Journal of Algebra **225**, 836–841 (2000)
doi:10.1006/jabr.1999.8160, available online at http://www.idealibrary.com on IDEAL®

Extending Characters and Lattices

Geoffrey R. Robinson

School of Mathematics and Statistics, University of Birmingham, Edgbaston,

Received April 20, 1999

INTRODUCTION

It is often the case that results about extending or lifting irreducible characters can be strengthened by insisting on rationality properties (e.g., improvements by Isaacs of the Fong–Swan theorem for p-solvable groups). We were motivated here by the fact that certain rationality properties can be explained by restricting the rings over which certain representations are realized. We also prefer to work at an "integral" level. We are especially interested in how behaviour with respect to different primes may be "glued together."

While our main theorem below is stated in somewhat more generality, it is motivated by the case of extending p-blocks of defect zero from normal subgroups. It is well known that if the normal subgroup has p-power index, then the unique irreducible complex character in the block of defect 0 of the normal subgroup extends. In fact,in this case, (as noted by M. Cabanes [2, 3]) the resulting block of the larger group is a nilpotent block, so its structure is very well understood by the work of Broué and Puig [1] and Puig [7] (in this special case, results of Brauer can be used to determine directly the usual invariants of the block). If p is odd, then it is easy to see, and well-known in this case, that there is a unique p-rational extension of the character of the normal subgroup, which may be regarded as the "canonical" choice of extension. If $p = 2$ there can be several p-rational extensions, and there seems to be no obvious reason to prefer one above another (it is also not clear at first sight that there need be *any* p-rational extension in this case.) We will see below that if we work at an integral level (over an appropriate local ring) there is a canonical choice of extension of a

0021-8693/00 $35.00

lattice affording the character in this case. We point out, however, that this canonical choice may depend on the choice of a prime ideal containing 2, so it is conceivable that the choice might not be canonical at the character level (i.e., a different choice of prime ideal might lead to a "canonical" lattice affording a different character). It turns out from the construction that the lattice is over a local subring of a cyclotomic field generated by odd order roots of unity, making the 2-rationality of its character transparent. The extension is canonical at the character level in the sense that it is invariant under any automorphism of the group which stabilizes the block of defect 0 of the normal subgroup.

B. Külshammer has informed us that it is also possible to derive the existence of a "canonical" 2-rational extension from results in Puig's paper [7]. We remark also that a "canonical" 2-rational extension is known to exist in the case of solvable groups by rather different methods of Isaacs.

1. EXTENDING LATTICES

THEOREM 1. *Let G be a finite group, π be a set of primes such that G/N is a π-group. Let μ be a G-stable irreducible character of N whose degree is divisible by $|N|_\pi$. Let h be the π'-part of the exponent of G, and let ω be a primitive complex h*th *root of unity. Let $R = \{\frac{\alpha}{\beta} : \alpha \in \mathbb{Z}[\omega], \beta \in \mathbb{Z}[\omega]\backslash\rho$ for each prime ideal divisor ρ of $|G|_\pi\}$. Then*:

(a) *μ extends to a π-rational irreducible character of G and furthermore, this extension is unique up to multiplication by linear characters of order 2.*

(b) *Each extension of μ in part (a) may be afforded by an R-free RG-module which is determined up to isomorphism by its character.*

Proof. We note that R is a Dedekind domain as it contains the Dedekind domain $\mathbb{Z}[\omega]$ and is contained in its field of fractions $\mathbb{Q}[\omega]$. Furthermore, R has only finitely many prime ideals, so it is a principal ideal domain (using the Chinese Remainder Theorem, for example). Hence μ may be afforded by an R-free RN-module V, which is unique up to isomorphism (by the results of [8], for example). Let σ: $N \to GL(\mu(1), R)$ be the associated matrix representation. We first claim that if a π-rational extension of μ, say $\tilde{\mu}$, exists, then (b) and the second part of (a) will follow. First, a π-rational extension of μ is unique up to multiplication by a π-rational linear character of G/N. Since G/N is a π-group, this linear character is rational-valued, hence has order at most 2. Let $\tilde{\sigma}$ be the extension of σ (to a complex representation of G to begin with) affording $\tilde{\mu}$. Choose a coset xN of N in G. Since $\mathrm{Res}_N^G(\tilde{\mu})$ is irreducible, we have $\sum_{n\in N} |\tilde{\mu}(xn)|^2 = |N|$, so we may suppose that $\tilde{\mu}(x) \neq 0$. Then we note that $\sum_{n\in C_N(x)\backslash N}[n, x]\sigma$ is either the zero matrix, or else intertwines the irreducible representations σ and σ^x of N. In any case, it is a scalar multiple of $x\tilde{\sigma}$. Taking traces, we see that it is in fact

