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# CONJUGATE PROCESSES AND THE SIMULATION OF RUIN PROBLEMS

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A general method is developed for giving simulation estimates of boundary crossing probabilities for processes related to random walks in discrete or continuous time. Particular attention is given to the probability $\psi(u, T)$ of ruin before time $T$ in compound Poisson risk processes. When the probability law $P$ governing the given process is imbedded in an exponential family $(P_\theta)$, one can write $\psi(u, T) = E_\theta R_\theta$ for certain random variables $R_\theta$ given by Wald's fundamental identity. Using this to simulate from $P_\theta$ rather than $P$, it is possible not only to overcome the difficulties connected with the case $T=\infty$, but also to obtain a considerable variance reduction. It is shown that the solution of the Lundberg equation determines the asymptotically optimal value of $\theta$ in heavy traffic when $T=\infty$, and some results guidelining the choice of $\theta$ when $T<\infty$ are also given. The potential of the method in different situations is illustrated by two examples.

risk reserve process * ruin probability * simulation * conjugate distributions * importance sampling * heavy traffic * fundamental identity of sequential analysis * Lundberg equation * periodic queues

## 1. Introduction

In a great number of applied probability areas like sequential analysis, queueing theory, storage and dam models, insurance risk and so on, a basic question is to assess the values of probabilities of the form $\psi(u) = P(\tau<\infty)$, $\psi(u, T) = P(\tau<T)$ where

$$\tau = \tau(u) = \inf\{t \geq 0: X_t > u(t)\}$$

is the time of the first crossing of some process $\{X_t\}$ over a boundary specified by $u(t)$. Unfortuantely this problem is far from easy. Even for such a simple case as $u(t) = u$ and $\{X_t\}$ a discrete time random walk, much ingenuity is required and the resulting expressions are not straightforward to implement numerically.

At least from the point of view of assessing numerical values to the $\psi(u)$, $\psi(u, T)$, it is therefore appealing to rely on simulation. The simplest example of this would be crude (or straightforward) simulation of $\psi(u, T)$ by running $N$ replicates of $\{X_t\}_{t\leq T}$ and estimate $\psi(u, T)$ by the fraction of runs with $\tau<T$. This method is of course so simple and standard that a discussion of its principles from the theoretical point of view very quickly exhausts the subject. However, some specific features suggest to take a closer look at the problem. In particular, $\psi(u)$ cannot be simulated

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in finite time and also the $\psi(u)$, $\psi(u, T)$ are in main cases small so that the relative error on the estimates becomes large.

The purpose of the present paper is the study of a general method, which is applicable and efficient at least in some basic cases, and also seems to contain some promises on adaptability to more complex situation. The basic idea is quite simple and comes from one of the classical tools in risk theory and sequential analysis, conjugate distributions. These may be thought of as arising from an imbedding of the probability law $P = P_{\theta_0}$ governing the given process in a conjugate family $(P_\theta)$, and a given $\psi(u, T)$ can by means of Wald's fundamental identity be expressed as $\psi(u, T) = E_\theta R_\theta$ for certain random variables $R_\theta$ (thus $R_{\theta_0}$ is simply the indicator of ruin before $T$). The point is that for suitable choice of $\theta$ the $R_\theta$ can be simulated in finitely many steps even when $T = \infty$, and that their vairances are small compared to $R_{\theta_0}$. This idea relates to the concept of importance sampling and has been studied within the framework of two-barrier problems in sequential analysis ([28] and references there), but somewhat surprisingly not in risk- and queueing theory where the random walk problem is a one-barrier one. In fact, for queues the more advanced literature is heavily orientated towards the method of regenerative simulation, cf. [8], [9], [15], [6, Ch. 6], whereas in the case of insurance risk the references (e.g. [5, 23]) that we know are few and do not go deep into the methodology of the subject.

## 2. Compound Poisson risk processes and conjugate families

The details of the paper will be worked out within the classical setting of insurance risk models (though at the end we exemplify some extensions to different settings, e.g. periodic queues).

Assume that the claims arrive incurred by an insurance company according to a Poisson process $\{N(t)\}_{t\geq 0}$ with intensity $\alpha$, that the claim sizes $Y_1, Y_2, \ldots$ are independent of $\{N(t)\}$ and i.i.d. with common moment generating function $\phi(s) = Ee^{sY}$, and that premiums come in at rate $p$ per unit time. That is, the interclaim times $Z_1, Z_2$ are i.i.d. with $P(Z > z) = e^{-\alpha z}$ and

$$X(t) = \sum_{n=1}^{N(t)} Y_n - pt$$

represents the net pay-out at time $t$. Thus if $u$ is the initial risk reserve, then in the usual terminology $\tau = \tau(u) = \inf\{t \geq 0: X_t > u\}$ is the time of ruin, $\psi(u) = P(\tau < \infty)$ is the probability of ultimate ruin and $\psi(u, T) = P(\tau < T)$ is the probability of ruin before time $T$. We assume that the safety loading $\eta = (p - \alpha EY)/\alpha EY$ is $>0$ which is equivalent to $\psi(u) < 1$ for all $u \geq 0$. This set-up is extensively studied in the literature. General references are [7, 21, 27] whereas for alternatives to simulation we refer to [24, 26, 32, 33, 2] for numerial methods and to [27, 2] for surveys of approximations.

We define the basic conjugate family $(P_\theta)$ of risk processes exactly as in [2]. We cite only the most basic facts and formulas and refer to [2] for a more complete discussion and references (to which we take here the opportunity to add [3], [18]). Let $\theta_0<0$ be defined by $\alpha\phi'(-\theta_0)=p$ and consider for each $\theta$ satisfying $\phi(\theta-\theta_0)$ a risk process governed by say $P_\theta$, corresponding to premium $p_\theta=p$, arrival intensity $\alpha_\theta=\alpha\phi(\theta-\theta_0)$ and m.g.f. $\phi_\theta(s)=E_\theta\,\mathrm{e}^{sY}=\phi(s+\theta-\theta_0)/\phi(\theta-\theta_0)$. Then the given process corresponds to $\theta=\theta_0$ and the cumulant generating functions satisfy

$$\kappa_\theta(s)=\log E_\theta\,\mathrm{e}^{sX(t)}/t=\alpha_\theta(\phi_\theta(s)-1)-ps=\kappa_{\theta'}(s+\theta-\theta')-\kappa_{\theta'}(\theta-\theta'). \tag{2.1}$$

The choice of origin (or equivalently $\theta_0$) ensures that $E_\theta X(t)\gtreqless 0$ exactly when $\theta\gtreqless 0$. This is well-known to imply that in particular ruin occurs a.s. when $\theta\geqslant 0$. I.e., $P_\theta(\tau(u)<\infty)=1$, $\theta\geqslant 0$. Besides $\theta_0$, a very important quantitity in risk theory is the solution $\gamma>0$ of the Lundberg equation $\kappa(\gamma)=0$, and we let $\theta_1=\gamma+\theta_0$.

