction hypothesis and from (a) that

(i) the first row of the tableau $P_{X(k+1)}$ has at least $\mu_1 + j - 1$ cells, and exactly $j - 1$ cells if and only if L_j^1 enters into the valid domain through a point $(k+1, y)$,
(ii) the $(\mu_1 + j - 1)$th cell of this row (if $j > 1$) is either unlabeled (if L_{j-1}^1 does not intersect the line $x = k+1$) or labeled by $Y(y_{j-1}) < Y(y_j)$,
(iii) the $(\mu_1 + j)$th cell of this row (if it exists) is labeled by $Y(z_j) > Y(y_j)$.

If $(k+1, y_j)$ is an A-point (i.e. a SW-corner of L_j^1), it follows from the definition of Φ that the pair $(X(k+1), Y(y_j)) \in \pi$ and that $P_{X(k+1)-1} = \mathrm{ExtIns}(P_{X(k+1)}, Y(y_j))$. On the other hand, we deduce from (i), (ii) and (i–ii) that the external insertion of $Y(y_j)$ in $P_{X(k+1)}$ labels the $(\mu_1 + j)$th cell of the first row with $Y(y_j)$, creating it if and only if L_j^1 enters into the valid domain through a point $(k+1, y)$, and according to rule (3) the result holds in this case.

If $(k+1, y_j)$ is not an A-point, it follows from the definition of the shadow lines and from the Algorithm 3 that $y_j = 0$, that B_1 contains at least j B-points, and that the jth B-point (x, y) of B_1 satisfies $x = k+1$ and $y \geqslant z_j$, which implies that the $(\mu_1 + j)$th cell of the first row of U is labeled by $X(k+1)$. Hence, during the $X(k+1)$th step of Φ, the $(\mu_1 + j)$th cell of the first row of $P_{X(k+1)}$ is modified by an internal insertion (if this cell exists), that remove its label, or an empty cell insertion if this cell does not exist in $P_{X(k+1)}$, and, according to rule (4) and points (i) and (ii), the result holds in this case. Finally, we can notice that this cell will not be modified by the last steps of Φ, which agrees with the rules of Theorem 4.5. □

Proof of Theorem 4.5. The proof of the theorem for all the rows of the tableaux $P_n, \ldots, P_0$ is based on the same principle as the proof of the first row and on the following facts:

- In Φ, a label removed from a cell of the first row by an internal or an external insertion labels a cell of the second row.
- All the NE-corners of the shadow lines of L^{i-1} belonging to the valid domain are the A-points used to define the shadow lines of L^i.
- The B-points of B_i being given by the ith rows of T and U. □

Remark 4.8. In the case where P is equivalent to a pair of standard tableaux having the same shape, our result is equivalent to the result of Viennot, as described in [14].

Now, we deduce from the previous theorem some combinatorial properties of the Robinson–Schensted correspondence for skew oscillating tableaux. One of the beautiful properties of the Robinson–Schensted correspondence for standard tableaux relies on the exchange of the tableaux P and Q. Schützenberger showed [17] that if (P,Q) is in bijection with a permutation σ, then (Q,P) is in bijection with σ^{-1}, and so, the Robinson–Schensted correspondence is a bijection between involutions and standard tableaux. Later Viennot [20] proved these results using simple properties of its geometric construction. In the same way, with our construction we can prove similar results for skew oscillating tableaux.

Definition 4.9. Let n be an integer and P a tableau belonging to $\mathrm{PT}(\lambda/\mu)$ (resp. $\overline{\mathrm{PT}}(\lambda/\mu)$), such that all these labels are less than or equal to n. We define the tableau P^c of $\overline{\mathrm{PT}}(\lambda/\mu)$ (resp. $\mathrm{PT}(\lambda/\mu)$) by $P^c(u,v)=n+1-P(u,v)$ for each cell $(u,v)\in\lambda/\mu$.

Definition 4.10. We denote IC_n the set of involutions on $[n]$ such that every cycle (a,b) $(a\neq b)$ and every fixed point (a) can be of two types (colors), called bold or normal.

The following property of the shadow lines associated to a triple (π,T,U) is a direct consequence of the previous definitions and of the definition of the shadow lines.

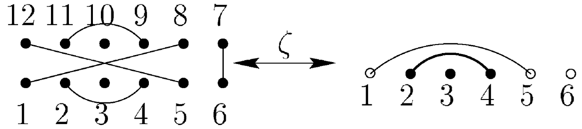
Claim 4.11. *The shadow lines of* (π^{-1},U^c,T^c) *and its valid domain are obtained from those of* (π,T,U) *by reflecting in the line* $y=x$.