$$\frac{[N : C_N(x)]\tilde{\mu}(x^{-1})}{\mu(1)} x\tilde{\sigma}.$$

Since $[n, x]\sigma \in GL(\mu(1), R)$ for all $n \in N$, we conclude that $(xn)\tilde{\sigma} \in GL(\mu(1), \mathbb{Q}[\omega])$ for all $n \in N$. To show that $x\tilde{\sigma} \in GL(\mu(1), R)$, it suffices to find, for each prime ideal ρ of $\mathbb{Z}[\omega]$ containing a rational prime $p \in \pi$, an element $y \in xN$ (which may vary with ρ) with $y\tilde{\sigma} \in GL(\mu(1), \mathbb{Z}[\omega]_\rho)$. Since $\sum_{n\in N} |\tilde{\mu}(xn)|^2/\tilde{\mu}(1) = |N|/\mu(1) \notin \rho$, we may choose $y \in xN$ so that

$$\frac{[N : C_N(y)]\tilde{\mu}(y^{-1})}{\mu(1)} \notin \rho.$$

Arguing as above, $\sum_{n\in C_N(y)\backslash N}[n, y]\sigma$ is $([N : C_N(y)]\tilde{\mu}(y^{-1})/\mu(1))y\tilde{\sigma}$, so that $y\tilde{\sigma} \in GL(\mu(1), \mathbb{Z}[\omega]_\rho)$, as required.

Notice that this argument shows that the extension $\tilde{\sigma}$ is uniquely determined by σ and the character $\tilde{\mu}$, so that (b) follows, as the RN-module V is unique up to isomorphism.

Now we consider the case that G/N is a p-group. Let ρ be a fixed prime ideal of $\mathbb{Z}[\omega]$ containing p. Let η be a primitive $|G|_p$th root of unity (in $\mathbb{C}$). There is a unique prime ideal of $\mathbb{Z}[\omega, \eta]$ containing ρ, say ρ', and this ideal is invariant under $Gal(\mathbb{Q}[\omega, \eta]/\mathbb{Q}[\omega])$. Choose $\alpha \in Gal(\mathbb{Q}[\omega, \eta]/\mathbb{Q}[\omega])$. Let $\hat{R}$ denote $\mathbb{Z}[\omega, \eta]_{\rho'}$. By a theorem of Heller ([4, 76.29]), the Krull–Schmidt theorem holds for $\hat{R}H$-modules for every section, H, of G. Now α induces a ring automorphism of $\hat{R}$ which fixes the subring R elementwise. We will prove that V has a unique extension to an RG-module $\tilde{V}$ such that the action of the vertex of $\tilde{V} \otimes_R \hat{R}$ on its source is unimodular. A simpler argument would work for odd p, but as the more subtle approach is necessary for the prime 2 in any case, we use it for all primes.

Let b be the block of $\hat{R}N$ containing μ (so that μ is the unique complex irreducible character "in" b by hypothesis). Let B be the block of $\hat{R}G$ covering b (it is unique, as G/N is a p-group). Let D be a defect group for B. Now B contains a unique (modular) simple module, which is the extension of a simple module for N. The simple module for G necessarily has vertex D and has height 0 by standard results, so that $[G : D]_p = |N|_p$, as the simple module for N "lifts" to an $\hat{R}N$-module affording character μ. Now B contains a complex character of height 0, say μ', which extends μ. Since $\hat{R}$ is a principal ideal domain whose field of fractions is a splitting field for G, μ' may be afforded by an $\hat{R}$-free $\hat{R}G$-module, U. Furthermore, as b has defect 0, $\mathrm{Res}^G_N(U)$ is (up to isomorphism) the unique $\hat{R}N$-module affording μ. Since $V \otimes_R \hat{R}$ is an $\hat{R}N$-module affording μ, we may assume that U affords a representation σ': $G \to GL(\mu(1), \hat{R})$ which extends σ: $N \to GL(\mu(1), R)$.