Now if $\tau^*$ is any stopping time and $\mathscr{F}_{\tau^*}$ the usual stopping time $\sigma$-algebra, then the $P_\theta$ are mutually equivalent on $\mathscr{F}_{\tau^*}\cap\{\tau^*<\infty\}$ with densities given by

$$\frac{\mathrm{d}P_{\theta'}}{\mathrm{d}P_\theta}=\exp\{(\theta'-\theta)X(\tau^*)-\tau^*\kappa_\theta(\theta'-\theta)\}. \tag{2.2}$$

In particular, if we let $\tau^*=\tau$, $\theta'=\theta_0$ and integrate over $\{\tau<T\}$ we get $\psi(u,T)=E_\theta R_\theta$ where

$$R_\theta=\exp\{(\theta_0-\theta)X(\tau)-\tau\kappa_\theta(\theta_0-\theta)\}I(\tau<T). \tag{2.3}$$

Note that, as one would expect, $R_{\theta_0}=I(\tau<T)$. If $T=\infty$ and $\theta\geqslant 0$, then $\tau<\infty$ a.s. so that $I(\tau<T)$ in (2.3) is vacuous. A further simplification occurs for the Lundberg value $\theta_1$. Hence $\kappa_{\theta_1}(\theta_0-\theta_1)=0$ so that the last term in the bracket vanishes. In view of the particular role of $\theta_1$ it will frequently be convenient to represent a general $\theta\geqslant 0$ as $\theta=\theta_1(1+\Delta)$. Then by (2.1),

$$R_{\theta_1(1+\Delta)}=\exp\{-(\gamma+\theta_1\Delta)X(\tau)+\tau\kappa_{\theta_1}(\theta_1\Delta)\}I(\tau<T) \tag{2.4}$$

and, if we let $v_\theta=\mathrm{Var}_\theta\,R_\theta$,

$$\begin{aligned} v_{\theta_1(1+\Delta)}&=E_{\theta_1(1+\Delta)}R^2_{\theta_1(1+\Delta)}-\psi(u)^2\\ &=E_{\theta_1}\exp\{-(2\gamma+\theta_1\Delta)X(\tau)+\tau\kappa_{\theta_1}(\theta_1\Delta)\}I(\tau<T)-\psi(u)^2. \end{aligned} \tag{2.5}$$

It is also frequently convenient to write $X(\tau)=u+B(u)$ where $B(u)$ is the overshot.

## 3. Performance measures. Empirical examples

The simulation method to be studied in the rest of the paper can now easily be explained: Instead of performing crude ($P_{\theta_0}$-) simulation, we can for each $\theta$ simulate a risk process with parameters corresponding to $P_\theta$ and observe the response $R_\theta$.

Replicating the experiment a suitable number $N_\theta$ of times, one can then in a standard manner use the empirical mean of the observed responses to estimate $\psi(u, T) = E_\theta R_\theta$ and their empirical variance to estimate $v_\theta = \mathrm{Var}_\theta R_\theta$ and thereby the standard error $(v_\theta/N_\theta)^{1/2}$ on the estimate. The case $\theta = \theta_1$ will play a particular role and is henceforth referred to as *Lundberg simulation.*

A number of problems arise immediately. For example, which value of $\theta$ should one choose and how does the method compare to crude simulation or other alternatives? To answer such questions, we need to define appropriate performance measures. From the point of view of optimal allocation of computer time, it is clearly important that both $v_\theta$, the variance on the response, and $i_\theta$, the expected CPU time needed to create one replicate of $R_\theta$, should be small. To balance these requirements, we suggest to *look for a $\theta$ minimizing the product $i_\theta v_\theta$.* This is motivated from the observation that if we simulate in some fixed large amount $t$ of time, we get approximately $N_\theta = t/i_\theta$ replicates and hence approximately a variance $v_\theta/N_\theta \simeq i_\theta v_\theta/t$ on the estimate [related remarks are in [22, p. 405] and [28]; that the randomness of $N_\theta$ is immaterial follows from Anscombe's theorem much in the same way as in [9, p. 54].

We start by some examples illustrating these points and take $T = \infty$ for simplicity. For empirical purposes, we have considered only the Poisson/Exponential ($P/E$) or $M/M/1$ case $P(Y > y) = \mathrm{e}^{-\beta y}$. Of course, this is hardly realistic but even though the case of a more geneal $Y$ causes no intrinsic difficulties, it has the advantage that a great number of functions can be evaluated explicitly, giving guidelines for the general case as well as checks of the simulation results.

**Example 3.1.** Looking first at $v_\theta$ alone, consider the $P/E$ case with $\beta = 1/EY = 1$, $p = 1$, $\alpha = 0.85$ and $\psi(u) = \alpha\, \mathrm{e}^{-(1-\alpha)u} = 5\%$, i.e., $u = 18.9$ (this set of parameters could be argued to be typical and will be used repeatedly in the paper). Crude simulation of $\psi(u)$ is not possible in the strict sense but could be implemented approximately by choosing an appropriate stopping criterion (e.g. large $t$ or small $X(t)$). Then $v_{\theta_0} \simeq 0.05(1-0.05) = 0.0475$ whereas for the Lundberg case we get

$$v_{\theta_1} = \mathrm{Var}_{\theta_1} \mathrm{e}^{-\gamma X(\tau)} = \mathrm{e}^{-2\gamma u}\, \mathrm{Var}_{\theta_1} \mathrm{e}^{-\gamma B(u)} = 0.0000575$$

(using the easily checked relations $\gamma = 1-\alpha$, $B(u) =^{\mathscr{D}} Y$). Needless to say that this is a dramatic reduction. Heuristic considerations indicate that also $i_{\theta_0} > i_{\theta_1}$. In fact, in crude simulation the runs with ruin will have about the same length as in the Lundberg process ([1], slightly adapted) whereas the ones without ruin will be longer since we must wait until our stopping criterion is met.

