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A Model of Porous Catalyst Accounting for Incipiently Non-isothermal Effects*

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An approximate model accounting for incipiently non-isothermal effects is derived from a well-known model of porous catalyst for appropriate, realistic limiting values of the parameters. In this limit, the original model is a singularly [illegible] boundary of the spatial domain. Some limiting cases are still considered in the approximate model that lead to further simplifications.

1. INTRODUCTION

The reaction–diffusion system

$$L\partial u/\partial t = \Delta u - \phi^2 f(u, v), \qquad \partial v/\partial t = \Delta v + \beta\phi^2 f(u, v) \qquad \text{in } \Omega, \tag{1.1}$$

has received great attention in the literature, as a prototype for several physical problems dealing with an exothermic, irreversible chemical reaction in a spatial domain Ω. Here $u > 0$ and $v > 0$ are the non-dimensional reactant concentration and temperature, respectively, $L > 0$ (*Lewis number*) is a ratio of thermal to material diffusivity, ϕ^2 (*Damköhler number*) is a measure of the reaction rate relative to the diffusion rate, and $\beta > 0$ is the non-dimensional, *chemical heat release* (βL is the ratio of the heat of reaction to the thermal energy in the domain Ω). If f is as given in Eq. (1.4) below then the system (1.1) is the simplest *thermo-diffusive model* for a premixed flame in Combustion theory [1]; in this case Ω is usually an unbounded cylinder (to model the burner) and the relevant solutions are *travelling waves* propagating along the axis of the cylinder. This model also

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applies in *porous catalyst theory* [2]. In this case Ω is usually bounded (to model a catalyst particle), and the following boundary conditions are imposed,

$$\partial u/\partial n = \sigma(1-u), \qquad \partial v/\partial n = \nu(1-v) \qquad \text{at } \partial\Omega, \tag{1.2}$$

to model mass and heat exchange with the outer unreacted fluid. Here n is the outward unit normal to the smooth boundary of the domain Ω and the *material and thermal Biot numbers*, $\sigma > 0$ and $\nu > 0$, are the ratios of the rates of mass and heat transfer between the surface of the catalyst and the external fluid to the corresponding rates of mass and heat transfer within the catalyst. The appropriate initial conditions are

$$u = u_0(x) > 0, \qquad v = v_0(x) > 0 \qquad \text{in } \Omega, \quad \text{at } t = 0, \tag{1.3}$$

and the relevant solutions are the attractors as $t \to \infty$, which may be steady states, limit cycles, and quasi-periodic and more complex chaotic attractors. The nonlinearity f depends on the type of *global kinetic law* that is assumed to model the several physico–chemical processes (*adsorption* of the reacting species at the internal surface of the porous body, *chemical reaction* in the adsorbed state, and *desorption*; see [2]) that are present (in addition to inertia and diffusion). The usual *Arrhenius* and *Langmuir–Hinshelwood* (also named after *Michaelis* and *Menten* in the mathematical Biology literature; the model above is also appropriate for *enzymes*) kinetic laws lead to the following nonlinearities,

$$f(u, v) = u^p \exp(\gamma - \gamma/v), \tag{1.4}$$

$$f(u, v) = u^p[u + k\exp(\gamma_a - \gamma_a/v)]^{-q} \exp(\gamma - \gamma/v), \tag{1.5}$$

where $p > 0$ and $q > 0$ are *reaction orders*, $k > 0$ is the *adsorption–desorption constant* and $\gamma \geqslant 0$ and $\gamma_a \geqslant 0$ are the *activation energies*.

Precise statements about the large time behavior of (1.1)–(1.3.) (with f as given in (1.4) or (1.5)) require direct numerical simulation except in some limiting cases (fortunately, those of practical interest) when simpler asymptotic submodels apply that are more amenable to purely analytical treatment. This paper deals with the rigorous derivation of one such submodel, which is posed by

$$\partial v/\partial t = \Delta v \qquad \text{in } \Omega, \qquad \partial v/\partial n = \nu(1-v) + \beta\phi \int_{-\infty}^{0} f(u, v)\, d\xi \qquad \text{at } \partial\Omega; \tag{1.6}$$

where, at each point $p \in \partial\Omega$, the function $u = u(\xi, t)$ is given by

$$(L/\phi^2)\,\partial u/\partial t = \partial^2 u/\partial \xi^2 - f(u, v) \qquad \text{in} \quad -\infty < \xi < 0, \tag{1.7}$$

$$u = 0 \qquad \text{at} \quad \xi = -\infty, \qquad \partial u/\partial \xi = (\sigma/\phi)(1-u) \qquad \text{at} \quad \xi = 0, \tag{1.8}$$

where the function $v = v(t)$ is in (1.7) the temperature v at p. The new rescaled variable ξ is

$$\xi = \phi\eta, \tag{1.9}$$

where η is a coordinate along the outward unit normal to $\partial\Omega$ at p. Thus this submodel consists of the heat Eq. (1.6) coupled (through the nonlinear boundary condition (1.6)) with infinitely many 1-D semilinear parabolic equations (i.e., the Eqs. (1.7)), one for each point of $\partial\Omega$. At first sight this (somewhat non-standard) model seems to be more involved than the original model (1.1)–(1.2), but this is not really so and in fact the submodel exhibits several advantages that will be explained in Section 4.

A formal derivation, via singular perturbation techniques, of (1.6)–(1.8), was given in [3], in the distinguished limit

$$\beta \to 0, \qquad \phi \to \infty, \qquad \sigma \to \infty, \qquad \beta\sigma\phi/(\phi+\sigma) \sim v \sim L/\phi^2 \sim 1. \tag{1.10}$$

In fact, in this paper we shall derive the submodel (1.6)–(1.8) in a range of limiting values of the parameters wider than that in (1.10); see assumption (1.11) below. That limit is realistic because the parameters appearing in (1.1) vary in the range [2]

$$10^{-2} < \beta < 1, \qquad 10^{-6} < \phi^2 < 2500, \qquad 5 < \sigma < 10^2,$$

$$10^{-2} < v < 5, \qquad 10^{-3} < L < 10^2.$$

Inside this parameter range, β and v/σ are frequently small because porous catalysts usually exhibit a large thermal conductivity.

The main interest of the submodel (1.6)–(1.8) is that it exhibits a large variety of codimension-two and -three bifurcations [3] that predict complex large-time dynamics. This is in contrast to other submodels of (1.1). For instance, if $\beta \to 0$, $\beta\phi\sigma/(\phi+\sigma) \to 0$, and $v \to 0$ then one obtains two *isothermal submodels* (for $\phi = O(1)$ and $\phi \to \infty$), first considered in [4, 5], which seem to exhibit no more complex attractors than steady states and limit cycles; see [5–7] for the (first formal and then rigorous) derivation of these submodels, [5, 8, 9] for several properties of the submodels concerning the steady states, local bifurcations, and global stability properties, and [10–13] for related submodels of general reaction–diffusion problems.

Let us now explain intuitively how this submodel is obtained in the limit (1.10). Since ϕ^2 is large the chemical reaction is quite strong and, after some time, the reactant (is consumed and its) concentration u becomes very small in Ω except in a thin boundary layer near $\partial\Omega$ (where it cannot be that small because of the boundary condition (1.2)). Since in addition, $f(0, v)=0$, the reaction term becomes also quite small outside the boundary layer and the temperature v thus evolves according to the heat equation. The appropriate boundary condition (1.8), to be imposed at the internal edge of the boundary layer, accounts for the heat flux through this internal edge, which equals the heat exchange with the external medium (the first term in the right hand side) plus the total heat produced by the chemical reaction, in the boundary layer, along each normal to $\partial\Omega$ (the second term). Notice that this balance relies on three approximations, namely, (i) a *quasi-steady approximation* for the evolution of the temperature in the boundary layer that requires the thermal inertia v_t to be appropriately bounded, (ii) a *quasi-one-dimensional approximation* that requires the thermal diffusion along each normal to $\partial\Omega$ (essentially, $\partial^2 u/\partial\eta^2$) to dominate the transversal diffusion ($\tilde{\Delta} v$, where $\tilde{\Delta}$ stands for the Laplacian along the hypersurfaces parallel to $\partial\Omega$) for, otherwise heat exchange with other neighboring normals to $\partial\Omega$ should also be taken into account in the above-mentioned heat–flux balance, and (iii) a *quasi-isothermal approximation* along each normal in the boundary layer that requires the thermal gradient along the normals to be appropriately bounded. The evolution of the reactant concentration u in the boundary layer, at each normal to $\partial\Omega$, is given by (1.7)–(1.8) if, again, a quasi-one-dimensional approximation (requiring $\tilde{\Delta} u$ to be appropriately controlled) holds.

The *main object of this paper is to provide a rigorous derivation of* (1.6)–(1.8), which will be made in Section 2. More precisely, we shall prove that, after some time T, (i) u is quite small except in a thin boundary layer near $\partial\Omega$, and (ii) the concentration u in the boundary layer and the temperature v satisfy (1.6)–(1.8) in first approximation, *uniformly* in $t \geqslant T$. The fact that the remainders are uniformly small as $t \to \infty$ is essential if we pretend that our model provides the large-time dynamics of (1.1)–(1.2) in first approximation. The approximate model (1.6)–(1.8) will be briefly analyzed in Section 3.

Let us now state precisely the assumptions to be used below. We shall consider the limit

$$\begin{aligned}
&\phi \to \infty, \qquad \beta\phi\sigma/[(\phi+\sigma)\,v] = O(1),\\
&\sigma^{-1} = O(1), \qquad v = O(1), \qquad \phi^{-1/3} = O(\beta\phi\sigma/(\phi+\sigma)), \qquad (1.11)\\
&\log(1+\phi/L) = O(\phi).
\end{aligned}$$

The first two conditions are essential for the asymptotic model to apply, as we explain now. ϕ must be large for the boundary condition near $\partial\Omega$ to develop, and $\beta\phi\sigma/[(\phi+\sigma)\,v]$ must be bounded for the temperature v to be uniformly bounded; in order to physically explain the latter, notice that the total heat produced in the boundary layer (i.e., the second term in the right hand side of (1.6), which is of the order of $\beta\phi\sigma/(\phi+\sigma)$, must be compensated by the heat loss through the boundary, $v(v-1)$, in order to control the temperature inside Ω. The last four conditions of (1.11) are only imposed for technical reasons and could be deleted if a more involved way of deriving the asymptotic model (than that below) were followed. The domain Ω and the nonlinearity f will be assumed to be such that

(H.1) $\Omega \subset \mathbb{R}^m$ ($m \geqslant 1$) is a bounded domain, with a $C^{4+\alpha}$ (for some $\alpha > 0$) boundary. Notice that then Ω satisfies uniformly the interior and exterior sphere conditions: there are two constants, $\rho_1 > 0$ and $\rho_2 > 0$, such that, for each $x \in \partial\Omega$, two hyperspheres of radii ρ_1 and ρ_2, S_1 and S_2 are tangent to $\partial\Omega$ at x and satisfy $S_1 \subset \Omega$ and $\bar{S}_2 \cap \Omega = \{x\}$ (with overbars standing hereafter for the closure).

(H.2) The C^1-function $f\colon [0, \infty[\times [0, \infty[\to \mathbb{R}$ is such that $f(0, v) = 0$ for all $v \geqslant 0$, and $f(u, v) > 0$ whenever $u > 0$ and $v > 0$.