We claim that $D \cap N = 1$. For if $D \cap N \neq 1$, then $Z(D) \cap N \neq 1$. By a well-known result of Brauer, no irreducible character χ in B vanishes identically on the p-section of an element z of $Z(D)$ (for the convenience of the reader, this can be seen by considering $[G : C_G(zy)]\chi(zy)/\chi(1)$ for y a p-regular element of $C_G(D)$ whose class sum is not annihilated by the central character associated to B). However, for any $z \in Z(D) \cap N$, the p-section of z in G consists of p-singular elements of N, and μ vanishes on all such elements. Hence $D \cap N = 1$ and D is a complement to N in G. Thus $N_G(D) = DC_N(D)$. Let T be a source of U (note that U has vertex D.) Since T is $N_G(D)$-stable, all indecomposable summands of $\mathrm{Res}_D^G(U)$ which have vertex D are isomorphic to T. Furthermore, $t = \mathrm{rank}_{\hat{R}}(T)$ is not divisible by p by standard results, as $\mathrm{rank}_{\hat{R}}(U)_p = [G : D]_p$ (see [5, Sections 19 and 20]). Taking the determinant of the action of D on T affords a linear character, say λ, of D which inflates to a linear character of G with N in its kernel. Choose an integer m with $mt \equiv 1 \,(\mathrm{mod}\, |G|_p)$. Then we may tensor U with a rank 1 $\hat{R}G/N$-module and replace μ' by $\lambda^{-m} \otimes \mu'$ if necessary (this does not change the endomorphism ring) and assume that the action of D on T is unimodular. Since T has p'-rank, under this assumption, it is also clear that tensoring U with a nontrivial rank 1 $\hat{R}G/N$-module leads to an $\hat{R}G$-module such that D no longer has unimodular action on its source.

We recall our element $\alpha \in Gal(\mathbb{Q}[\omega, \eta]/\mathbb{Q}[\omega])$. Then $(\sigma')^\alpha$ also extends σ, and for each $x \in D$ and each $n \in N$, we have

$$\begin{aligned}(x\sigma')^{-1}(n\sigma)(x\sigma') &= (x^{-1}nx)\sigma' \\ &= (x^{-1}nx)\sigma'^{\alpha} \\ &= (x\sigma'^{\alpha})^{-1}(n\sigma'^{\alpha})(x\sigma'^{\alpha}) \\ &= (x\sigma'^{\alpha})^{-1}(n\sigma)(x\sigma'^{\alpha}).\end{aligned}$$

Hence $x\sigma'^{\alpha}$ is a scalar multiple of $x\sigma'$. Since these two matrices each have p-power order, the scalar multiple must be of the form η^j for some integer j. However, D must still have unimodular action on a source of U^α as this source is (isomorphic to) T^α. This forces $\eta^j = 1$, as T has p'-rank. In particular, $x\sigma' \in GL(\mu(1), \mathbb{Q}[\omega])$ as α was arbitrary in $Gal(\mathbb{Q}[\omega, \eta]/\mathbb{Q}[\omega])$. This shows that σ' maps into $GL(\mu(1), \mathbb{Q}[\omega])$, so that μ' is a p-rational extension of μ. The argument used at the beginning of the proof now shows that σ' maps into $GL(\mu(1), R)$, hence that U "comes from" an RG-module, say $\tilde{V}$, which is uniquely determined by the character μ'. This completes the proof in the case that G/N is a p-group. We remark, as a matter of general interest, that $\tilde{\mu}$ is invariant under any automorphism of G which stabilizes both N and μ. For, by a Frattini argument, it suffices to consider an automorphism β which normalizes D. Then D has unimodular action on T^β, which is a source of U^β, so by the uniqueness properties established above, $U \cong U^\beta$.