**Example 3.2.** To judge the behaviour of $v_\theta$ for $\theta \neq \theta_1$ is somewhat more complicated, but at least too large values of $\theta$ cannot be appropriate since it may be deduced from (2.2), (2.5) and [31] that

$$v_\theta = \infty \quad \text{if } \kappa_{\theta_1}(\theta - \theta_1) > -\kappa_{\theta_0}(-\theta_0). \tag{3.1}$$

On the contrary, $i_\theta$ may be expected to decrease with $\theta$ since the $P_\theta$-distribution of $\{X(t)\}$ is readily seen to be stochastically increasing with $\theta$ and hence $\tau$ decreasing. To illustrate these phenomena, we return to the $P/E$ example $\alpha = 0.85$, $\psi(u) = 5\%$. Computer simulation were performed for $\theta_1$, and for larger as well as smaller $\theta$. For each value of $\theta$, the computer time allowed was the same, one second CPU time. That is, the number $N_\theta$ of runs was finalized the first time the CPU time at the end of a run exceeded one second. The estimates and asymptotic 95% confidence bands are depicted in Figure 1.

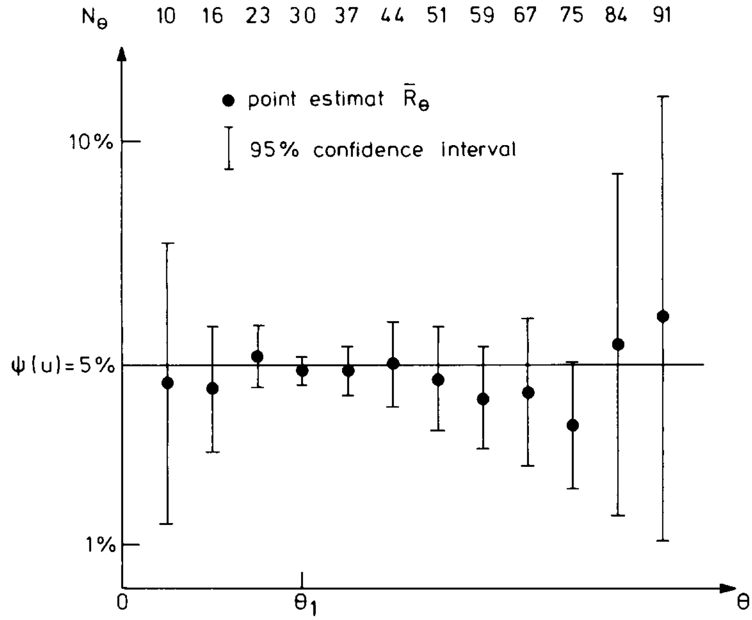


Figure 1. Simulation results from one second CPU time simulation from $P_{\theta_1(1+\Delta)}$. $P/E$ case, $\beta = p = 1$, $\psi(u) = 5\%$, $\Delta = -0.75(0.25)2.00$.

The simulations were carried out in Pascal at the Regional Computing Center, University of Copenhagen, on their Univac 1181 Machine. Uniform random numbers were produced by N.A.G. routine S05CAF, initialized by S05CBF($I$) with $I = 17$ for any single simulation estimate reported in the paper. The algorithm generating $R_\theta$ is extremely simple and considers the times of claims only as follows:

(1) Put $X = T = 0$;

(2) Generate a claim size $Y$ and an interarrival time $Z$ according to $P_\theta$; put $X = X + Y - pZ$, $T = T + Z$;

(3) If $X < u$, return to (2). Otherwise let $R = \exp\{(\theta_0 - \theta)X - T\kappa_\theta(\theta_0 - \theta)\}$.

It is seen from Fig. 1 that as expected $N_\theta$ increases with $\theta$ and that the width of the confidence band is minimal when $\theta = \theta_1$.

**Example 3.3.** It is well-known [20, 21, 25] that if for convenience the time scale is chosen such that $p=1$, then $\psi(u)=P(V>u)$ where $V$ is the virtual waiting time in a stationary $M/G/1$ queue with arrival intensity $\alpha$ and service times distributed as $Y$. This could suggest to apply instead regenerative simulation, cf. [9], [6, Ch. 6]. To compare these two approaches, we performed Lundberg simulation and regenerative simulation in each one second CPU time for various sets of parameters. The results are summarized in Table 1 and indicate that Lundberg simulation is superior in a wide range of parameters (not only in the tail though the difference becomes more marked there).

Table 1
Empirical variance on simulation estimates of $\psi(u)$ obtained within one second CPU time. $P/E$ model with $\beta=p=1$

| $\alpha$ | $\psi(u)(\%)$ | Regenerative | $P_{\theta_1}$ | Ratio |
|---|---|---|---|---|
| 0.50 | 5 | $1.6_{10}-4$ | $1.8_{10}-6$ | 91.8 |
| 0.50 | 50 | $2.0_{10}-4$ | $6.2_{10}-5$ | 3.2 |
| 0.85 | 5 | $9.8_{10}-4$ | $2.2_{10}-6$ | 436 |
| 0.85 | 50 | $2.8_{10}-3$ | $3.2_{10}-5$ | 86.4 |
| 0.85 | 85 | $5.0_{10}-4$ | $3.4_{10}-5$ | 14.6 |

These examples clearly indicate that simulation from a conjugate process may lead to considerable variance reduction compared to traditional approaches, and thereby motivate a closer study of the method.

We shall first look at the optimal choice of $\theta$, considering the cases $T=\infty$ and $T<\infty$ separately. E.g., for $T=\infty$ we show that the Lundberg value $\theta=\theta_1$ is asymptotically optimal under heavy traffic conditions, thereby providing an explanation of Fig. 1. A related result is given in [28], where the optimality of $\theta_1$ is shown for fixed $\theta_0$ in the case of two barriers $-v<0<u$ where $u\uparrow\infty$. This would correspond here to $\psi(u)\to 0$, i.e., in the queueing terminology of Example 3.1 to pass to the tail of the waiting time distribution. However, obviously the heavy traffic limit seems more relevant in many cases and also Example 3.1 indicates that the present method may be worthwhile not only for tail probabilities.