(H.3) There is a continuous, increasing function $g_1 : [0, \infty[\to \mathbb{R}$ such that

$$f(u, v) \leqslant g_1(u) \qquad \text{if} \quad u \geqslant 0 \qquad \text{and} \qquad v \geqslant 0.$$

(H.4) There are two strictly positive constants, k_1 and k_2, and a positive, continuous, decreasing function, $g_2 : [0, \infty[\to \mathbb{R}$, such that

$$k_2 u \leqslant f(u, v) \leqslant k_1 u \qquad \text{if} \quad 0 \leqslant u \leqslant 2 \qquad \text{and} \qquad v \geqslant 1/2,$$

$$0 < u g_2(u) \leqslant f(u, v) \qquad \text{if} \quad u > 0 \qquad \text{and} \qquad v \geqslant 1/2.$$

(H.5) There are three constants, $k_3 > 0$, $k_4 > 0$, and $k_5 > 0$ such that

$$k_3 < f'_u(u, v) < k_4, \qquad |f'_v(u, v)| \leqslant k_5 u$$

$$\text{if} \quad 0 \leqslant u \leqslant \sigma/(\sigma + \phi\sqrt{k_2/2m}) \qquad \text{and} \qquad v \geqslant 1/2.$$

In addition, the initial conditions (1.3) will be assumed be such that

(H.6) $\|u_0\|_{C(\bar{\Omega})} = O(1)$ and $\|v_0\|_{C(\bar{\Omega})} = O(1)$ in the limit (1.11).

The assumptions (H.1)–(H.6) are the same as those imposed in [7] to derive the second quasi-isothermal submodel, and deserve the same remarks made there, which are not repeated here for the sake of brevity.

2. MATHEMATICAL DERIVATION OF THE APPROXIMATE MODEL

Under the assumptions (H.1)–(H.3) the parabolic problem (1.1)–(1.3) is readily seen to have a unique classical solution in $0 \leqslant t < \infty$. In order to derive the asymptotic model (1.6)–(1.8) we could proceed in a somewhat straightforward manner, following the main ideas in our intuitive justification given above, as follows. We would first prove that *after some time*:

(i) The concentration u (and thus the reaction term $\phi^2 f(u, v)$; see assumption (H.2)) becomes quite small except in a thin boundary layer (whose thickness is of the order of ϕ^{-1}) near $\partial\Omega$, and the temperature v becomes uniformly bounded by a $O(1)$ quantity in $\bar{\Omega}$.

(ii) $|v_t|$ is bounded by a $O(1)$ quantity in $\bar{\Omega}$, and the first derivative of v along the normals to $\partial\Omega$ is bounded by a $O(1)$ quantity in the boundary layer.

(iii) The first derivatives of u and v along the normals to $\partial\Omega$, and the first and second derivatives of u and v along the hypersurfaces parallel to $\partial\Omega$, become small as compared to the corresponding second derivatives of u and v along the normals to $\partial\Omega$, in the boundary layer.

Notice that properties (ii) and (iii) would justify the *quasi-steady and quasi-isothermal approximations* for v and the *quasi-one-dimensional approximations* for u and v (in the boundary layer) that were mentioned above. Properties (i)–(iii) would allow us to readily obtain the asymptotic model (1.6)–(1.8) for (a) the heat Eq. (1.6) would apply (in first approximation) in the bulk (i.e., outside the boundary layer) according to property (i); (b) the (1-D) Eq. (1.7) would apply (in first approximation) to u in the boundary layer, according to property (iii); and (c) the boundary condition (1.6), at the internal edge of the boundary layer (which coincides with $\partial\Omega$ in first approximation) would be readily obtained upon integration of the second Eq. (1.1) along each normal to $\partial\Omega$, from the internal edge of the boundary layer up to the boundary (notice that, according to properties (ii) and (iii), only the second derivative along the normals to $\partial\Omega$ and the nonlinear term need to be considered in first approximation). Now, property (i) readily comes from Lemma 2.1 below, which is a straighforward extension of results already proven in [7]. But, in order to prove properties (ii) and (iii) we would need to follow a fairly involved and technical process. Notice that the problem is singularly perturbed and usual *a priori* estimates do not directly provide the required results; these estimates provide bounds for the derivatives of u and v that are much weaker than needed (see Lemma 2.1 below). Thus we shall not pursue the ideas above. Instead, for the sake of brevity, we shall follow a somewhat tricky and *ad hoc* approach, which relies on the following decomposition of the temperature v for $t \geqslant T_1$ (with T_1 as defined in Lemma 2.1 below),

$$v = v_1 + V, \tag{2.1}$$

where v_1 and V are defined as

$$\begin{aligned} \partial v_1/\partial t &= \Delta v_1 - \phi^{2/3} v_1 + \beta\phi^2 f(u, v) && \text{in } \Omega, \\ \partial v_1/\partial n &= -\phi v_1 && \text{at } \partial\Omega, \end{aligned} \tag{2.2}$$

$$\begin{aligned} \partial V/\partial t &= \Delta V + \phi^{2/3} v_1 && \text{in } \Omega, \\ \partial V/\partial n &= \nu(1 - V) + (\phi - \nu) v_1 && \text{at } \partial\Omega, \end{aligned} \tag{2.3}$$

for $t > T_1$, with initial conditions

$$v_1(\cdot, T_1) = 0, \qquad V(\cdot, T_1) = v(\cdot, T_1) \qquad \text{in } \bar{\Omega}. \tag{2.4}$$

The main idea in this decomposition is connected with the main difficulty of obtaining close bounds on the derivatives of v from the second Eq. (1.1), namely, that the nonlinearity f has the bad sign in this equation, and the spatial derivatives of f in the boundary layer are quite large because $|\nabla u|$ is quite large there; thus, the usual a priori estimates applied to the equations giving the derivatives of v (which are obtained upon derivation of the second Eq. (1.1)) give results that are not good enough for our purposes. In our decomposition of v, the nonlinearity f appears only in the equation giving v_1; but because of the dissipative terms we have introduced in both the equation and the boundary condition (2.2) (namely, $-\phi^{2/3} v_1$ and $-\phi v_1$, respectively) we can show that both v_1 and their derivatives are appropriately small. Of course, there is a price for the introduction of these dissipative terms, namely, that they appear as forcing terms in (2.13); but these forcing terms are not too strong and both V and their derivatives can be controlled.

A further simplification in the analysis below will result from our use of local time averages and local spatial averages along the hypersurfaces parallel to $\partial\Omega$ when bounding both the remainders that are neglected in the asymptotic model and the difference between their solutions and those of the original model (in the proof of Theorem 2.4 below). Then we shall not obtain optimal results (because of the loss of precision associated with the averaging process) but the derivation will be greatly simplified because we shall only need to obtain bounds of the first-order spatial derivatives of u, v_1, and V and of the (1/2)-temporal-Hölder oscillation of u and v_1 (instead of the bounds of the second-order spatial derivatives and first-order temporal derivatives that will be needed in order to obtain optimal results).

This section is organized as follows. In Section 2.1 we first give (in Lemmata 2.1–2.3) the above-mentioned bounds on u, v, v_1, and V, and on their first-order spatial derivatives and (1/2)-temporal-Hölder oscillation. With these bounds at hand, we shall obtain the asymptotic model in Theorem 2.4, which is the main result of the paper. For the sake of clarity, we omit in Section 2.1 the proofs of Lemmata 2.1–2.3, which are given in Section 2.2 along with the statements and proofs of some additional, purely technical results that are also needed.

2.1. *Derivation of the Model*

Let us begin with some notation. If ρ_1 is as defined in assumption (H.1) (at the end of Section 1), we consider the domain $\Omega_1 \subset \Omega$, defined as

$$\Omega_1 = \{x \in \Omega : d(x) < \rho_1/2\}, \tag{2.5}$$

where $d(x)$ stands hereafter for the distance from x to $\partial\Omega$. Notice that the hypersurfaces parallel to $\partial\Omega$ are well defined and smooth in Ω_1, where we can use a curvilinear coordinate system based on these hypersurfaces and on their common normals. Also, the intrinsic gradient operator,

$$\tilde{\nabla} = \text{gradient along the hypersurfaces parallel to } \partial\Omega, \tag{2.6}$$

which will be frequently used below, is well-defined in Ω_1. For convenience, we shall use below the Hölder, temporal, local oscillation bound, defined as

$$\langle w \rangle_t^{(\alpha)} = \sup\{|w(x, t') - w(x, t'')|/|t' - t''|^\alpha : t \leqslant t' < t'' \leqslant t+1\}, \quad \text{for} \quad 0 \leqslant \alpha \leqslant 1, \tag{2.7}$$

with $0 < \alpha \leqslant 1$. Notice that $\langle w \rangle_t^\alpha$ depends on x and t, and that its lower, upper bound in $x \in \bar{\Omega}$ is the usual α-Hölder temporal seminorm in $\bar{\Omega} \times [t, t+1]$; see, e.g., [14].

LEMMA 2.1. *Under the assumptions of* (H.1)–(H.4) *and* (H.6) (*at the end of Section* 1) *every solution of* (1.1)–(1.3) *is such that*

$$\mu_0 \exp[-\sqrt{2k_1}\, \phi d(x)] < u < \mu_1 \exp[-\sqrt{k_2/m}\, \phi d_1(x)], \qquad 1/2 < v < 1 + \beta\phi\mu_1/v, \tag{2.8}$$

$$\begin{aligned} (L/\phi)\, \|u\|_{C^{0,1/2}(\bar{\Omega}\times[t,t+1])} + |\nabla u| &< \phi\mu_1 \exp[-\sqrt{k_2/m}\, \phi\, d_1(x)], \\ \|v\|_{C^{1,1/2}(\bar{\Omega}\times[t,t+1])} &< 1 + \beta\phi\mu_1 + \beta\mu_1\phi^{(2m+5)/(m+3)}, \end{aligned} \tag{2.9}$$

whenever $x \in \bar{\Omega}$ *and* $t \geqslant T_1$, *where* ρ_1, k_1, *and* k_2 *are as defined in assumptions* (H.1) *and* (H.4), $d(x)$ *is the distance from* x *to* $\partial\Omega$, $d_1(x) = \min\{d(x), \rho_1\}$ *and* T_1, μ_0, *and* μ_1 *satisfy*

$$T_1 = O(1 + (L/\phi^2)\log(2 + \sigma/\phi + \phi/\sigma) + (1/v)\log(2 + \beta\phi^2/v))$$

$$\mu_0^{-1} = O((\sigma + \phi)/\sigma), \qquad \mu_1 = O(\sigma/(\sigma + \phi)),$$

in the limit (1.11). *Also*, μ_0 *and* μ_1 *depend only on the quantities*

$$\phi, \sigma, \beta, v, \tag{2.10}$$

and T_1 *depends only on these quantities*, L, $\|u_0\|_{C(\bar{\Omega})}$, *and* $\|v_0\|_{C(\bar{\Omega})}$.

LEMMA 2.2. *Under the assumptions* (H.1)–(H.6) (*at the end of Section* 1) *let* (u, v) *be a solution of* (1.1)–(1.3), *and let* v_1 *and* T_1 *be as defined in* (2.2), (2.4), *and Lemma* 2.1. *Then we have*

$$\begin{aligned} &|v_1| < \mu_2 \quad \text{in } \bar{\Omega}; \\ &|\nabla v_1| < \mu_2/\delta(x), \qquad \langle v_1 \rangle_t^{(1/2)} < \mu_2/\delta(x)^{5/4} \qquad \text{in} \quad \bar{\Omega}_1, \end{aligned} \tag{2.11}$$

$$|\nabla v_1| + \langle v_1 \rangle_t^{(1/2)} < \mu_2 \qquad \text{in } \Omega \backslash \Omega_1, \tag{2.12}$$

for all $t \geqslant T_1 + 1$, *where* $\delta(x) = \phi^{-1}$ *or* $\phi^{-2/3}$ *depending on whether* $d(x) < \phi^{-2/3}$ *or* $d(x) \geqslant \phi^{-2/3}$ ($d(x) =$ *distance from* x *to* $\partial\Omega$, *as above*) *and* μ_2 *depends only on the quantities* (2.10), $\mu_2 = O(\beta\sigma/(\phi + \sigma))$ *in the limit* (1.11).