Now we proceed to the general case. We define a projective representation (in Schur's sense) σ': $G \to GL(\mu(1), R)$ as follows: for each prime $p \in \pi$, we choose a Sylow p-subgroup S_p of G. We let B_p be the unique p-block of NS_p covering the p-block of N containing μ. We let D_p be a defect group of B_p and extend σ to the distiguished representation σ_p: $ND_p \to GL(\mu(1), R)$ constructed as in the case that G/N is a p-group. We then set $x\sigma' = x\sigma_p$ for each $x \in ND_p$. We claim that this extends σ' in a well-defined way on cosets xN of N in G such that xN has odd prime power order in G/N, and determines $x\sigma'$ up to multiplication by ± 1 in the case that xN has 2-power order. This is because σ has a unique extension to a representation of $\langle x \rangle N$ over R in the case of x an element of odd prime order by the p-group case, while for $p = 2$, the extension is only unique up to tensoring with a linear character of order 2. It is possible for x, an arbitrary 2-element of G, that two different conjugates of x lie in S_2.

Let us turn for the moment to the case that G/N is cyclic, say $\langle gN \rangle$, where $g \in G$ is a π-element. We set $g\sigma' = \Pi_{p \in \pi}(g_p\sigma')$, using the above definitions for each g_p. Notice that this places $g\sigma'$ in $GL(\mu(1), R)$. We claim that this extends the representation σ to a true representation of G in this special case. For it at least allows us to define a projective representation of G extending σ. The subgroup of $GL(n, R)$ generated by the $g_p\sigma'$'s is cyclic (mod scalars), so is Abelian. Since each $g_p\sigma'$ is a p-element, we see that $g\sigma'$ is a π-element, and that σ' is a true representation of G in this special case. Notice once more that $g\sigma'$ is uniquely determined in the case that gN has odd order, but that is only really determined up to a multiple of ± 1 when gN has even order.

Now let us return to the general case. The sceptical reader is invited to fill in some cohomological details in the argument below, bearing in mind that we are dealing with the case that G/N is a π-group and that we have a projective representation over R (which only contains roots of unity of the form $\pm\omega^m$). We use the case G/N cyclic to define $g\sigma'$ for each π element $g \in G$ (up to multiples of ± 1 in the case that gN has even order). Fixing one choice of $g\sigma'$ for each coset gN of even order then produces a 2-cocycle of G/N which takes values in $\{1, -1\}$. Since σ does extend to a representation of S_2N, it follows from a well-known result of Gaschütz ([6]) that we may "kill" this 2-cocycle to produce a true representation of G (this only involves replacing $g\sigma'$ by $-g\sigma'$ for certain g with gN of even order, so does not affect realizability or rationality questions).

Remark. We remark in passing that it is relatively easy to prove, using Brauer's characterization of characters, that as long as we define σ' above on 2-elements so that $g \to \text{trace}(g\sigma')$ is a class function, then this class function is an irreducible character realizable over R (we omit the details, but this is ultimately because in the case that G/N is Brauer elementary *all* 2-rational extensions of μ to S_2N have canonical τ-rational extensions to G, where $\tau = \pi \backslash \{2\}$).

REFERENCES

1. M. Broué and L. Puig, A Frobenius theorem for blocks, *Invent. Math.*, **56** (1980), 117–128.
2. M. Cabanes, Extensions of p-groups and construction of characters, *Comm. Algebra*, **15**, No. 6, (1986), 1297–1311.
3. M. Cabanes, A note on extensions of p-blocks by p-groups and their characters, *J. Algebra*, **115** (1988), 445–449.
4. C. Curtis and I. Reiner, Representation Theory of Finite Groups and Associative Algebras, Wiley Interscience, New York, 1962.
5. C. Curtis and I. Reiner, Methods of Representation Theory, Vol. I, Wiley Interscience, New York, 1981.
6. W. Gaschütz, Zur Erweiterungstheorie der endlichen Gruppen, *J. Reine Angew. Math.*, **190** (1952), 93–107.
7. L. Puig, Nilpotent blocks and their source algebras, *Invent. Math.*, **93**, No. 1 (1988), 77–116.
8. G. R. Robinson and R. Staszewski, On the representation theory of π-separable groups, *J. Algebra*, **119** (1988), 226–232.