For the asymptotic considerations, it is necessary to put the somewhat unprecise definition of $i_\theta$ into a form more suitable for theoretical analysis. A look at the algorithm above seems to suggest that the main time consuming factor when generating a single $R$ is the repetitions of step (2), the number of which is

$$\boldsymbol{n}=\inf\left\{n\geq 1:\ \sum_{k=1}^{n}\{Y_k-pZ_k\}>u\right\}.$$

The time needed for each step is of course machine- and programming language dependent but does not significantly vary with $\theta$, and in the following we shall therefore replace $i_\theta$ by $E_\theta\boldsymbol{n}$. That is, our object of study is

$$f(\Delta)=E_{\theta_1(1+\Delta)}\boldsymbol{n}\cdot v_{\theta_1(1+\Delta)} \tag{3.2}$$

## 4. Diffusion approximations in heavy traffic

We shall consider the same limiting procedure as in [29], [2, Section 5] (which is not restricted to the $P/E$ case). That is, we think of $P_0$ (i.e., of $p$, $\alpha_0$, $\phi_0$) as the fixed parameter and consider the limit

$$\theta_0 \uparrow 0,\ u \uparrow \infty \quad \text{in such a way that } \theta_0 u \to -\xi \tag{4.1}$$

for some $\xi > 0$ (note that in [2] we write $\theta_0 u \to \xi$ with $\xi < 0$). As explained in [2] (the argument is essentially the same as in [10, 11, 29]), it holds subject to (4.1) that

$$\frac{\tau(u)\alpha_0 E_0 Y^2}{u^2} \to \tau_{-\xi} \quad \text{in } P_{\theta_0}\text{-distribution} \tag{4.2}$$

where $\tau_\xi$ is the time of first passage of Brownian motion with unit variance and drift $\xi$ to level 1 (thus $\tau_\xi$ is defective when $\xi < 0$). The distribution of $\tau_\xi$ is the so-called *inverse Gaussian distribution* and has density, cumulative d.f., resp. moment generating function

$$g(t;\xi) = \frac{1}{\sqrt{2\pi}} t^{-3/2} \exp\left\{\xi - \tfrac{1}{2}\left(\frac{1}{t} + \xi^2 t\right)\right\}, \quad t > 0, \tag{4.3}$$

$$G(t,\xi) = P(\tau_\xi < t) = 1 - \Phi(t^{-1/2} - \xi t^{1/2}) + e^{2\xi}\Phi(-t^{-1/2} - \xi t^{1/2}), \quad t > 0, \tag{4.4}$$

$$E\, e^{\lambda \tau_\xi} = \begin{cases} \infty, & \lambda > \xi^2/2, \\ \exp\{\xi - \sqrt{\xi^2 - 2\lambda}\}, & \lambda \leq \xi^2/2. \end{cases} \tag{4.5}$$

See [30, Ch. 7] or [16, Ch. 15] for more detail.

We quote some consequences of (4.1), (4.2) from the above references. First

$$\psi(u) = P_{\theta_0}(\tau < \infty) \cong P(\tau_{-\xi} < \infty) = G(\infty, -\xi) = e^{-2\xi}, \tag{4.6}$$

$$\psi(u, Tu^2/\alpha_0 E_0 Y^2) = P_{\theta_0}(\tau < Tu^2/\alpha_0 E_0 Y^2) \cong P(\tau_{-\xi} < T) = G(T; -\xi). \tag{4.7}$$

Next, since $\theta_0 < 0$, $\theta_1 > 0$ are connected by $\kappa_0(\theta_0) = \kappa_0(\theta_1)$, it follows by Taylor expansion that (4.1) is equivalent to

$$\theta_1 u \to \xi. \tag{4.8}$$

From this relations similar to (4.7) for the $P_{\theta_1(1+\Delta)}$-distribution of $\tau$ follow by replacing $-\xi$ by $\xi(1+\Delta)$. Furthermore:

**Lemma 4.1.** *Subject to* (4.8), *it holds that $B(u) \to B(\infty)$ in $P_{\theta_1}$-distribution. Here $B(\infty)$ has the limiting $P_0$-distribution of $B(u)$ as $u \to \infty$, viz.*

$$E_0\, e^{\lambda B(\infty)} = \sum_{k=0}^{\infty} \frac{\lambda^k}{(k+2)!} \frac{2E_0 Y^{k+2}}{E_0 Y^2}. \tag{4.9}$$

*Furthermore $E_{\theta_1} e^{\lambda B(u)} \to E_0\, e^{\lambda B(\infty)}$ in a neighbourhood of zero.*

Indeed, the first statement is contained in [29], the formula (4.9) in the proof of Lemma 5.1 of [2] whereas the last statement as well as some further estimates to be used in the sequel requires some uniform integrability arguments. As example of how to carry out these, we give the proof of the following Lemma:

**Lemma 4.2.** *Subject to* (4.8), *it holds that*

$$E_{\theta_1} \exp\{\lambda\tau\alpha_0 E_0 Y^2/u^2\}\begin{cases} = \infty \text{ ultimately}, & \lambda > \xi^2/2, \\ \to \exp\{\xi - \sqrt{\xi^2 - 2\lambda}\}, & \lambda < \xi^2/2. \end{cases}$$

(We have not investigated the case $\lambda = \xi^2/2$.)

**Proof.** Since $\tau \to \tau_\xi$ in $P_{\theta_1}$-distribution, the result is trivial for $\lambda \leqslant 0$, and for $0 < \lambda < \xi^2/2$ it suffices by standard uniform integrability arguments to show

$$\overline{\lim}\, E_{\theta_1} \exp\{\lambda'\tau\alpha_0 E_0 Y^2/u^2\} < \infty \tag{4.10}$$

for some $\lambda' > \lambda$. Choose $\lambda' < \xi^2/2$ and let $\varepsilon$, $p$, $q$ satisfy $0 < \varepsilon < 1$, $p > 1$, $1/p + 1/q = 1$, $\lambda' < (2\varepsilon - \varepsilon^2)\xi^2/2p < \xi^2/2$. Since

$$-\kappa_{\theta_1}(-\theta_1\varepsilon) = \kappa_0(\theta_1) - \kappa_0(\theta_1(1-\varepsilon)) \cong \frac{\kappa_0''(0)}{2}\{\theta_1^2 - \theta_1^2(1-\varepsilon)^2\}$$

$$\cong \alpha_0 E_0 Y^2/u^2(2\varepsilon - \varepsilon^2)\xi^2/2 \tag{4.11}$$

we can then bound (4.10) by

$$\begin{gathered} \overline{\lim}\, E_{\theta_1} \mathrm{e}^{-\tau\kappa_{\theta_1}(-\theta_1\varepsilon)/p} = \overline{\lim}\, E_{\theta_1}(A_{\theta_1} B_{\theta_1}), \\ A_{\theta_1} = \mathrm{e}^{-\theta_1\varepsilon X(\tau)/p - \tau\kappa_{\theta_1}(-\theta_1\varepsilon)/p}, \qquad B_{\theta_1} = \mathrm{e}^{\theta_1\varepsilon X(\tau)/p}. \end{gathered} \tag{4.12}$$

Here $E_{\theta_1} A_{\theta_1}^p = 1$ by a standard martingale identity, whereas $B(u) \to B(\infty)$ implies

$$E_{\theta_1} B_{\theta_1}^q = \mathrm{e}^{\theta_1\varepsilon u q/p} E_{\theta_1} \mathrm{e}^{\theta_1\varepsilon B(u) q/p} \cong \mathrm{e}^{\xi\varepsilon q/p} \cdot 1.$$

Hence (4.12) is finite by Hölder's inequality.