LEMMA 2.3. *Under the assumptions of Lemma* 2.2, *let* $\tilde{\nabla}$ *be the gradient along the hypersurfaces parallel to* $\partial\Omega$ *and let* $\langle \cdot \rangle_t^{(1/2)}$ *be as defined in* (2.7). *If* (u, v) *is a solution of* (1.1)–(1.3), *and* v_1 *and* V *are as defined in* (2.2)–(2.4) *then the following estimates hold*,

$$\begin{aligned} &|\tilde{\nabla} u| + \langle u \rangle_t^{(1/2)} < [\mu_3 \phi\sigma/(\phi + \sigma)] \exp[-\sqrt{k/m}\, \phi d(x)/2], \\ &|\tilde{\nabla} v_1| + \langle v_1 \rangle_t^{(1/2)} < \mu_3, \qquad |\nabla V| < \mu_3 \phi, \end{aligned} \tag{2.13}$$

if $x \in \bar{\Omega}_1$ *and* $t \geqslant T_2$, *where* Ω_1 *is as defined in* (2.5), $k = \min\{k_2, k_3\}$, *with* k_2 *and* k_3 *as defined in assumptions* (H.4)–(H.5), $d(x)$ *is as in Lemma* 2.1, *the quantity* μ_3 *depends only on the quantities* (2.10), *and* $\mu_3 = O(\beta\sigma\phi^{1/3}/(\phi + \sigma))$ *and* $0 \leqslant T_2 - T_1 = O(1 + L/\phi^2)\log(\phi + \phi^2/L)$ *in the limit* (1.11).

The results in lemmata above provide the ingredients for the derivation of the asymptotic model (1.6)–(1.8) that will be made in the proof of

Theorem 2.4 below. For convenience, let us write down the asymptotic model again as

$$\frac{\partial V}{\partial t} = \Delta V + \psi_1(x, t) \qquad \text{in } \Omega, \tag{2.14}$$

$$\frac{\partial V}{\partial n} = v(1 - V) + \beta\phi \int_{-\infty}^{0} f(U, V)\, d\xi + \psi_2(s, t) \qquad \text{at } \partial\Omega, \tag{2.15}$$

where at each point $s \in \partial\Omega$, U satisfies

$$\frac{L}{\phi^2}\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial \xi^2} - f(U, V(s, t)) \qquad \text{in} \quad -\infty < \xi < 0, \tag{2.16}$$

$$U = 0 \qquad \text{at} \quad \xi = -\infty, \qquad \frac{\partial U}{\partial \xi} = \frac{\sigma}{\phi}(1 - U) \qquad \text{at} \quad \xi = 0. \tag{2.17}$$

THEOREM 2.4. *Under the assumptions* (H.1)–(H.6) (*at the end of Section* 1), *there are two constants*, $\lambda > 0$ *and* $\mu > 0$, *and for each solution of* (1.1)–(1.3), *there is a solution of* (2.14)–(2.17) *and a constant* $\tilde{T} \geqslant 0$ *such that*:

(i) λ *depends only on the domain* Ω *and on the nonlinearity* f, μ *depends only on* Ω, L, f *and on the quantities* (2.10), *and* $\tilde{T}$ *depends only on* Ω, L, f, *on the quantities* (2.10), *and on* $\|u_0\|_{C(\bar{\Omega})}$ *and* $\|v_0\|_{C(\bar{\Omega})}$.

(ii) μ *and* $\tilde{T}$ *are such that*

$$\mu = O(\beta\phi^{2/3}\sigma/(\phi + \sigma)), \tag{2.18}$$

$$\tilde{T} = O(L/\phi^2 + 1/v)\log(2 + \sigma/\phi + \phi/\sigma) + O(2 + L/\phi^2)\log(\phi + \phi^2/L), \tag{2.19}$$

in the limit (1.11).

(iii) *For all* $t \geqslant \tilde{T}$ *we have*

$$\begin{aligned} |V(x, t) - v(x, t)| < \mu, \ |\psi_1(x, t)| < \mu \qquad &\text{if} \quad x \in \bar{\Omega}, \\ |\psi_2(s, t)| < \mu \qquad &\text{if} \quad s \in \partial\Omega, \end{aligned} \tag{2.20}$$

$$|U(-\phi d(x), s, t) - u(x, t)| < [\mu\sigma/(\phi + \sigma)] \exp[-\lambda\phi d(x)] \qquad \text{if} \quad d(x) < \rho_1/2, \tag{2.21}$$

where $d(x)$ *is the distance from* x *to* $\partial\Omega$, $s \in \partial\Omega$ *is the point where such a distance is reached, and* ρ_1 *is as defined in assumption* (H.1).

Proof. For each $x \in \bar{\Omega}$ such that $d(x) < \rho_1/2$, let $H(x)$ be the hypersurface parallel to $\partial\Omega$ passing through x and let $\mathscr{B}(x)$ be defined as

$$\mathscr{B}(x) = \{ y \in H(x) : \delta(x, y) < \gamma(x) \}, \tag{2.22}$$

where δ is the geodesic distance along $H(x)$ and $\gamma(x)$ is selected such that the measure of $\mathscr{B}(x)$ equals $\phi^{-2(m-1)/3}$, that is,

$$\int_{\mathscr{B}(x)} dS = \phi^{-2(m-1)/3}. \tag{2.23}$$

Thus $\mathscr{B}(x)$ is a geodesic hypersphere of $H(x)$ centered at x, and the geodesic radius of $\mathscr{B}(x)$ is of the order of $\phi^{-2/3}$ as $\phi \to \infty$, uniformly in $d(x) < \rho_1/2$. Now, let us define the functions U_1 and V_1 as

$$U_1 = \phi^{2(m-1)/3} \int_{\mathscr{B}(x)} u(\cdot, t)\, dS, \tag{2.24}$$

$$V_1 = \phi^{2(m+1)/3} \int_t^{t+\phi^{-4/3}} \left[\int_{\mathscr{B}(x)} v_1(\cdot, t)\, dS \right] d\tau. \tag{2.25}$$

Notice that, according to Lemma 2.3 and the mean value theorem we have

$$\begin{aligned} &|V_1 - v_1| \leqslant C_1 \mu_3 \phi^{-2/3}, \\ &|U_1 - u| \leqslant C_1 [\mu_3 \phi^{1/3} \sigma/(\phi + \sigma)] \exp[-\lambda \phi d(x)] \\ &\qquad \text{if} \quad d(x) > \rho_1/2 \qquad \text{and} \qquad t \geqslant \tilde{T}, \end{aligned} \tag{2.26}$$

where μ_3 is as in Lemma 2.3, $C_1 > 0$ depends only on the domain Ω, and

$$\lambda = \sqrt{k/8m}, \qquad \tilde{T} = T_2, \tag{2.27}$$

with k and T_2 as in Lemma 2.3. Also, λ and $\tilde{T}$ satisfy the stated properties (see Lemmata 2.1–2.3).

Now, for each point $s \in \partial\Omega$, let η be a coordinate along the outward unit normal to $\partial\Omega$ at s. If we take into account the expression for the Laplacian operator in Lemma 2.8 below, and the equations and boundary conditions (1.1)–(1.2) and (1.4), then U_1 and V_1 are seen to satisfy

$$\partial^2 V_1/\partial\eta^2 + \beta\phi^2 f(U_1, V(s, t)) = \varphi_1(\eta, s, t), \tag{2.28}$$

$$L\, \partial U_1/\partial t - \partial^2 U_1/\partial\eta^2 + \phi^2 f(U_1, V(s, t)) = \varphi_2(\eta, s, t)$$

$$\text{if} \quad -\rho_1/2 < \eta < 0, \tag{2.29}$$

$$
\begin{aligned}
\partial V_1/\partial\eta + \phi V_1 &= \varphi_3(s, t), \\
\partial U_1/\partial\eta - \sigma(1-U_1) &= \varphi_4(s, t) \qquad \text{at} \quad \eta = 0,
\end{aligned}
\tag{2.30}
$$

$$
\partial V_1/\partial\eta = \varphi_5(s, t) \qquad \text{at} \quad \eta = -\phi^{-2/3}, \tag{2.31}
$$

where $V = V(x, t)$ is as defined by (2.3)–(2.4) and

$$
\begin{aligned}
\varphi_1(\eta, s, t) = {} & \phi^{2(m+1)/3} \Bigg[\int_{\mathscr{B}(\eta)} [v_1(\cdot, t+\phi^{-4/3}) - v_1(\cdot, t)] \, dS \\
& - (m-1) \int_t^{t+\phi^{-4/3}} \frac{\partial}{\partial\eta} \int_{\mathscr{B}(\eta)} M v_1 \, dS \, d\tau \Bigg] \\
& + \phi^{2(m+1)/3} \int_t^{t+\phi^{-4/3}} \Bigg[\int_{\mathscr{B}(\eta)} (\phi^{2/3} v_1 - \beta\phi^2 f(u, v)) \, dS \\
& - \int_{\partial\mathscr{B}(\eta)} \tilde{\nabla} v_1 \cdot \tilde{n} \, dS_1 \Bigg] d\tau + \beta\phi^2 f(U_1, V(s, t)),
\end{aligned}
\tag{2.32}
$$

$$
\begin{aligned}
\varphi_2(\eta, s, t) = {} & \phi^{2(m-1)/3} \Bigg[\int_{\mathscr{B}(\eta)} \Bigg((m-1) \frac{\partial}{\partial\eta} (Mu) - (m-1)^2 M^2 u \\
& - \phi^2 f(u, v) \Bigg) dS - \int_{\partial\mathscr{B}(\eta)} \tilde{\nabla} u \cdot \tilde{n} \, dS_1 \Bigg] \\
& + \phi^2 f(U_1, V(s, t)),
\end{aligned}
\tag{2.33}
$$

$$
\begin{aligned}
\varphi_3(s, t) &= -(m-1) \, \phi^{2(m+1)/3} \int_t^{t+\phi^{-4/3}} \int_{\mathscr{B}(0)} M v_1 \, dS, \\
\varphi_4(s, t) &= -(m-1) \, \phi^{2(m-1)/3} \int_{\mathscr{B}(0)} M u \, dS,
\end{aligned}
\tag{2.34}
$$

$$
\varphi_5(s, t) = \phi^{2(m+1)/3} \int_t^{t+\phi^{-4/3}} \int_{\mathscr{B}(-\phi^{-2/3})} [\partial v_1/\partial\eta - (m-1) M v_1] \, dS \, d\tau.
\tag{2.35}
$$

Here $\tilde{\nabla}$ and $\tilde{n}$ are the gradient operator and the outward unit normal to $\partial\mathscr{B}(\eta)$ along the hypersurfaces parallel to $\partial\Omega$ and M is the mean curvature of such hypersurfaces and, in addition to integrating by parts, we have taken into account Eq. (2.52) below. If, in addition, we use the assumption (H.5) (at the end of Section 1) and the results in Lemmata 2.1, 2.2, and 2.3, and apply the mean value theorem, then we obtain the following estimates,

$$\left| \int_{-\phi^{-2/3}}^{0} \varphi_1(\eta, s, t)\, d\eta \right| < \mu_4, \tag{2.36}$$

$$|\varphi_2(\eta, s, t)| < [\phi^2 \mu_4 \sigma/(\phi + \sigma)] \exp(\lambda \phi \eta),$$

$$|\varphi_3(s, t)| < \mu_4, \qquad |\varphi_4(s, t)| < C_2 \sigma/(\phi + \sigma),$$

$$|\varphi_5(s, t)| < \mu_4, \qquad \text{if} \quad -\rho_1/2 < \eta < 0, \quad s \in \partial\Omega \quad \text{and} \quad t \geqslant \tilde{T}, \tag{2.37}$$

where $C_2 > 0$ depends only on the domain Ω, and μ_4 depends only on Ω, f and the quantities (2.10) and satisfies $\mu_4 = O(\beta\phi^{2/3}\sigma/(\phi + \sigma))$ in the limit (1.11).

If we now integrate Eq. (2.28) in $-\phi^{-2/3} < \eta < 0$, take into account (2.30) and (2.31), and add $\phi v_1(s, t)$ to both sides of the resulting equation, we obtain the following expression for $v_1(s, t)$,

$$\phi v_1(s, t) = \beta\phi^2 \int_{-\phi^{-2/3}}^{0} f(U_1, V(s, t))\, d\eta + \varphi_6(s, t), \tag{2.38}$$

where the remainder $\varphi_6(s, t)$ is given by

$$\varphi_6(s, t) = \phi[v_1(s, t) - V_1(s, t)] + \varphi_3(s, t) - \varphi_5(s, t) - \int_{-\phi^{-2/3}}^{0} \varphi_1(\eta, s, t)\, d\eta,$$

and, according to (2.26), (2.36), and (2.37) satisfies

$$|\varphi_6(s, t)| < \mu_5 \qquad \text{if} \quad s \in \partial\Omega \qquad \text{and} \qquad t \geqslant \tilde{T},$$

with μ_5 depending only on Ω, f, and the quantities (2.10), and satisfying $\mu_5 = O(\beta\phi^{2/3}\sigma/(\phi + \sigma))$ in the limit (1.11).