Theorem 4.12 (Dulucq and Sagan [5]). *Let* $n=2m$ *be an integer and* (π,T,U) *a triple of* $\mathrm{BW}_n\times\mathrm{PT}(\alpha/\mu)\times\overline{\mathrm{PT}}(\alpha/\mu)$. *If* $\mathrm{RS}_{\mathrm{SO}}(\pi,T,U)=(P,Q)$ *then* $\mathrm{RS}_{\mathrm{SO}}(\pi^{-1},U^c,T^c)=(Q,P)$.

Proof. This result follows in a straightforward way from Claim 4.11 and Theorem 4.5. □

Proposition 4.13. *There is a bijection* $\zeta:\{\pi\in\mathrm{BW}_{2n}\,|\,\pi=\pi^{-1}\}\to\mathrm{IC}_n$.

Example 4.14.



We then have $\zeta(\pi)=(\mathbf{3})(\mathbf{2},\mathbf{4})(1,5)(6)$: the cycle $(\mathbf{2},\mathbf{4})$ and the fixed point $(\mathbf{3})$ are bold, the cycle $(1,5)$ and the fixed point (6) are normal.

As an immediate consequence of Theorem 4.12 and Proposition 4.13, we have the following result.

Corollary 4.15 (Dulucq and Sagan [5]). *Let* $n=2m$. $\mathrm{RS}_{\mathrm{SO}}$ *induces a bijection between the skew oscillating tableaux of* $\mathrm{SO}_m(\alpha,\beta)$ *and the pairs* (σ,T) *such that* $\sigma\in\mathrm{IC}_m$, $T\in\mathrm{PT}(\alpha/\mu)$ *and* $\zeta^{-1}(\sigma)\dot{\cup}T\dot{\cup}T^c=[n]$.

Furthermore, with the bijection between involutions and standard tableaux, Schützenberger [18] proved that the number of odd height columns in a standard tableau is the number of fixed points of the involution corresponding to this tableau, a result which has a direct geometric proof using the construction of Viennot. Sagan and Stanley [15] give an analogous result in the case of the correspondence for skew tableaux. To conclude this section, we extend these results to the case of skew oscillating tableaux. For a given Ferrers diagram α, we denote by $\mathrm{odd}(\alpha)$ the number of odd height columns in α.

Theorem 4.16. *Let* σ *be an involution of* IC_m, $T\in\mathrm{PT}(\alpha/\mu)$ *and* $P\in\mathrm{SO}_m(\alpha,\beta)$, *such that* (σ,T) *corresponds to* P *as in Corollary* 4.15. *Then* $\mathrm{odd}(\beta)=\mathrm{odd}(\mu)+\mathrm{fix}_{\mathrm{N}}(\sigma)$, *where* $\mathrm{fix}_{\mathrm{N}}(\sigma)$ *the number of normal fixed points in* σ.

In order to prove this result, we need the following lemma.

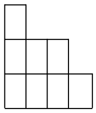
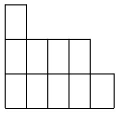
Lemma 4.17. *The* i*th row of the tableau* P_m *has exactly* μ_i+k *cells if and only if exactly* k *shadow lines belonging to* L^i *intersect the line* $y=x$ *in the valid domain.*

Proof. The proof is divided into two cases (we recall that $n=2m$).

(a) If during the mth step a cell is added to P_m, there is $l\in[m]$ such that $X(l)=m$. It follows from Theorem 4.5 that the number of cells in the ith row of P_m is μ_i plus the number of shadow lines of L^i whose intersection with $x=l$ is entirely in the valid domain or whose termination point is $(x,0)$ with $x<l$. Let y_{VD} be the minimal height of the boundary of the valid domain at $x=l$. If $X(l)=m$, then there are $l-1$ strictly positive ordinates y_i such that $Y(y_i)<m$. It follows that $x=l$ intersects $x=y$ at the point $(l,y_{\mathrm{VD}}+1)$. Thus, every shadow line intersecting $x=l$ in the valid domain or leaving it through a point $(x,0)$ such that $x\leqslant l$ intersects the line $x=y$ in the valid domain and conversely.