Suppose finally $\lambda > \xi^2/2$ and let

$$\beta = \frac{\lambda\alpha_0 E_0 Y^2}{u^2}, \qquad \beta_0 = \kappa_0(-\theta_0) \cong \frac{\alpha_0 E_0 Y^2}{u^2}\cdot\frac{\xi^2}{2}.$$

Then $\beta > \beta_0$ ultimately and similarly to (3.1) one gets $E_{\theta_1} \mathrm{e}^{\beta\tau} = \infty$ whenever $\beta > \beta_0$. □

## 5. Asymptotic optimality of Lundberg simulation for $T = \infty$

We can now easily obtain the limiting behaviour of (3.2):

**Proposition 5.1.** *Subject to* (4.1), (4.8), *it holds that* $f(\Delta)=\infty$ *ultimately if* $\Delta>\sqrt{2}-1$ (*i.e.*, $\Delta^2+2\Delta>1$). *If* $-1<\Delta<\sqrt{2}-1$, *then*

$$f(\Delta)\cong\frac{u^2\,\mathrm{e}^{-4\xi}}{E_0Y^2\xi}\frac{1}{1+\Delta}\{\mathrm{e}^{-\Delta\xi+\xi-\xi(1-\Delta^2-2\Delta)^{1/2}}-1\}. \tag{5.1}$$

*More precisely, for* $\Delta=0$ *it holds that*

$$f(0)\to\frac{2\xi\,\mathrm{e}^{-4\xi}}{9(E_0Y^2)^3}\{3E_0Y^4E_0Y^2-2(E_0Y^3)^2\}. \tag{5.2}$$

Clearly, this result contains the asymptotic optimality of $\theta_1$ since $\{\cdots\}$ in (5.1) vanishes for $\Delta=0$ and is necessarily always $\geqslant 0$ as limit of non-negative quantities (this is also easily proved directly). Proposition 5.1 contains, however, some further information: Since $f(\Delta)/f(0)\cong cu^2$ for $\Delta\neq 0$, the difference between $\theta_1$ and $\theta_1(1+\Delta)$ becomes more and more marked as the traffic increases. It is also seen from (5.2) that the efficiency of Lundberg simulation does not deteriorate in the limit. This is in marked contrast to regenerative simulation, which at a number of places in the literature (e.g. [19]) has been noted to behave badly under heavy traffic conditions.

**Proof of Proposition 5.1.** Using Wald's identity, we first note that

$$E_{\theta_1(1+\Delta)}\,n=\frac{E_{\theta_1(1+\Delta)}\tau}{E_{\theta_1(1+\Delta)}Z}\cong\frac{u^2/\alpha_0E_0Y^2E\tau_{\xi(1+\Delta)}}{E_0Z}=\frac{u^2}{E_0Y^2\xi(1+\Delta)} \tag{5.3}$$

(the estimate for $E\tau$ requires some uniform integrability argument along the lines of Lemma 4.2. We omit the details). In the remaining factor (2.5), $\psi(u)^2\cong\mathrm{e}^{-4\xi}$ by (4.6) Furthermore

$$(2\gamma+\theta_1\Delta)X(\tau)\cong\theta_1(4+\Delta)(u+B(\infty))\cong(4+\Delta)\xi, \tag{5.4}$$

$$\kappa_{\theta_1}(\theta_1\Delta)\cong\alpha_0E_0Y^2/u^2(\Delta^2+2\Delta)\xi^2/2, \tag{5.5}$$

cf. (4.11). Therefore

$$\exp\{-(2\gamma+\theta_1\Delta)X(\tau)+\tau\kappa_{\theta_1}(\theta_1\Delta)\}\to\exp\{-(4+\Delta)\xi+(\Delta^2+2\Delta)\xi^2/2\cdot\tau_\xi\}$$

in distribution. By Lemma 4.2 and standard results on weak convergence, the expectations converge as well with limit

$$\exp\{-(4+\Delta)\xi+\xi-\xi\sqrt{1-\Delta^2-2\Delta}\}$$

given by (4.5). Combining the above estimates, (5.1) follows.

For (5.2), we need to estimate $v_{\theta_1}=\mathrm{Var}_{\theta_1}\,\mathrm{e}^{-\gamma X(\tau)}$ more precisely. However,

$$|\mathrm{e}^{-\gamma B(u)}-1+\gamma B(u)|\leqslant\gamma^2B(u)^2\,\mathrm{e}^{\gamma B(u)}.$$

Hence by Lemma 4.1

$$v_{\theta_1}\cong\mathrm{e}^{-2\gamma u}\gamma^2\,\mathrm{Var}_{\theta_1}\,B(u)\cong\mathrm{e}^{-4\xi}\frac{4\xi^2}{u^2}\,\mathrm{Var}_0\,B(\infty).$$

But according to (4.9),

$$\mathrm{Var}_0\, B(\infty) = \frac{E_0 Y^4}{6E_0 Y^2} - \left(\frac{E_0 Y^3}{3E_0 Y^2}\right).$$

Combining with (5.3), (5.2) follows. Finally the assertion for $\Delta^2 + 2\Delta > 1$ is an easy consequence of (5.4), (5.5) and Lemma 4.2. □

One might note that the discussion of [29, 2] suggests that the approximations given in Section 4 and underlying Proposition 5.1 are not terribly accurate until the term of next order ($O(u^{-1})$) are added. Presumably such refinements in the asymptotic form of $f(\Delta)$ could be made and thereby provide a better approximation to the value $\Delta_{\min}$ minimizing $f(\Delta)$ than just $\Delta_{\min} \cong 0$. We have not carried this out since some explicit calculations for the $P/E$ case suggest that the resulting variance reduction would be small.