Now, for each $s \in \partial\Omega$, let $U = U(\eta, s, t)$ be the unique solution of

$$\begin{aligned} L\partial U/\partial t &= \partial^2 U/\partial \eta^2 - \phi^2 f(U, V(s, t)) && \text{in} \quad -\infty < \eta < 0, \\ \partial U/\partial \eta &= \sigma(1 - U) && \text{at} \quad \eta = 0, \end{aligned} \tag{2.39}$$

if $t \geqslant \tilde{T}$, with initial conditions

$$U(\eta, s, \tilde{T}) = U_1(\eta, s, \tilde{T}) \qquad \text{if} \quad -\rho_1/2 \leqslant \eta \leqslant 0, \tag{2.40}$$

$$U(\eta, s, \tilde{T}) = U_1(-\rho_1/2, s, \tilde{T}) \exp[\sqrt{k_1/2m}\, \phi(\eta + \rho_1/2)] \qquad \text{if} \quad -\infty < \eta < -\rho_1/2, \tag{2.41}$$

where k_1 and $\tilde{T}$ are as defined in assumption (H.4) and Eq. (2.27). Since $k_2 U < f(U, V) < k_1 U$ (assumption (H.4)) and $0 < U(\eta, s, \tilde{T}) < [\sigma/(\phi\sqrt{k_2/m} + \sigma)] \exp[-\sqrt{k_2/2m}\,\phi d(x)]$ (see Eqs. (2.8), (2.24), and (2.40)–(2.41)) maximum principles readily imply that

$$0 < U(\eta, s, t) < [\sigma/(\phi\sqrt{k_2/m} + \sigma)] \exp(\sqrt{k_2/2m}\,\phi\eta)$$
$$\text{if} \quad -\infty < \eta \leqslant 0, \quad s \in \partial\Omega, \quad \text{and} \quad t \geqslant \tilde{T}, \tag{2.42}$$

and assumption (H.5) and the mean value theorem readily imply that

$$f(U_1, V) - f(U, V) = h(\eta, s, t)(U_1 - U), \qquad \text{with} \quad h \geqslant k_3$$
$$\text{if} \quad -\rho_1/2 < \eta < 0, \quad s \in \partial\Omega, \quad \text{and} \quad t \geqslant \tilde{T}. \tag{2.43}$$

Then if λ is as defined in (2.27) and μ_6 is defined as

$$\mu_6 = \max\{2\mu_4\sigma/[k_3(\phi + \sigma)], C_2\sigma/[\lambda\phi(\phi + \sigma)],$$
$$[\sigma(\phi\sqrt{k_2/m} + \sigma)] \exp(-\sqrt{k_2/2m}\,\phi\rho_1/4)\}, \tag{2.44}$$

the functions $w_{\pm} = \mu_6 \exp(\lambda\phi\eta) \pm (U_1 - U)$ are readily seen to satisfy (see (2.29)–(2.30), (2.36)–(2.37), and (2.39)–(2.43)) for all $s \in \partial\Omega$,

$$\partial w_{\pm}/\partial t - \partial^2 w_{\pm}/\partial\eta^2 + \phi^2 h(\eta, s, t)\, w_{\pm} > 0$$
$$\text{in} \quad -\rho_1/2 < \eta < 0, \qquad \text{if} \quad t > \tilde{T},$$
$$w_{\pm} > 0 \qquad \text{at} \quad \eta = -\rho_1/2,$$
$$\partial w_{\pm}/\partial\eta + \sigma w_{\pm} > 0 \qquad \text{at} \quad \eta = 0, \qquad \text{if} \quad t > \tilde{T},$$
$$w_{\pm} > 0 \qquad \text{in} \quad -\rho_1/2 < \eta < 0 \qquad \text{if} \quad t = \tilde{T},$$

and maximum principles readily imply that $w_{\pm} > 0$, i.e., that

$$|U_1 - U| \leqslant \mu_6 \exp(\lambda\phi\eta) \qquad \text{if} \quad -\rho_1/2 \leqslant \eta \leqslant 0, \qquad s \in \partial\Omega \qquad \text{and} \qquad t \geqslant \tilde{T}. \tag{2.45}$$

Also, according to the definition (2.44), μ_6 depends only on Ω, f, and the quantities (2.10), and satisfies $\mu_6 = O(\beta\phi^{2/3}\sigma/(\phi + \sigma) + \phi^{-1})\,\sigma/(\phi + \sigma)$ in the limit (1.11).

Now, Eqs. (2.16) and (2.17) are readily obtained from (2.39) when using the rescaled variable

$$\xi = \phi\eta, \tag{2.46}$$

and the estimate (2.21) readily follows from the estimates (2.26) and (2.45), with the upper bound μ satisfying the stated properties. Also, Eqs. (2.14) and (2.15) are obtained from Eqs. (2.3) and (2.38) if the variable (2.46) is used, with

$$\psi_1(x, t) = \phi^{2/3} v_1,$$

$$\psi_2(s, t) = -\nu v_1 + \beta\phi \left[\int_{-\phi^{1/3}}^{0} f(U_1, V(s, t))\, d\xi - \int_{-\infty}^{0} f(U, V(s, t))\, d\xi \right] + \varphi_6(s, t)).$$

Finally, when using Lemma 2.2, assumption (H.5), and Eqs. (2.26) and (2.42), and applying the mean value theorem, $|\psi_1|$, $|\psi_2|$, and $|v - V| = |v_1|$ are seen to satisfy the inequalities (2.20), with the upper bound μ satisfying the stated properties. Thus the proof is complete.

Remarks 2.5. *Four Remarks about Theorem* 2.4. (a) The result in Theorem 2.4 shows that, after an initial transient $0 \leqslant t \leqslant \tilde{T}$, the solution of (1.1)–(1.3) becomes close to a solution of the asymptotic model (2.14)–(2.17) in the sense of the estimates (2.20) and (2.21). Notice that in the limit (1.11), $\mu \ll \beta\phi\sigma/(\phi + \sigma)$ and $\mu \ll 1$. Since, on the other hand (see Lemma 2.1 and Eqs. (2.20)–(2.21)), $U \sim \sigma/(\phi + \sigma)$, $|V - 1| \sim \beta\phi\sigma/(\phi + \sigma)\, \nu$, and $|\Delta V|$ is, at least, of the order of $\nu\, |V - 1|$, we have $|u - U| \ll U$, $|v - V| \ll V$, $|\psi_1| \ll |\Delta V|$, and $|\psi_2| \ll \nu\, |V - 1|$. Thus, both the errors, $|u - U|$ and $|v - V|$, and the remainders, $|\psi_1|$ and $|\psi_2|$, are appropriately small.

(b) The heat Eq. (2.14) applies in the whole domain Ω, and not in a slightly smaller sub-domain (i.e., outside the boundary layer) as our physical explanation in Section 1 suggested. This has been so because Eq. (2.14) applies to V and not to the original variable v.

(c) A close look at the proofs of Lemmata 2.2 and 2.3 and Theorem 2.4 shows that our estimates are not optimal. But, as pointed out at the beginning of this section, in order to avoid a too technical and lengthy derivation, we did not pretend any optimality.

(d) The estimates (2.20)–(2.21) hold uniformly in $\tilde{T} < t < \infty$. This is essential, for if, according to these estimates, we consider the distance associated with the norm

$$\|(U(\cdot, \cdot, t), V(\cdot, t))\| = \sup\{U(\xi, s, t)\exp(-\lambda\xi)\colon -\rho_1\phi/2 < \xi < 0,\ s \in \partial\Omega\} + \sup\{|V(x, t)|\colon x \in \Omega\},$$

then the solutions of the original problem and of the asymptotic model remain close in finite time intervals (after the initial transient), as readily seen via maximum principles. As a consequence, *the exponential attractors as $t \to \infty$ of both problems are close to each other*; of course, non-exponential attractors need not be close. This is the sense in which the asymptotic behavior as $t \to \infty$ of the original model may be approximated by that of the asymptotic model.

2.2. *The Proof of Lemmata* 2.1–2.3 *and Some Technical Results*

In order to prove Lemmata 2.1–2.3, which is the object of this subsection, we need four technical results, which are considered first. The following result provides the key ingredient to obtain the estimates (2.5) in Lemma 2.1.

LEMMA 2.6. *Under the assumptions* (H.1)–(H.4) *and* (H.6) (*at the end of Section* 1), *there is a constant* T_0, *depending on* $\|u_0\|_{C(\bar{\Omega})}$, $\|v_0\|_{C(\bar{\Omega})}$, L, *and the quantities* (2.10) *such that* (i) $T_0 = O((L/\phi^2)\log(2+\sigma/\phi+\phi/\sigma) + (1/v)\log(2+\beta\phi^2/v))$ *in the limit* (1.11), *and* (ii) *the solution of* (1.1)–(1.3) *satisfies*

$$u_1 < u(\cdot, t) < u_2, \qquad 1/2 < v(\cdot, t) < 2 + \hat{v}_1 \qquad \text{in} \quad \Omega, \qquad \text{if} \quad t \geqslant T_0,$$

where u_1, u_2, *and* $\hat{v}_1$ *are the unique solutions of the following linear problems*,

$$\begin{aligned} &\Delta u_1 = 2k_1\phi^2 u_1 && \text{in } \Omega, && \partial u_1/\partial n = \sigma(1-u_1) && \text{at } \partial\Omega \\ &\Delta u_2 = k_2\phi^2 u_2/2 && \text{in } \Omega, && \partial u_2/\partial n = \sigma(1-u_2) && \text{at } \partial\Omega \\ &\Delta \hat{v}_1 + \beta k_1\phi^2 u_2 = 0 && \text{in } \Omega, && \partial \hat{v}_1/\partial n = -v\hat{v}_1 && \text{at } \partial\Omega \end{aligned}$$

with the constants $k_1 > 0$ *and* $k_2 > 0$ *as defined in assumption* (H.4).

Proof. [7, Lemma 2.2]. Notice that the scaling of the time variable here is different from that in [7].

In order to bound the functions u_1, u_2, and v_1 appearing in Lemma 2.6 we shall need the following technical result, which will be also needed in the sequel.

LEMMA 2.7. *Under the assumption* (H.1) (*at the end of Section* 1), *let* $\tilde{u}$ *and* $\tilde{v}$ *be the solutions of*

$$\Delta\tilde{u} = \Lambda^2 u \qquad \text{in } \Omega, \qquad \partial\tilde{u}/\partial n = \sigma(1-\tilde{u}) \qquad \text{at } \partial\Omega, \tag{2.47}$$

$$\Delta\tilde{v} + \varepsilon\Lambda^2\tilde{u} = 0 \qquad \text{in } \Omega, \qquad \partial\tilde{v}/\partial n = -\gamma\tilde{v} \qquad \text{at } \partial\Omega, \tag{2.48}$$

where Λ, ε, σ, and γ are strictly positive. Then the following estimates hold,

$$[\sigma/(\sigma+\delta_1)] \exp[-\delta_1 d(x)] \leqslant \tilde{u}(x) \leqslant [\sigma/(\sigma+\delta_2)] \cosh[\delta_2(\rho_1 - d_1(x))]/\cosh(\delta_2 \rho_1),$$
$$0 < \tilde{v}(x) < \delta_3,$$

for all $x \in \bar{\Omega}$, where ρ_1 is as defined in assumption (H.1), *$d(x)$ is the distance from x to $\partial\Omega$, $d_1(x) = \min\{d(x), \rho_1\}$, and the positive constants δ_1, δ_2, and δ_3 satisfy*

$$\begin{aligned} &\delta_2 = \Lambda/\sqrt{m}, \qquad |\delta_1 - \Lambda| = O(\Lambda^{-1}) \\ &\delta_3 = O(\varepsilon\sigma(\Lambda+\gamma)/((\sigma+\Lambda)\gamma))) \qquad \text{as} \quad \Lambda \to \infty, \end{aligned} \tag{2.49}$$

uniformly in $\varepsilon > 0$, $\sigma > 0$, and $\gamma > 0$.