(b) Else, during the mth step, a cell is removed from P_m, which implies that there is $l\in[m]$ such that $Y(l)=m$, and $l'\in\{0,\ldots,m\}$ such that $X(l')>m$ and $X(l'+1)>m$ or $l'=m$. Following Theorem 4.5, the number of cells in the ith row of P_m is μ_i plus the number of shadow lines of L^i whose intersection with $x=l$ is entirely in the valid domain or whose termination point is $(x,0)$ with $x<l$. We can notice that the point (l',l') is in the valid domain, the point $(l'+1,l')$ (if $l'<m$) is not in the valid domain and it follows that the line $x=y$ intersects the boundary of the valid domain on a vertical segment of this boundary. Hence, a shadow line of L^i intersects $x=y$ in the valid domain if and only if it intersects the line $x=l'$ (we recall that we consider the lowest intersection) on a point (x,y) such that $y\leqslant l'$, which implies the result. $\square$

Proof of Theorem 4.16. Let σ, T and P be as defined in Theorem 4.16. First, for a positive integer j, we denote by k_j the greatest integer i such that the set L^i of shadow lines has at least j lines. Next, we define an operation of extension of a Ferrers diagram λ in the following way: if i is a positive integer, then $\text{ext}(\lambda, i)$ is the diagram obtained by adding a cell at the end of every row λ_j of λ such that $1 \leqslant i \leqslant k_j$. For example, if $k_1 = 2$, then

$\lambda =$  $\Rightarrow \text{ext}(\lambda, 1) =$  .

If this operation adds k cells to λ, we have $\text{odd}(\text{ext}(\lambda, i)) - \text{odd}(\lambda) = k \bmod 2$. Indeed, for any odd integer i, adding a cell at the end of rows i and $i+1$ of λ does not modify the number of odd height columns. Moreover, if we remark that $\mu = \text{ext}((\ldots\text{ext}(\text{ext}(\mu, 1), 2), \ldots), |L^1|)$ (where $|L^1|$ denotes the number of lines of L^1), it remains to show that the number of $j \in [k_1]$, such that k_j is odd, is the number of normal fixed points of σ. For this, we first deduce from the symmetry of the shadow diagram corresponding to (σ, T) (Claim 4.11) that every shadow line has a SW-corner or a NE-corner on the line $x = y$. Next, we remark that a normal fixed point (x) of σ corresponds to a pair $(n+1-x, x)$ in π and that the point $(X^{-1}(n+1-x), Y^{-1}(x))$ lies on the line $x = y$, which implies a one-to-one correspondence between the SW-corners of the lines belonging to L^1 and the normal fixed points of σ. Finally, it suffices to remark that

- a NE-corner of L^i_j in the valid domain and on the line $y = x$ induces a SW-corner in the valid domain and on the line $y = x$ of L^{i+1}_j,
- a SW-corner of L^i_j in the valid domain and on the line $y = x$ induces either no corner of L^{i+1} on the line $x = y$ or a NE-corner of L^{i+1}_j in the valid domain and on the line $y = x$ (and, by the previous point, a SW-corner of L^{i+2}_j in the valid domain and on the line $y = x$). $\square$

5. Conclusion

In this paper, we gave extensions of classical combinatorial properties of the Robinson–Schensted correspondence for the family of skew oscillating tableaux: the geometric construction of Viennot, the algorithm of Beissinger and a property concerning the number of odd columns in the final shape of a skew oscillating tableau.

There are other properties of this correspondence that can be extended to the families of oscillating and skew oscillating tableaux. For example, in [3,2], we extend the notion of Knuth class (defined by Knuth [9]) to the family of oscillating tableaux and, partially, to the family of skew oscillating tableaux, and the algorithm of Beissinger, which gives us an alternative proof of Theorem 4.16. Moreover, all the results presented here and in [3,2] can easily be extended to the family of generalized oscillating and generalized skew oscillating tableaux (see [8,12] for the notions of generalized insertion and generalized suppression and [12] for a description of a correspondence for the family of generalized skew oscillating tableaux).

We should note that in [12,11], Roby describes a geometric version of the work of Fomin [7], and then a geometric version of the Robinson–Schensted correspondence for skew oscillating tableaux. It would be interesting to relate his construction to our construction.

Finally, there are some combinatorial properties of the Robinson–Schensted correspondence that we could not extend to the family of skew oscillating tableaux, like the "Jeu de taquin" of Schützenberger [18] or his "vidage-remplissage" [17,6] (although an analogue of this construction has been defined for the family of oscillating tableaux in [4]). On the other hand, it could be interesting to study the families of shifted oscillating and shifted skew oscillating tableaux (see [13,21] for the notion of shifted tableaux).

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