## 6. Simulation of ruin probabilities in finite time

An analysis similar to the one in Section 5 can be carried out also for $T < \infty$. We first need to redefine $i_\theta$. Since simulation goes on until the risk reserve becomes negative or time $T$ has passed, the appropriate choice appears to be $i_\theta = E_\theta \boldsymbol{n} \wedge \boldsymbol{n}_T$ where

$$\underline{n}_T = \inf\left\{ n \geq 1 \colon \sum_{k=1}^{n} Z_k > T \right\}.$$

We then have the following extension of Proposition 5.1:

**Propostion 6.1.** *Suppose that* $T\alpha_0 E_0 Y^2 / u^2 \to T_0 \in (0, \infty)$ *subject to* (4.1), (4.8). *Then for all* $\Delta > -1$,

$$i_{\theta_1(1+\Delta)} \cong \frac{u^2}{E_0 Y^2} E\tau_{\xi(1+\Delta)} \wedge T_0 \tag{6.1}$$

*where*

$$E\tau_\xi \wedge T_0 = \frac{1}{\xi}\{G(T_0; \xi) - 2\,\mathrm{e}^{2\xi}\Phi(-T_0^{-1/2} - \xi T_0^{1/2})\} + T_0\{1 - G(T_0; \xi)\}, \tag{6.2}$$

$$v_{\theta_1(1+\Delta)} \cong \mathrm{e}^{-4\xi}[\mathrm{e}^{-\Delta\xi} E\,\mathrm{e}^{(\Delta^2+2\Delta)\xi^2/2\tau_\xi} I(\tau_\xi < T_0) - G(T_0; \xi)^2]. \tag{6.3}$$

It should be noted that it is no longer required that $\Delta^2 + 2\Delta < 1$. This is simply because $\tau_\xi$ when restricted to $\{\tau_\xi < T_0\}$ is bounded and hence has exponential moments of all order. A slight simplification in (6.3) occurs, however, if $\Delta^2 + 2\Delta < 1$ in view of the formula

$$E\,\mathrm{e}^{\beta\tau_\xi} I(\tau_\xi < T_0) = \mathrm{e}^{\xi - \sqrt{\xi^2 - 2\beta}} G(T_0; \sqrt{\xi^2 - 2\beta}), \quad \beta \leq \frac{\xi^2}{2}, \tag{6.4}$$

which is immediate by an exponential family argument. We have not been able to find closed expressions if $\beta > \xi^2/2$.

**Proof of Proposition 6.1.** If $C(T)$ is the wating time until the next claim following $T$, then

$$\tau \wedge T = Z_1 + \cdots + Z_{\underline{n} \wedge \underline{n}_T} - C(T) I(\underline{n}_T \le \underline{n}). \tag{6.5}$$

Clearly, $P_\theta(C(T) > c) = e^{-\alpha_\theta c}$. Therefore the last term in (6.5) vanishes in the limit and using Wald's identity, we get

$$E_{\theta_1(1+\Delta)} \underline{n} \wedge \underline{n}_T \cong E_{\theta_1(1+\Delta)} \tau \wedge T / E_{\theta_1(1+\Delta)} Z \cong \frac{u^2}{\alpha_0 E_0 Y^2} E\tau_{\xi(1+\Delta)} \wedge T_0 \cdot \alpha_0$$

proving (6.1). Obviously (6.2) is equivalent to

$$E(\tau_\xi | \tau_\xi < T_0) = \frac{1}{\xi}\left\{1 - \frac{2\,e^{2\xi}\Phi(-T_0^{-1/2} - \xi T_0^{1/2})}{G(T_0;\xi)}\right\}. \tag{6.6}$$

Now the class of distributions of $\tau_\xi$ given $\{\tau_\xi < T_0\}$ form an exponential family with canonical parameter $\mu = -\xi^2/2$ and densities

$$\frac{e^\xi}{G(T_0;\xi)} e^{\mu t} \quad (0 < t < T_0) \quad \text{w.r.t.} \frac{1}{\sqrt{2\pi}} t^{-3/2} e^{-(1/2)t}\,dt.$$

Hence [4, Theorem 8.1]

$$E(\tau_\xi | \tau_\xi < T_0) = \frac{d}{d\mu} \log \frac{G(T_0;\xi(\mu))}{e^{\xi(\mu)}} = \frac{\frac{\partial G}{\partial \xi}\frac{\partial \xi}{\partial \mu}}{G(T_0;\xi)} - \frac{\partial \xi}{\partial \mu}.$$

Using (4.4), it is easily verified that

$$\partial G/\partial \xi = 2\,e^{2\xi}\Phi(-T_0^{-1/2} - \xi T_0^{1/2})$$

and since $\xi = \sqrt{-2\mu}$, we have $\partial \xi/\partial \mu = -1/\xi$ and (6.2) follows.

In (6.3), we get from (4.7) that

$$(E_{\theta_1(1+\Delta)} R_{\theta_1(1+\Delta)})^2 = \psi(u)^2 \cong G(T_0; -\xi)^2 = e^{-4\xi} G(T_0;\xi)^2.$$

and (6.3) now follows immediately from (5.4), (5.5) since

$$\begin{aligned} E_{\theta_1(1+\Delta)} R^2_{\theta_1(1+\Delta)} &= E_{\theta_1} \exp\{-(2\gamma + \theta_1\Delta)X(\tau) + \tau\kappa_{\theta_1}(\theta_1\Delta)\} I(\tau < T) \\ &\cong \exp\{-(4+\Delta)\xi\} E e^{\tau_\xi(\Delta^2 + 2\Delta)\xi^2/2} I(\tau_\xi < T_0). \qquad \square \end{aligned}$$

In the same way as for $T = \infty$, we are concerned with finding the value $\Delta_{\min}$ of $\Delta$ for which

$$g(\Delta) = \lim i_{\theta_1(1+\Delta)} v_{\theta_1(1+\Delta)} / u^2$$

is minimized.

As the first main consequence of Proposition 6.1, it is immediately observed that it is no longer true that $\Delta_{\min}=0$. That is, one can do better than to apply Lundberg simulation.

To find closed forms for $\Delta_{\min}$ does not look easy. A tabulation of $g(\Delta)$ (using (6.4) when possible and otherwise numerical integration) seemed to indicate that indeed a well-defined minimum of $g(\Delta)$ exists. Some values of $\xi$ and $T_0$ which we consider typical were selected, and $\Delta_{\min}$ computed numerically, cf. Table 2.

Table 2
$\Delta_{\min}$ as function of selected values of $\xi$, $T_0$

| $T_0\xi$ \ $\xi$ | 0.5 | 1 | 2 | 5 |
|---|---|---|---|---|
| 0.5 | 2.717 | 1.395 | 0.636 | 0.148 |
| 1 | 2.067 | 0.927 | 0.332 | 0.038 |
| 2 | 1.640 | 0.614 | 0.148 | 0.005 |
| 5 | 1.306 | 0.356 | 0.033 | 0.000 |

It is seen that for $T$ small is $\Delta_{\min}$ not only significantly different from zero but also larger than the value $\sqrt{2}-1$ which is critical when $T=\infty$. As expected, $\Delta_{\min}$ approaches zero as $T_0\to\infty$ with $\xi$ fixed.