Proof. The estimates for $\tilde{u}$ and the lower estimate for $\tilde{v}$ are proven in [7, Lemma 2.1]. The upper estimate for $\tilde{v}$ is readily obtained when taking into account the fact that $\tilde{v}_1 = \tilde{v} - \varepsilon[\sigma/(\sigma+\delta_2) - \tilde{u}]$ satisfies $\Delta\tilde{v}_1 = 0$ in Ω, $\partial\tilde{v}_1/\partial n = -\gamma\tilde{v}_1 - \varepsilon\gamma[\sigma/(\sigma+\delta_2) - \tilde{u}] + \varepsilon\sigma(1-\tilde{u})$ at $\partial\Omega$. But, according to the estimates for $\tilde{u}$, we have $\sigma/(\sigma+\delta_1) \leqslant u \leqslant \sigma/(\sigma+\delta_2)$ at $\partial\Omega$, and thus when maximum principles [15] are applied, we get $\tilde{v}_1 \leqslant \varepsilon\sigma\delta_1/((\sigma+\delta_1)\gamma)$ in $\bar{\Omega}$, or $\tilde{v} \leqslant \varepsilon\sigma/(\sigma+\delta_2) + \varepsilon\sigma\delta_1/((\sigma+\delta_1)\gamma)$ and the upper estimate for $\tilde{v}$ readily follows. Thus the proof is complete.

The following result provides a decomposition of the Laplacian operator in terms of the derivatives along the outward unit normal to the boundary and the intrinsic Laplacian operator along the hypersurfaces parallel to the boundary.

LEMMA 2.8. *Under the assumption* (H.1) (*at the end of Section* 1), *the Laplacian of a function $w \in C^2(\Omega_1)$ at $p \in \Omega_1$ is given by*

$$\Delta w = \partial^2 w/\partial\eta^2 - (m-1)\, M(p)\, \partial w/\partial\eta + \tilde{\Delta} w \qquad \text{at } p,$$

where η is a coordinate along the outward unit normal to $\partial\Omega$, $M(p)$ is the mean curvature of the hypersurface H passing through p (*with the sign of M chosen according to the outer unit normal to H*) *and $\tilde{\Delta}$ is the* (*intrinsic*) *Laplacian operator along H.*

Proof. Let $x = x_0(\eta^2, ..., \eta^m)$ be a C^3-regular parametric representation of a neighborhood of p in H such that the associated parametric lines are orthogonal everywhere and $x_{0\eta^k} \cdot x_{0\eta^l} = \delta_{kl}$ (= the Kronecker symbol) at p. If $n = n(\eta^2, ..., \eta^m)$ is the outward unit normal to H, then

$$x = \eta^1 n(\eta^2, ..., \eta^m) + x_0(\eta^2, ..., \eta^m)$$

defines a C^2-coordinate system in a neighborhood $\mathcal{N}$ of p in $\mathbb{R}^m$ such that the hypersurfaces $\eta^1 = \text{constant}$ are precisely those parallel to H (and to $\partial\Omega$), and

$$\eta^1 = \eta - d(p). \tag{2.50}$$

Also, the co- and contra-variant components of the associated metric tensor are such that

$$\begin{aligned} &g_{11} = g^{11} = 1, \qquad g_{kl} = g^{kl} = 0 \qquad \text{if} \quad k \neq l \qquad \text{in } \mathcal{N}; \\ &g_{kk} = g^{kk} = 1 \qquad \text{at} \quad p \qquad \text{for all } k. \end{aligned} \tag{2.51}$$

If G is the determinant of the $m \times m$ matrix (g_{ij}), then we have

$$\partial G/\partial \eta^1 = 2 \sum_{k=2}^{m} \bar{n}_{\eta^k} \cdot \bar{x}_{0\eta^k} = -2 \sum_{k=2}^{m} L_{kk} = -2(m-1)\, M(p), \qquad \text{at } p. \tag{2.52}$$

Here we have used (2.51), the definition of the mean curvature [18], and the Weingarten equations (see [18, p.115] or [19, Vol. III, p.11]) applied to H at p that, according to (2.51), may be written as

$$n_{\eta^k} = -\sum_{l=2}^{m} g^{ll} L_{kl}\, x_{0\eta^l} \qquad \text{at } p,$$

where L_{kl} are the coefficients of the second fundamental form associated with the parametric representation $x = x_0(\eta^2, ..., \eta^m)$.

Now, the Laplacian of w in $\mathcal{N}$ is given by

$$\Delta w = G^{-1/2} \frac{\partial}{\partial \eta^1}\left(G^{1/2} \frac{\partial w}{\partial \eta^1}\right) + G^{-1/2} \sum_{i,j=2}^{m} \frac{\partial}{\partial \eta^i}\left(G^{1/2} g^{ij} \frac{\partial w}{\partial \eta^j}\right),$$

where we have used (2.51). Then we only need to use (2.50) and (2.52), and take into account the fact that the last term in the right hand side is precisely $\tilde{\Delta} w$; notice that the stated result does not depend on the coordinate system $(\eta^2, ..., \eta^m)$. Thus the proof is complete.

The following result was proven in [7] and provides the key ingredient for obtaining that part of the estimates (2.13) dealing with the gradient along the hypersurfaces parallel to $\partial\Omega$.

LEMMA 2.9. [7, Lemma 2.7]. *Under the assumptions* (H.1) (*at the end of Section* 1) *let* $\tilde{t}_0$ *be a unit vector that is tangent to a hypersurface* H, *parallel to* $\partial\Omega$, *at* $p \in \bar{\Omega}_1$. *Then there are a neighborhood* $\mathcal{N}$ *of* p *in* $\mathbb{R}^m$, *a* C^3-*vector field* $\tilde{t}: \mathcal{N} \to \mathbb{R}^m$, *two vectors* a_1 *and* a_2, *and two scalars* b_1 *and* b_2, *such that the following properties hold*:

(i) a_1 *and* a_2, b_1 *and* b_2 *depend continuously on* p *and* $\tilde{t}_0$.

(ii) $\tilde{t} = \tilde{t}_0$ *at* p, $\tilde{t} \cdot \tilde{t} = 1$ *in* $\mathcal{N}$, *and, for each* $q \in \mathcal{N} \cap \Omega_1$, $\tilde{t}(q)$ *is tangent to the hypersurface parallel to* $\partial\Omega$ *passing through* q.

(iii) *If* $I \subset \mathbb{R}$ *is an open interval and* $w: (\mathcal{N} \cap \bar{\Omega}_1) \times I \to \mathbb{R}$ *is* C^3-*function satisfying*

$$\partial w/\partial t = \Delta w + \varphi \qquad \text{in} \quad (\mathcal{N} \cap \bar{\Omega}_1) \times I,$$

then the $C^{2,1}$-*function* $w_1 = \nabla w \cdot \tilde{t}$ *satisfies*

$$\partial w_1/\partial t = \Delta w_1 + a_1 \cdot \nabla w_1 + a_2 \cdot \nabla w + b_1 w_1 + \nabla\varphi \cdot \tilde{t} \qquad \text{at } p, \quad \text{for all} \quad t \in I, \tag{2.53}$$

$$\partial w_1/\partial n = \nabla(\partial w_1/\partial n) \cdot \tilde{t} + b_2 w_1 \qquad \text{at } p, \qquad \text{for all} \quad t \in I, \tag{2.54}$$

where n *is the outward unit normal to* H.

Now we have the ingredients to prove Lemmata 2.1–2.3. The first two proofs are based on Lemmata 2.6 and 2.7, and on standard estimates up to the boundary and imbedding theorems, which are used in conjunction with appropriate rescalings of the space and time variables.

Proof of Lemma 2.1. Let us define $T_1 = 1 + L/\phi^2 + T_0$, where T_0 is as defined in Lemma 2.6. The estimates (2.8) readily follow when Lemmata 2.6 and 2.7 above are applied. The first estimate (2.9) is obtained precisely as in [7, Lemma 2.6] (recall that the scaling for the time variable here is different from that in [7] and that when the time variable is rescaled as $t = \varepsilon\tau$ then $\|u\|_{C^{0,\alpha}(\bar{\Omega} \times [t, t+1])} \leqslant \varepsilon^{-1} \|u\|_{C^{0,\alpha}(\bar{\Omega} \times [\tau, \tau+1])}$ whenever $0 < \varepsilon \leqslant 1$ and $0 \leqslant \alpha \leqslant 1$), and the second estimate (2.9) readily follows upon application of L_p estimates up to the boundary [16, p.133] and imbedding theorems [14, p. 80] to the second Eq. (1.1) and the boundary condition (1.2).

Proof of Lemma 2.2. Let u_2 be as in Lemma 2.6, and let us define the function $v_2 > 0$ as the unique solution of

$$\Delta v_2 + k_1 \beta \phi^2 u_2 = 0 \qquad \text{in } \Omega, \qquad \partial v_2/\partial n = -\phi v_2 \qquad \text{at } \partial\Omega,$$

which, according to Lemma 2.7, satisfies

$$(0 <)\, v_2 \leqslant O(\beta\sigma/(\phi + \sigma)) \qquad \text{uniformly in } x \qquad \text{in } \bar{\Omega}. \tag{2.55}$$

Now, according to assumption (H.4) and Eq. (2.8) we have $|f(u, v)| < k_1 u_2$ if $x \in \bar{\Omega}$ and $t > T_1$ and thus v_2 is a super-solution of (2.2). When applying maximum principles to (2.2), (2.4) we obtain $0 \leqslant v_1(x, t) < v_2(x)$ if $x \in \bar{\Omega}$ and $t \geqslant T_1$, and the first estimate (2.11) readily follows from (2.55).

In order to obtain the remaining estimates (2.11), let (x_0, t_0) be such that $x_0 \in \bar{\Omega}_1$ and $t_0 \geqslant T_1 + 1$, and let us consider the stretched variables $\xi = (x - x_0)/\delta(x_0)$ and $\tau = (t - t_0)/\delta(x_0)^2$, to rewrite (2.2) as

$$\partial v_1/\partial\tau = \Delta_\xi v_1 - \delta(x_0)^2\,\phi^{2/3} v_1 + \beta\phi^2\,\delta(x_0)^2\, f(u, v)$$

$$\text{if} \quad x_0 + \delta(x_0)\xi \in \Omega,$$

$$\partial v_1/\partial n = -\delta(x_0)\,\phi v_1 \qquad \text{if} \quad x_0 + \delta(x_0)\,\xi \in \partial\Omega.$$

Since, in addition, according to (H.4), Eq. (2.8) and the first estimate (2.11),

$$|-\delta(x_0)^2\,\phi^{2/3} v_1 + \beta\phi^2\,\delta(x_0)^2\, f(u, v)|$$

$$= O(\beta\sigma/(\phi + \sigma)) \qquad \text{if} \quad |\xi| \leqslant 2 \qquad \text{and} \qquad -1 \leqslant \tau \leqslant 1,$$

uniformly in $x_0 \in \bar{\Omega}_1$ and $t_0 \geqslant T_1 + 1$, we only need to apply local L_p estimates (up to the boundary if $d(x_0) < \phi^{-1/2}$, and interior estimates otherwise) and imbedding theorems to obtain $|\nabla_\xi v_1| + \langle v_1 \rangle_\tau^{(4/5)} = O(\beta\sigma/(\phi + \sigma))$ at $\xi = 0$, $\tau = 0$, uniformly in $x_0 \in \bar{\Omega}_1$, $t_0 \geqslant T_1 + 1$. When coming back to the original variables we readily get $|\nabla v_1| < \delta(x_0)^{-1}\, O(\beta\sigma/(\phi + \sigma))$ and $\langle v_1 \rangle_t^{(4/5)} < \delta(x_0)^{-2}\, O(\beta\sigma/(\phi + \sigma))$, uniformly in $x_0 \in \bar{\Omega}_1$ and $t_0 \geqslant T_1 + 1$, and the second estimate (2.11) follows. In order to obtain the third estimate (2.11), we consider the following interpolation inequality (which is stated in terms of Hölder norms in [14, p. 80], but is readily seen to apply also to the local oscillation $\langle \cdot \rangle_t^{(\alpha)}$ defined in (2.7)), $\langle v_1 \rangle_t^{(1/2)} \leqslant c_1 \varepsilon^{3/10} \langle v_1 \rangle_t^{(4/5)} + c_2 \varepsilon^{-1/2}\, |v_1|$, which holds whenever $0 < \varepsilon < 1$, with $c_1 > 0$ and $c_2 > 0$ independent of v_1; if we now take $\varepsilon = \delta(x_0)^{5/2}$ then we readily obtain $\langle v_1 \rangle_t^{(1/2)} = \delta(x_0)^{-5/4}\, O(\beta\sigma/(\phi + \sigma))$ (uniformly in $x_0 \in \bar{\Omega}_1$ and $t_0 \geqslant T_1 + 1$), and the third inequality (2.11) follows.

In order to obtain (2.12), let x_0 be a point of Ω such that $d(x_0) > \rho_1/3$ and let B_1 be the hypersphere of radious $\rho_1/6$ centered at x_0; notice that $\partial B_1 \subset \Omega$. Let us consider the function $v_3\colon \bar{B}_1 \to \mathbb{R}$ defined as

$$v_3 = [A_1 + A_2(\phi)]\,\cosh(\phi^{1/3} r)/\cosh(\phi^{1/3}\rho_1/6),$$

where

$$A_1 = \max\{|v_1(x, t)| : x \in \bar{\Omega},\, t \geqslant T_2\},$$

$$A_2(\phi) = 2k_1\beta\phi^{4/3}\mu_1\cosh(\phi^{1/3}\rho_1/6)\exp(-\sqrt{k_2/m}\,\phi\rho_1/6)$$

with k_1, k_2, and μ_1 as defined in assumptions (H.4)–(H.5) and Lemma 2.1. That function is readily seen to satisfy

$$\Delta v_3 - \phi^{2/3} v_3 + \beta\phi^2 f(u, v) < 0 \qquad \text{in } B_1,$$
$$v_3 > v_1 \qquad \text{at } \partial B_1 \qquad \text{if} \quad t \geqslant T_2.$$

Thus v_3 is a super-solution of (2.2) in $\bar{B}_1$ and thus $v(x, t) \leqslant v_3$ if $x \in B_1$ and $t \geqslant T_1$. In particular, this inequality holds at x_0 and consequently, at $x = x_0$ we have,

$$|-\phi^{2/3} v_1 + \beta\phi^2 f(u, v)| \leqslant |\phi^{2/3} v_3| = \phi^{2/3}(A_1 + A_2)/\cosh(\phi^{1/3}\rho_1/6)$$
$$= O(\beta\sigma/(\phi + \sigma))$$

in the limit (1.11), uniformly in $d(x_0) > \rho_1/3$ and $t \geqslant T_1$. Then we only need to apply (to (2.2)) local, interior L_p estimates and imbedding theorems to obtain (2.12), and the proof is complete.

Now we give the proof of Lemma 2.3, which is the most involved one in this section.

Proof of Lemma 2.3. Let us first consider the following quantities,

$$P(T', T'') = \sup\{|\nabla V| + \langle V \rangle_t^{(1/2)}: x \in \bar{\Omega},\ T' \leqslant t \leqslant T''\}, \tag{2.56}$$

$$Q(T', T'') = \sup\{|\tilde{\nabla} v_1| + \langle v_1 \rangle_t^{(1/2)}: x \in \bar{\Omega}_1,\ T' \leqslant t \leqslant T''\}, \tag{2.57}$$

$$R(T', T'') = \sup\{[|\tilde{\nabla} u| + \langle u \rangle_t^{(1/2)}] \exp[\sqrt{k/m}\,\phi d(x)/2]: x \in \bar{\Omega}_1,\ T' \leqslant t \leqslant T''\}, \tag{2.58}$$

$$S(T', T'') = \sup\{|\nabla v| + \langle v \rangle_t^{(1/2)}: x \in \bar{\Omega},\ T' \leqslant t \leqslant T''\}, \tag{2.59}$$

where $T_1 + 1 \leqslant T' < T'' < \infty$, with T_1 as defined in Lemma 2.1. Notice that according to the estimates (2.9), (2.11) and the definition (2.1), the four quantities appearing in (2.56)–(2.59) are bounded (and thus the definitions make sense) and

$$S(T', T'') \leqslant P(T', T'') + Q(T', T'') + \mu_2 \phi \tag{2.60}$$

whenever $T_1 + 1 \leqslant T' < T'' \leqslant \infty$. The proof proceeds in four steps.

Step 1. *The following estimate holds if* $T \geqslant T_1 + 1$ *and* $0 < \varepsilon < \varepsilon_0$,

$$P(T+1, \infty) \leqslant C_1[\phi\varepsilon^{1/2} Q(T, \infty) + \varepsilon^{-3/2}], \tag{2.61}$$

where C_1 *and* $\varepsilon_0 < 1$ *are constants* (*depending only on* Ω), *and* μ_1 *and* μ_2 *are as in Lemmata* 2.1 *and* 2.2.

In order to obtain this estimate we apply L_p estimates up to the boundary [16, p. 133] to the problem (2.3), to obtain

$$\|V\|_{W_p^{2,1}(\Omega\times]t+1,t+2[)} \leqslant \tilde{C}_1[\phi^{2/3}\|v_1\|_{L_p(\Omega\times]t,t+2[)} + \|V\|_{L_1(\Omega\times]t,t+2[)}] \\ + \tilde{C}_2|\phi-v|\,[\|v_1\|_{C(\bar{\Omega}\times[t,t+2])} + Q(t,t+2)], \tag{2.62}$$

where the constants $\tilde{C}_1$ and $\tilde{C}_2$ depend only on Ω and $p>1$ (recall that v remains bounded, see (1.11) and, in the second term in the right hand side, we are using straightforward bounds on Sobolev-type norms of functions defined on $\partial\Omega$. Also, when using the imbedding estimate in [14, p.80] we get

$$\|V\|_{C^{1,1/2}(\bar{\Omega}\times[t+1,t+2])} \leqslant \tilde{C}_3\varepsilon^{1-(m+2)/p}\|V\|_{W_p^{2,1}(\Omega\times]t+1,t+2[)} \\ + \tilde{C}_4\varepsilon^{-1-(m+2)/p}\|V\|_{L_p(\Omega\times]t+1,t+2[)}, \tag{2.63}$$

whenever $p>(m+2)$ and $0<\varepsilon<\varepsilon_0$, where the constants $\tilde{C}_3$, $\tilde{C}_4$, and ε_0 depend only on Ω and p. Since, in addition (Lemmata 2.1 and 2.2 and Eq. (2.1)), $|v_1|\leqslant\mu_2=O(\phi^{-1})$, $|V|\leqslant 1+\beta\phi\mu_1/v=O(1)$ in $\bar{\Omega}$, whenever $t\geqslant T_1+1$, and $\varepsilon<1$, the estimates (2.62)–(2.63) with $p=2(m+2)$ yield

$$P(t+1,t+2)\leqslant C_1[\phi\varepsilon^{1/2}Q(t,t+2)+\varepsilon^{-3/2}]$$

if $t\geqslant T_1+1$, and we only need to use the definitions (2.56)–(2.57) to obtain the stated result and complete this step.

Step 2. *There are three constants*, C_2, C_3, *and* ϕ_0, *such that*

$$R(t,\infty)\leqslant 2R(T,\infty)\exp[-k\phi^2(t-T)/4mL]+2\mu_1[C_2S(T,\infty)+C_3/\phi], \tag{2.64}$$

if $t\geqslant T\geqslant T_1+1$ *and* $\phi>\phi_0$, *where* μ_1 *is as defined in Lemma* 2.1.

Let the constants C_2 and C_3 be defined as

$$C_2=4k_5/k_3, \qquad C_3=\max\{4|a_2(x)|/k_3 : x\in\bar{\Omega}_1\}, \tag{2.65}$$

and let the constant ϕ_0 be such that the following inequalities hold if $x\in\bar{\Omega}_1$ and $\phi>\phi_0$,

$$[(m-1)|M(x)|+|a_1(x)|]\sqrt{k/m}/\phi_0+|b_1(x)|/\phi_0^2<k_3/4, \\ \sigma+\sqrt{k/m}\,\phi_0/2>|b_2(x)|, \tag{2.66} \\ C_3\exp(\sqrt{k/m}\,\phi\rho_1/2)>\phi^2(1+\phi/L),$$

where the vectors a_1 and a_2, the scalars b_1 and b_2, and the mean curvature M are as in Lemmata 2.8 and 2.9, $k=\min\{k_2, k_3\}$ as above, and ρ_1, k_2, k_3, and k_5 are as in assumptions (H.1), (H.4), and (H.5) (at the end of Section 1). Notice that, since a_1, a_2, b_1, b_2, and M depend continuously on x in the compact set $\bar{\Omega}_1$ and, according to (1.11), $|\log(1+\phi/L)|/\phi$ is bounded, Eqs. (2.65) and (2.66) do define C_2, C_3, and ϕ_0 as finite constants.

Now, let us consider the function $\tilde{w}$ defined as

$$\tilde{w}(x, t) = \{R(T, \infty)\exp[-k\phi^2(t-T)/4mL] + C_2\mu_1 S(T, \infty) + C_3\mu_1/\phi\} \times \exp[\sqrt{k/m}\,\phi\eta/2], \tag{2.67}$$

in $x \in \bar{\Omega}_1$ and $t \geqslant T_1$, where η is a coordinate along the outward unit normal to $\partial\Omega$. When taking into account Eqs. (2.65) and (2.66), assumption (H.5), the expression for $\Delta\tilde{w}$ in Lemma 2.8, and the estimates (2.9), the function $\tilde{w}$ is seen to be such that

$$L\partial\tilde{w}/\partial t > \Delta\tilde{w} + a_1\cdot\nabla\tilde{w} + a_2\cdot\nabla u + b_1\tilde{w} - k_3\phi^2\tilde{w} + k_5\phi^2 S(T, \infty)\,u, \tag{2.68}$$

$$L\partial\tilde{w}/\partial t > \Delta\tilde{w} - k_3\phi^2\tilde{w} + k_5\phi^2 S(T, \infty)\,u \quad \text{if} \quad x\in\Omega_1 \quad \text{and} \quad t \geqslant T \geqslant T_1+1; \tag{2.69}$$

$$\partial\tilde{w}/\partial n > (b_2-\sigma)\,\tilde{w}, \qquad \partial\tilde{w}/\partial n > -\sigma\tilde{w} \quad \text{if} \quad x\in\partial\Omega \quad \text{and} \quad t \geqslant T \geqslant T_1+1; \tag{2.70}$$

$$\tilde{w} > |\tilde{\nabla}u|, \qquad \tilde{w} > \langle u\rangle_t^{(1/2)} \quad \text{if} \quad d(x)=\rho_1/2 \quad \text{and} \quad t \geqslant T_1+1. \tag{2.71}$$

Now, let us see that if $0 < a \leqslant 1$, then

$$|\tilde{\nabla}u| < \tilde{w} \quad \text{and} \quad |u(\cdot, t+a) - u(\cdot, t)|/a^{1/2} < \tilde{w} \quad \text{in } \bar{\Omega}_1, \quad \text{if} \quad t \geqslant T \geqslant T_1+1. \tag{2.72}$$

In order to obtain the first estimate first notice that, according to the definition of R, it holds at $t=T$. Assume *for contradiction* that there are a first value of t, t_0, and a point $x_0 \in \bar{\Omega}_1$ such that

$$|\tilde{\nabla}u(x_0, t_0)| = \tilde{w}(x_0, t_0), \qquad |\tilde{\nabla}u| < \tilde{w} \quad \text{if} \quad x\in\bar{\Omega}_1 \quad \text{and} \quad T\leqslant t\leqslant t_0. \tag{2.73}$$