For a comparison of simulations with $\theta=\theta_1$ or $\theta=\theta_1(1+\Delta_{\min})$ it is straightforward to compute $g(\Delta_{\min})/g(0)$ and we obtain a table of the asymptotic variance reduction (Table 3). It is also of interest to compare the two parameters to crude ($\theta=\theta_0$) simulation. Here

$$v_{\theta_0}=\psi(u,T)-\psi(u,T)^2\cong G(T_0;-\xi)(1-G(T_0;-\xi)),$$

Table 3
Asymptotic variance reduction $g(\Delta_{\min})/g(0)$

| $T_0\xi$ \ $\xi$ | 0.5 | 1 | 2 | 5 |
|---|---|---|---|---|
| 0.5 | 0.15 | 0.21 | 0.35 | 0.69 |
| 1 | 0.15 | 0.27 | 0.49 | 0.87 |
| 2 | 0.14 | 0.32 | 0.66 | 0.98 |
| 5 | 0.08 | 0.40 | 0.87 | 1.00 |

and it follows exactly as above that

$$i_{\theta_0}=E_{\theta_0}\boldsymbol{n}\wedge\boldsymbol{n}_T\cong\frac{u^2}{E_0Y^2}E\tau_{-\xi}\wedge T_0.$$

Now

$$\begin{aligned}E\tau_{-\xi}\wedge T_0&=E(\tau_{-\xi}\wedge T_0|\tau_{-\xi}<\infty)P(\tau_{-\xi}<\infty)+T_0P(\tau_{-\xi}=\infty)\\&=E\tau_{\xi}\wedge T_0\,\mathrm{e}^{-2\xi}+T_0(1-\mathrm{e}^{-2\xi}).\end{aligned}$$

Combining with (6.2), one can thus compute $g_c = \lim i_{\theta_0} v_{\theta_0}/u^2$ and Tables 4 and 5 give the corresponding asymptotic variance reductions $g(0)/g_c$, resp. $g(\Delta_{\min})/g_c$ when passing from $\theta = \theta_0$ to $\theta = \theta_1$, resp. $\theta = \theta_1(1+\Delta_{\min})$.

Table 4
Asymptotic variance reduction $g(0)/g_c$

| $T_0\xi$ \ $\xi$ | 0.5 | 1 | 2 | 5 |
|---|---|---|---|---|
| 0.5 | $4.4_{10}-2$ | $4.1_{10}-2$ | $2.3_{10}-2$ | $3.5_{10}-3$ |
| 1 | $1.9_{10}-2$ | $1.6_{10}-2$ | $6.4_{10}-3$ | $3.4_{10}-4$ |
| 2 | $3.0_{10}-3$ | $2.6_{10}-3$ | $6.4_{10}-4$ | $6.3_{10}-6$ |
| 5 | $8.6_{10}-6$ | $7.8_{10}-6$ | $7.2_{10}-7$ | $9.0_{10}-11$ |

Table 5
Asymptotic variance reduction $g(\Delta_{\min})/g_c$

| $T_0\xi$ \ $\xi$ | 0.5 | 1 | 2 | 5 |
|---|---|---|---|---|
| 0.5 | $6.5_{10}-3$ | $8.6_{10}-3$ | $8.0_{10}-3$ | $2.4_{10}-3$ |
| 1 | $2.9_{10}-3$ | $4.4_{10}-3$ | $3.2_{10}-3$ | $2.9_{10}-4$ |
| 2 | $4.3_{10}-4$ | $8.3_{10}-4$ | $4.3_{10}-4$ | $6.1_{10}-6$ |
| 5 | $6.9_{10}-7$ | $3.1_{10}-6$ | $6.2_{10}-7$ | $9.0_{10}-11$ |

It is seen that $\theta_1$ is much preferable to $\theta_0$. In some cases the further variance reduction by passing on to $\theta_1(1+\Delta_{\min})$ is considerable, in others not.

As an illustration of how results of the above type may be applied in practice and of the accuracy of the approximations, we shall give a final example.

Consider again the $P/E$ case with $\beta = p = 1$, $\alpha = 0.85$ and let $u = 15$, $T = 100$. Here $\gamma = 0.15$, $\theta_1 = \alpha^{1/2} - \alpha = 0.072$ so according to (4.8) we let $\xi = \theta_1 u = 1.079$. Finally let $T_0 = T\alpha_0 E_0 Y^2/u^2 = 2T/\alpha^{1/2}u^2 = 0.964$.

It was found numerically that $\Delta_{\min} = 0.8408$. Simulations were performed with $\theta = \theta_0$, $\theta = \theta_1$ and $\theta = \theta_1(1+\Delta_{\min})$, each $\theta$ allowed 5 seconds CPU time and the results can be summarised as in Table 6.

Table 6
Simulation estimates and empirical and asymptotic variances obtained by 5 seconds CPU time $P_\theta$-simulation, $P/E$ model, $\beta = p = 1$, $u = 15$, $T = 100$

| | $\bar{R}$ | $s^2/N$ | $s^2/N$ in % of $\theta_0$-value | asymptotical (%) |
|---|---|---|---|---|
| crude $\theta = \theta_0$ | 0.068 | $3.1_{10}-4$ | 100 | $g_c/g_c = 100$ |
| Lundberg $\theta = \theta_1$ | 0.064 | $6.4_{10}-6$ | 2.0 | $g(0)/g_c = 1.4$ |
| optimal $\theta = \theta_1(1+\Delta_{\min})$ | 0.059 | $3.1_{10}-6$ | 1.0 | $g(\Delta_{\min})/g_c = 0.4$ |

As a comparison, approximation (5.8) of [2] gave $\psi(u, T) = 0.062$ where according to the numerical evidence of [2] all figures should be correct.

It is seen that our theoretical results are supported qualitatively from Table 6: $\theta_1$ is much preferable to $\theta_0$ and $\theta_1(1+\Delta_{\min})$ even somewhat better. The quantitative agreement of the empirical and theoretical results (i.e., of the two last columns) occurs also reasonable at least when absolute (rather than relative) deviations are considered.

## 7. Two more complex examples

It might appear that the scope of the method developed so far is somewhat narrow by relying quite heavily on the structure of stationary independent increments of both signs, in particular for the definition of the exponential family $(P_\theta)$ and the Lundberg parameter $\theta_1$. Without claiming to demonstrate the opposite in any great generality, we therefore considered it worthwhile to sketch in some examples how one in fact can cope with problems which at a first sight look rather different and where simulation presumably is the only possibility.