Let $\tilde{\imath}_0$ be a unit vector, tangent at x_0 to the hypersurface parallel to $\partial\Omega$ passing through x_0, such that $\nabla u(x_0, t_0)\cdot\tilde{\imath}_0 = |\tilde{\nabla}u(x_0, t_0)|$, let $\tilde{\imath}$ be the unit vector field defined in Lemma 2.9, and let $w_1 = \nabla u \cdot \tilde{\imath}$. Notice that, according to (2.73) and the definition of w_1 we have

$$w_1(x_0, t_0) = \tilde{w}(x_0, t_0) \quad \text{and} \quad w_1 < \tilde{w} \quad \text{if} \quad x \in \bar{\Omega}_1 \quad \text{and} \quad T \leqslant t \leqslant t_0. \tag{2.74}$$

Now, according to (2.71) we have $d(x_0) < \rho_1/2$ and thus either $x_0 \in \Omega_1$ or $x_0 \in \partial\Omega$. But if $x_0 \in \Omega_1$ then w_1 satisfies (see (1.1) and (2.53))

$$L\partial w_1/\partial t = \Delta w_1 + a_1 \cdot \nabla w_1 + a_2 \cdot \nabla u + b_1 w_1 - \phi^2 f_u(u, v)\, w_1 - \phi^2 f_v(u, v)\, \nabla v \cdot \tilde{\imath}$$

at (x_0, t_0). But, according to assumption (H.5) and the definition (2.59), we have $-f_u(u, v)\, w_1 - f_v(u, v)\, \nabla v \cdot \tilde{\imath} \leqslant -k_3 w_1 + k_5 S(T, \infty)\, u$ at (x_0, t_0) and thus (see (2.68)) $L(\partial\tilde{w}/\partial t - \partial w_1/\partial t) > \Delta\tilde{w} - \Delta w_1$ at (x_0, t_0), and this is in contradiction with (2.74). Similarly, if $x_0 \in \partial\Omega$, then according to (2.47) and (2.54) we have $\partial w_1/\partial n = (b_2 - \sigma)\, w_1$ and (see (2.70) and (2.74)) $\partial\tilde{w}/\partial n > \partial w_1/\partial n$; and this is again in contradiction with (2.74). Thus the first estimate (2.72) has been obtained.

In order to obtain the second estimate (2.72) first notice that if $t \geqslant T_1$ then the function $w_2 = [u(\cdot, t+a) - u(\cdot, t)]/a^{1/2}$ satisfies

$$\begin{aligned} L\partial w_2/\partial t &= \Delta w_2 - \phi^2 h_1(x, t)\, w_2 - \phi^2 h_2(x, t)\, u && \text{in } \Omega_1, \\ \partial w_2/\partial n &= -\sigma w_2 && \text{at } \partial\Omega, \end{aligned} \tag{2.75}$$

where we have used the mean value theorem and, according to assumption (H.5) and the definition (2.59), we have

$$\begin{aligned} &h_1(x, t) \geqslant k_3 \quad \text{and} \\ &|h_2(x, t)| \leqslant k_5\, |v(x, t+a) - v(x, t)|/a^{1/2} \leqslant k_5 S(T, \infty). \end{aligned} \tag{2.76}$$

In addition, the definitions (2.7), (2.58), and (2.67) and Eq. (2.71) imply that

$$\begin{aligned} &|w_2| < \tilde{w} \quad \text{if} \quad x \in \bar{\Omega}_1 \quad \text{and} \\ &t = T \quad \text{or if} \quad d(x) = \rho_1/2 \quad \text{and} \quad t \geqslant T. \end{aligned} \tag{2.77}$$

And we only need to take into account (2.69)–(2.70) and (2.75)–(2.77), and apply maximum principles to obtain $\tilde{w} \pm w_2 > 0$ if $x \in \bar{\Omega}_1$ and $t \geqslant T$. Thus the second estimate (2.72) holds.

Finally, when taking into account the estimates (2.72) and the definitions (2.7), (2.58), (2.59), and (2.67), the estimate (2.64) readily follows and the step is complete.

Step 3. *There are three constants, C_4, C_5, and ϕ_1, such that*

$$Q(t, \infty) \leqslant 2Q(T, \infty)\exp[-\phi^{2/3}(t-T)/2] + 2C_4\beta[\mu_1 S(T, \infty) + R(T, \infty)] + 2C_5\beta\sigma\phi^{1/3}/(\phi+\sigma), \tag{2.78}$$

if $t \geqslant T \geqslant T_2 + 1$ and $\phi > \phi_1$, where μ_1 is as defined in Lemma 2.1.

Let the constants C_4, C_5, and ϕ be defined such that

$$C_4 = 8m(k_4+k_5)/k, \qquad C_5 \geqslant \mu_2(\phi+\sigma)(\phi^{2/3} + |a_2(x)|)/\beta\sigma\phi,$$
$$\phi_1 > (2/\rho_1)^3 + 4m[b_2^2(x) + 2\,|b_1(x)|]/k + 4\sqrt{m/k}\,[|a_1(x)| + (m-1)\,|M(x)|] \qquad \text{for all} \quad x \in \bar{\Omega}_1, \tag{2.79}$$

where the vectors a_1 and a_2, the scalars b_1, b_2, and M, and μ_3 are as in Lemmata 2.2, 2.8, and 2.9. Notice that, as in Step 2, $|a_1|$, $|a_2|$, $|b_1|$, $|b_2|$, and $|M|$ are bounded in $\bar{\Omega}_1$; also, the quantity $\phi\mu_2/v$ is bounded. Thus, the definitions (2.79) do define C_4, C_5, and ϕ_1 as finite constants.

If we now define the function $\tilde{w}$ as

$$\tilde{w}(x, t) = C_4\beta[\mu_1 S(T, \infty) + R(T, \infty)][1 - \exp(\sqrt{k/m}\,\phi\eta/2)] + C_5\beta\sigma\phi^{1/3}/(\phi+\sigma) + Q(T, \infty)\exp[-\phi^{2/3}(t-T)/2], \tag{2.80}$$

where η is a coordinate along the outward unit normal to $\partial\Omega$, then $\tilde{w}$ is seen to be such that

$$\partial\tilde{w}/\partial t > \Delta\tilde{w} - \phi^{2/3}\tilde{w} + \beta\phi^2[k_4 R(T, \infty) + k_5\mu_1 S(T, \infty)]\exp[\sqrt{k/m}\,\phi\eta/2],$$

$$\partial\tilde{w}/\partial t > \Delta\tilde{w} + a_1\cdot\nabla\tilde{w} + a_2\cdot\nabla v_1 + b_1\tilde{w} - \phi^{2/3}\tilde{w} + \beta\phi^2[k_4 R(T, \infty) + k_5\mu_1 S(T, \infty)]\exp[\sqrt{k/m}\,\phi\eta/2]$$
$$\text{if} \quad x \in \bar{\Omega}_1 \quad \text{and} \quad t \geqslant T \geqslant T_1 + 1;$$

$$\partial\tilde{w}/\partial n > -\phi\tilde{w}, \qquad \partial\tilde{w}/\partial n > (b_2 - \phi)\,\tilde{w}$$
$$\text{if} \quad x \in \partial\Omega \quad \text{and} \quad t \geqslant T_1 + 1;$$

$$\tilde{w} > |\tilde{\nabla} v_1| + \langle v_1\rangle_t^{(1/2)} \qquad \text{if} \quad d(x) = \rho_1/2$$
$$\text{and} \quad t \geqslant T_1 + 1.$$

Here we have used Eqs. (2.11) and (2.79), and the expression for $\Delta\tilde{w}$ in Lemma 2.8. Notice that, according to assumption (H.4), Eq. (2.8), and the definitions (2.58) and (2.59), we have

$$|f_u(u, v)|\,[|\tilde{\nabla} u| + \langle u\rangle_t^{(1/2)}] + |f_v(u, v)|\,[|\tilde{\nabla} v| + \langle v\rangle_t^{(1/2)}]$$
$$\leqslant [k_4 R(T, \infty) + k_5 \mu_1 S(T, \infty)] \exp[\sqrt{k/m}\,\phi\eta/2]$$
$$\text{if} \quad x \in \bar{\Omega}_1 \qquad \text{and} \qquad t \geqslant T_1 + 1.$$

With these inequalities at hand, we only need to proceed as in Step 2 (the argument is not repeated here for the sake of brevity) to obtain

$$|\tilde{\nabla} v_1| < \tilde{w}, \qquad |v_1(\cdot, t+a) - v_1(\cdot, t)|/a^{1/2} < \tilde{w} \qquad \text{if} \quad 0 < a \leqslant 1 \qquad \text{and}$$
$$t \geqslant T \quad (\geqslant T_1 + 1),$$

and, when taking into account (2.7), (2.57), and (2.80), the estimate (2.78) follows. Thus the step is complete.

Step 4. *The estimates* (2.13) *hold, with the quantities* μ_3 *and* T_2 *as stated.*

The estimates (2.13) are now obtained from the inequalities (2.60), (2.61), (2.64), and (2.78) as follows. Let the constant b be defined as

$$b = \max\{1, (4mL/k\phi^2) \log 6, 4\phi^{-2/3} \log 6\}, \tag{2.81}$$

and let the sequences $\{P_k\}$, $\{Q_k\}$, and $\{R_k\}$ be defined for $k \geqslant 0$ as

$$\begin{aligned} P_k &= P(T_1 + 2 + kb, \infty), \\ Q_k &= Q(T_1 + 2 + kb, \infty), \\ R_k &= (\phi + \sigma)\, R(T_1 + 2 + kb, \infty)/\sigma, \end{aligned} \tag{2.82}$$

where the functions P, Q, and R are as defined by (2.56)–(2.58). Since the functions $T \to P(T, \infty)$, $T \to Q(T, \infty)$, and $T \to R(T, \infty)$ are non-increasing by definition, Eqs. (2.60), (2.61), (2.64), and (2.78) imply that

$$\begin{aligned} P_{k+1} &\leqslant C_6(\phi \varepsilon^{1/2} Q_k + \varepsilon^{-3/2}), \\ Q_{k+1} &\leqslant Q_k/3 + C_6 \beta\sigma [P_k + Q_k + R_k + 1 + \phi^{1/3}]/(\phi + \sigma), \\ R_{k+1} &\leqslant R_k/3 + C_6(P_k + Q_k + 1), \end{aligned} \tag{2.82$'$}$$

whenever $k \geqslant 0$, $0 < \varepsilon < \varepsilon_0$ and $\phi > \max\{\phi_0, \phi_1\}$, where ε_0, ϕ_0, and ϕ_1 are as in Steps 1–3 and C_6 is any constant such that

$$C_6 \geqslant \max\{C_1, 2\mu_1(\phi+\sigma)[C_2(1+\mu_2\phi)+C_3/\phi]/\sigma,$$
$$2C_4\mu_1(\phi+\sigma)(1+\mu_2\phi)/\sigma, 4C_5, \mu_1, \mu_2\phi\}.$$

Notice that this inequality does define a bounded constant because the right hand side is bounded in the limit (1.11). Also, the estimates (2.9) and (2.11)–(2.12) imply that

$$R_0 \leqslant C_6\phi(1+\phi/L), \qquad Q_0 \leqslant C_6\phi^{1/4}. \tag{2.83}$$

Now, let us choose the constants $\varepsilon \in]0, \varepsilon_0[$ and $\phi_2 \geqslant \max\{\phi_0, \phi_1\}$ such that

$$24C_6\beta\sigma(1+6C_6)/(\phi+\sigma) < 1,$$
$$24C_6^2\varepsilon^{1/2}(1+6C_6)\,\beta\sigma\phi/(\phi+\sigma) < 1 \qquad \text{if} \quad \phi > \phi_2.$$