Our first example is motivated from a common objection to the compound Poisson risk model (with constant drift), viz. that it is unrealistic to assume that the company takes no action if the risk reserve $U(t) = u - X(t)$ becomes close to zero. Thus assume that the premium rate $p$ is a function of the current risk reserve $v = V(t)$, say $p = p(v)$. An example of the paths of the risk reserve process $\{V(t)\}_{t\geq 0}$ is given in Figure 2 for the case where $p$ has only two values $p_0 > p_\infty$, $p(v) = p_0$ when $v \leq v_0$ and $p(v) = p_\infty$ when $v > v_0$.

The input process $I(t)$ is again assumed to be compound Poisson, $I(t) = \sum_1^{N_t} Y_n$. As a reasonable set of general conditions on $p = p(v)$, assume that $p$ is nonincreasing with limits $p_0 < \infty$, $p_\infty > \alpha EY$ as $v \downarrow 0$, resp. $v \uparrow \infty$. Clearly, the risk reserve process

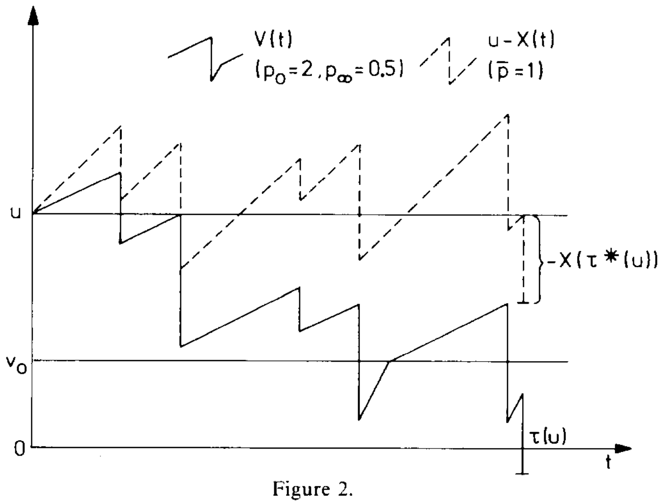


Figure 2.

satisfies the storage equation

$$V(T)=u-I(T)+\int_0^T p(V(t))\,\mathrm{d}t$$

as in [14], and the ruin probability is

$$\psi(u)=P(\tau^*(u)<\infty), \qquad \tau^*(u)=\inf\{t\geq 0\colon V(t)<0\}.$$

Since $p(v)\geq p(\infty)>\alpha EY$, $P(\tau(u)<\infty)<1$ for all $u$ and crude simulation does not apply (neither is it immediately clear how to apply regenerative simulation in this case). Instead we note that $\tau^*(u)$ is a stopping time not only w.r.t. $\{V(t)\}$ but also w.r.t. $\{I(t)\}$ and hence w.r.t. $\{X(t)\}_{t\geq 0}$ where $X(t)=I(t)-\bar{p}t$ with some arbitrary choice of $\bar{p}$. Thus (2.2) is applicable as before, and in summary, we suggest to proceed as follows:

(1) Choose $\bar{p}\in[p_\infty, p_0]$ and consider the compound Poisson model $X(t)=I(t)-\bar{p}t$;

(2) Solve the Lundberg equation for $X$;

(3) Lundberg simulate $I$ and keep track of both $V$ and $X$. Stop when at time $t=\tau^*(u)$ $V(t)<0$ and observe the response $R_{\theta_1}=\mathrm{e}^{-\gamma X(\tau^*(u))}$ (see Fig. 2 for an illustration). Replicate the experiment a suitable number of times.

The second example is a M/G/1 queue with an arrival intensity $\alpha(t)$ depending periodically upon time. It has been shown in [13] that if the traffic intensity $\rho$ (defined in an appropriate sense) is less than one and $W_1, W_2, \ldots$ denote the actual waiting times, then $\psi(u)=\lim_{n\to\infty}P(W_n>u)$ exists. However, it is not obvious how to compute $\psi(u)$ numerically and simulation may therefore be appropriate.

If $\alpha(t)$ is constant, then $\psi(u)$ is simply the ruin probability studied so far, cf. Section 5, and our method applies immediately. To show how one can deal also with the period case, one may first use an operational time argument to reformulate the model as a queue with stationary Poisson arrivals at unit rate and the server working at a periodic rate $\mathrm{d}B^*(t)/\mathrm{d}t$ where $B^*$ is the inverse function of $\int_0^t \alpha(s)\,\mathrm{d}s$ ($B^*$ may have jumps if $\alpha$ is identically zero on non-degenerate intervals). Let $Y_1, Y_2, \ldots$ be the service times, $V(t)$ the virtual waiting time at time $t$, $B(t)$ the periodic extension of $B^*(-t)$ to $[0,\infty)$ and $b$ the period of $B$, $B^*$. Then $\rho=bEY/B(b)$ and if $\rho<1$, it may be deduced from [13] after some manipulations that $\psi(u)=(1/b)\int_0^b \psi_a(u)\,\mathrm{d}a$ where

$$\psi_a(u)=\lim_{n\to\infty}P(V(a+nb)>u)=P\left(\max_{0\leq t<\infty} X_a(t)>u\right) \tag{7.1}$$

where

$$X_a(t)=\sum_{n=1}^{N_t} Y_n+B(a)-B(a+t).$$

If we write $B(t)=tB(b)/b+C(t)$, then $C$ has period $b$, and defining

$$X(t)=\sum_{n=1}^{N_t} Y_n-tB(b)/b, \qquad u_a(t)=u+C(a+t)-C(t),$$

$\tau_a = \inf\{t: X(t) > u_a(t)\}$, we may rewrite (7.1) as $\psi_a(u) = P(\tau_a < \infty)$. Furthermore $\{X(t)\}$ has the desired structure so that we may form $(P_\theta)$, compute the Lundberg value $\theta = \theta_1$ and the corresponding changed parameters. Because of $\rho < 1$, we have $P_{\theta_1}(\tau_a < \infty) = 1$ and we may Lundberg simulate with $R_{\theta_1} = \exp\{-\gamma X(\tau_a)\}$, cf. (2.2), to estimate $\psi_a(u)$. It is strongly suggested from Sect. 3 that this is superior to regenerative simulation which otherwise would be the approach suggested by [13]. For simulation of $\psi(u)$, we simply have in each replicate to draw the parameter $a$ uniformly on $[0, b]$.

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