Notice that, since $\beta\sigma\phi/(\phi+\sigma)$ is bounded in the limit (1.11), $\varepsilon > 0$ and $\phi_2 < \infty$ can be in fact selected such that these two inequalities hold. Then the sequences $\{P_k^*\}$, $\{Q_k^*\}$ and $\{R_k^*\}$, defined as

$$\begin{aligned} P_k^* &= 2C_6\phi\varepsilon^{1/2}A_0/2^k + A_1, \\ Q_k^* &= A_0/2^k + A_2, \\ R_k^* &= 6C_6(1+2C_6\phi\varepsilon^{1/2})\,A_0/2^k + A_3, \end{aligned} \tag{2.84}$$

where

$$\begin{aligned} &A_0 = C_6(\phi+\phi^2/L+\phi^{1/4}), \qquad A_1 = C_6(\varepsilon^{1/2}\phi A_2 + \varepsilon^{-3/2}), \\ &A_3 = 3C_6[A_2(1+C_6\phi\varepsilon^{1/2}) + (1+C_6\varepsilon^{-3/2})]/2, \\ &A_2 = 3C_6\beta\sigma[3C_6 + C_6(2+3C_6)\,\varepsilon^{-3/2} + 2(\phi^{1/3}+1)]/2(\phi+\sigma), \end{aligned} \tag{2.85}$$

are readily seen to satisfy precisely the inequalities opposite to those in (2.82′) (that is, with the $\leqslant$ sign replaced by $>$) for all $k \geqslant 0$, and to be such that

$$Q_0 < Q_0^* \qquad \text{and} \qquad R_0 < R_0^*.$$

Then an induction argument readily shows that

$$P_k < P_k^*, \qquad Q_k < Q_k^*, \qquad R_k < R_k^* \qquad \text{for all} \quad k \geqslant 1.$$

As a consequence, if

$$k > \tilde{k} = \log[A_0 \max\{2C_6\phi\varepsilon^{1/2}/A_1,\ 1/A_2,\ 6C_6(1+2C_6\phi\varepsilon^{1/2})/A_3\}]/\log 2,$$

then $P_k < 2A_1$, $Q_k < 2A_2$, and $R_k < 2A_3$. Thus, we only need to take into account the definitions (2.56)–(2.58) and (2.82) to obtain the inequalities (2.13), with $\mu_3 = \max\{A_1/\phi,\ A_2,\ A_3/\phi\}$ and $T_2 = T_1 + 3 + \tilde{k}b$, where, according to (2.81) and (2.85), $\mu_3 = O(\beta\sigma\phi^{1/3}/(\phi+\sigma))$ and $T_2 - T_1 = O(1 + L/\phi^2)\log(\phi + \phi^2/L)$ in the limit (1.11); also, μ_3 depends only on the quantities (2.10), as stated. Thus the proof is complete.

3. THE ASYMPTOTIC MODEL

The *asymptotic model* is posed by Eqs. (2.14)–(2.17) after we ignore the remainders ψ_1 and ψ_2. Let us also consider the *distinguished limit*

$$L/\phi^2 \to l, \qquad \sigma/\phi \to b, \qquad \beta\phi \to \Phi, \tag{3.1}$$

for some constants $l > 0$, $b > 0$ and $\Phi > 0$, to rewrite the model as

$$\begin{aligned} &\partial V/\partial t = \Delta V && \text{in } \Omega, \\ &\partial V/\partial n = v(1-V) + \Phi \int_{-\infty}^{0} f(U, V)\, d\xi && \text{at } \partial\Omega, \end{aligned} \tag{3.2}$$

where, at each $s \in \partial\Omega$, $U = U(\xi, s, t)$ is given by

$$l\partial U/\partial t = \partial^2 U/\partial \xi^2 - f(U, V(s,t)) \qquad \text{in} \quad -\infty < \xi < 0, \tag{3.3}$$

$$U = 0 \quad \text{at} \quad \xi = -\infty, \qquad \partial U/\partial \xi = b(1-U) \quad \text{at} \quad \xi = 0, \tag{3.4}$$

with appropriate initial conditions. In fact, according to the estimate (2.42), we shall only consider solutions of (3.2)–(3.4) such that

$$0 < U(\xi, s, t) < [b/(\sqrt{k_2/m} + b)] \exp[\sqrt{k_2/2m}\,\xi]$$
$$\text{in} \quad -\infty < \xi < 0, \qquad s \in \partial\Omega, \qquad t \geqslant 0. \tag{3.5}$$

The asymptotic model is more amenable to purely analytical treatment than the original model. For example, the steady states of the original model must be calculated numerically. The steady states of (3.2)–(3.4) instead, are given by

$$\Delta V = 0 \quad \text{in } \Omega, \qquad \partial V/\partial n = v(1-V) + \Phi Q(V) \quad \text{at } \partial\Omega, \tag{3.6}$$

where

$$Q(V) = \left[2 \int_0^{U_0} f(z, V)\, dz \right]^{1/2} \tag{3.7}$$

and $U_0 > 0$ is the unique solution of

$$\int_0^{U_0} f(z, V)\, dz = b^2(1 - U_0)^2/2. \tag{3.8}$$

Notice that if, in addition, Ω is a ball of $\mathbb{R}^m$ then (3.6) may be solved in closed form, and the linear stability of the solution of (3.6) (as steady states of (3.2)–(3.4)) may be also analyzed in closed form. This was done (for $m = 1$) in [3], where it was seen that, for appropriate values of the parameters l, b, ν, and ϕ, the asymptotic model exhibits quite complex behavior as $t \to \infty$.

Let us now consider some particular sublimits, when still simpler submodels apply.

3.1. *Sublimits of* (3.2)–(3.4)

Let us now consider the limits $b \to 0$, $b \to \infty$, and $l \to 0$, with Φ appropriate in each case and

$$\nu^{-1} = O(1). \tag{3.9}$$

In fact, the limit $\nu \to 0$ (with $\beta\phi\sigma/(\phi + \sigma)\, \nu = \Phi b/(1 + b)\, \nu = O(1)$) was considered in [7], where an asymptotic model was obtained that consists of a 1-D PDE (for the concentration u in the boundary layer) and an ODE (for the temperature v, which becomes spatially constant in first approximation after some time). That model can be also obtained from (3.2)–(3.4) but, for the sake of brevity, we shall not consider this limit here.

As

$$b \to 0 \tag{3.10}$$

U is small (see (3.5)) and the nonlinearity f may be written as

$$f(U, V) = f_U(0, V)\, U + O(|U|^2). \tag{3.11}$$

Thus $\int_{-\infty}^0 f(U, V)\, d\xi$ is small and if Φ is bounded above then $|V - 1|$ becomes small after some time (see (3.2)); this readily implies that the dynamics of the model is quite simple. If, instead, Φ is large, such that

$$b\Phi \to \Phi_1 \neq 0, \infty, \tag{3.12}$$

then the model (3.2)–(3.4) may be written as

$$\partial V/\partial t = \Delta V \qquad \text{in } \Omega,$$
$$\partial V/\partial n = \nu(1-V) + \Phi_1 f_1(V) \int_{-\infty}^{0} U_1\, d\xi + \varphi_1(s,t), \tag{3.13}$$

$$l\partial U_1/\partial t = \partial^2 U_1/\partial \xi^2 - f_1(V(s,t))\, U_1 + \varphi_2(\xi, s, t)$$
$$\text{in} \quad -\infty < \xi < 0, \tag{3.14}$$

$$U_1 = 0 \qquad \text{at} \quad \xi = -\infty,$$
$$\partial U_1/\partial \xi = 1 + \varphi_3(s,t) \qquad \text{at} \quad \xi = 0, \tag{3.15}$$

where

$$U_1 = U/b, \qquad f_1(V) = f_U(0, V),$$
$$\varphi_1(b,t) = \Phi \int_{-\infty}^{0} [f(bU_1, V) - f_1(V)\, U_1 b]\, d\xi$$
$$\varphi_2(s,t) = f_1(V(s,t))\, U_1 - f(bU_1, V(s,t))/b, \tag{3.16}$$
$$\varphi_3(s,t) = -bU_1(0, s, t).$$

Also, since $U = bU_1$ satisfies (3.5), the remainders, φ_1, φ_2, and φ_3, are such that

$$|\varphi_1(s,t)| + |\varphi_2(\xi, s, t)| \exp(-\sqrt{k_2/2m}\, \xi) + |\varphi_3(s,t)| = O(b)$$

uniformly in $-\infty < \xi < 0$, $s \in \partial\Omega$, and $t \geqslant 0$. If the remainders are ignored then the resulting submodel (3.13)–(3.15) is not (essentially) simpler than the original asymptotic model (3.2)–(3.4), except for the fact that the submodel depends only on three parameters (l,Φ_1, and ν).

In the limit

$$b \to \infty \tag{3.17}$$

we have

$$|\partial U(0, s, t))/\partial \xi| = \text{uniformly bounded in} \qquad s \in \partial\Omega, \qquad t \geqslant 0, \tag{3.18}$$

as readily obtained by an argument similar to that in the proof of [7, Lemma 2.6]. Then the boundary condition (3.4) may be written as

$$U = 0 \qquad \text{at} \quad \xi = -\infty, \qquad U = 1 + \varphi_1(s,t) \qquad \text{at} \quad \xi = 0, \tag{3.4$'$}$$

where $\varphi_1(s,t) = b^{-1}\, \partial U(0, s, t)/\partial \xi$ is such that $|\varphi_1(s,t)| = O(b^{-1})$ uniformly in $s \in \partial\Omega$ and $t \geqslant 0$ (see (3.18)). Again, if the remainder φ_1 is ignored, then the resulting submodel (3.2)–(3.3), (3.4′) is not essentially simpler than the original asymptotic submodel.

In the limit

$$l \to 0 \tag{3.19}$$

we have

$$|\partial U(\xi, s, t)/\partial t| \exp(-k_3 \xi) + |\partial V(x, t)/\partial t| = \text{uniformly bounded}, \tag{3.20}$$

in $-\infty < \xi < 0$, $s \in \partial\Omega$, $x \in \bar{\Omega}$, and $t \geqslant t_0$, where k_3 is as in assumption (H.5) (at the end of Section 1) and t_0 is a given constant; these estimates are readily obtained by an argument similar to (but simpler than) that in the proof of Lemma 2.3 above. Then Eq. (3.3) may be written as

$$\partial^2 U/\partial \xi^2 - f(U, V(s, t)) = \varphi_1(\xi, s, t) \qquad \text{in} \quad -\infty < \xi < 0, \tag{3.3′}$$

where $\varphi_1(\xi, s, t) = l\, \partial U/\partial t$ satisfies $|\varphi_1(\xi, s, t)| \exp(-\lambda \xi) = O(l)$, uniformly in $-\infty < \xi < 0$, $s \in \partial\Omega$, $t \geqslant t_0$ (see (3.20)). If the initial transient $0 < t < t_0$ and the remainder φ_1 are ignored, the resulting problem (3.3′)–(3.4) is solved and its solution is substituted into the boundary condition in (3.2), then the following submodel results,

$$\partial V/\partial t = \Delta V \quad \text{in} \quad \Omega, \qquad \partial V/\partial n = \nu(1 - V) + \Phi Q(V) \quad \text{at} \quad \partial\Omega, \tag{3.21}$$

where the nonlinearity Q is as given by (3.7). Standard dynamical systems theory [17] implies that the solution of the gradient-like problem (3.21) converges to the set of steady states as $t \to \infty$ and thus its dynamics is trivial. An interesting question arises: Does (3.21) posses non-constant stable steady states? Aronson [21] and Aronson and Peletier [20] solved that question for the heat equation with nonlinear boundary conditions in 1-D, and gave a precise characterization of the domains of attraction of the stable steady states; see also [22–24] for some partial results in the multidimensional case.

Similarly, as $l \to \infty$ we could use the new time variable $\tau = t/l$ and try to prove that $l^{-1}\, \partial V/\partial \tau$ can be ignored in first approximation. Unfortunately, in order to prove that property we would need that $f_v < 0$, while f_v is usually positive (see (1.4)–(1.5)). Thus, no simpler submodel seems to apply in this limit.

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