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The Method of Multiple Scales: Asymptotic Solutions and Normal Forms for Nonlinear Oscillatory Problems

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The method of multiple scales is implemented in Maple V Release 2 to generate a uniform asymptotic solution $O(\epsilon^r)$ for a weakly nonlinear oscillator. In recent work, it has been shown that the method of multiple scales also transforms the differential equations into normal form, so the given algorithm can be used to simplify the equations describing the dynamics of a system near a fixed point. These results are equivalent to those obtained with the traditional method of normal forms which uses a near-identity coordinate transformation to get the system into the "simplest" form. A few Duffing type oscillators are analysed to illustrate the power of the procedure. The algorithm can be modified to take care of systems of ODEs, PDEs and other nonlinear cases.

1. Introduction

Nonlinear differential equations describing oscillatory problems have played a fundamental role in science and engineering. The basic problem can be illustrated with a second-order ordinary differential equation (ODE). In this case, the objective is to determine the behavior of a weakly nonlinear system described by an equation of the type

$$\ddot{x} + \omega^2 x = \epsilon f(x, \dot{x}, t) \ , \tag{1.1}$$

where $0 < \epsilon \ll 1$ is a parameter. The analytical efforts to determine a solution for this equation can be traced back to Euler (1772), Lindstedt (1882), and Poincaré (1886) who, motivated by problems in celestial mechanics, sought solutions, typically in the form

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + \cdots \ , \tag{1.2}$$

with the condition that $x(t)$ and $\dot{x}(t)$ would be bounded functions of t, for $t \in \Re$. During the early to middle part of this century efforts by von Zeipel (1916), Krylov and Bogoliubov (1934), among others, helped to develop the methods of averaging which were effective for generating solutions and for setting bounds in truncation errors, and which

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are still in use today. During the late 1950s and early 1960s, efforts by Sturrock (1957), Frieman (1963), Kevorkian (1963), and Nayfeh (1965) led to the development of the method of multiple scales (MMS), which was shown to be equivalent to the methods of averaging (Morrison, 1966), and provided an alternative approach for determining solutions for nonlinear ODE's. The MMS is a popular technique and a renewed attention to nonlinear phenomena, combined with new symbolic computational power, has increased its use in science and engineering. We present a general algorithm to implement the MMS and show a number of applications processed with a Maple V Release 2 version of it. Nayfeh (1993) has shown that the results obtained by the method of normal forms are equivalent to those obtained with other perturbation methods. Therefore, the MMS can also be used to transform differential equations into their normal forms near a fixed point. These manipulations typically precede center manifold and bifurcation analyses.

2. The Method of Multiple Scales

The method of multiple scales, as presented by Nayfeh (1981), considers the expansion to be a function of multiple independent variables, or scales, instead of a single variable t. The independent variables are defined as

$$T_n = \epsilon^n t \quad for \;\; n = 0, 1, 2, \ldots \tag{2.1}$$

It is assumed that the solution of interest can be represented by an expression having the form

$$x(t;\epsilon) = x_0(T_0, T_1, T_2, \ldots) + \epsilon x_1(T_0, T_1, T_2, \ldots) + \epsilon^2 x_1(T_0, T_1, T_2, \ldots) + \cdots \tag{2.2}$$

where the number of independent time scales depends on the order to which the expansion is carried out. Substituting (2.2) into the governing differential equation and collecting coefficients of equal powers of ϵ generates a system of $n+1$ differential equations. To obtain a uniform solution, the system of ODEs needs to be solved sequentially for $k = 0, 1, \ldots, n-1$, eliminating secular terms, those terms that will become large when t increases, in the process at each order ϵ^k for $k = 1, 2, \ldots, n$. This will ensure that

$$x(t;\epsilon) = \sum_{k=0}^{n-1} x_k(T_0, T_1, \ldots, T_n) + O(\epsilon^n) \tag{2.3}$$

is a uniform $O(\epsilon^n)$ solution. For example, consider a Duffing oscillator of the type

$$\ddot{x} + \omega^2 x - \epsilon\alpha x^3 = 0. \tag{2.4}$$

A first-order analysis ($n = 1$) of (2.4) would generate the system of equations

$${D_0}^2 x_0 + \omega^2 x_0 = 0 \tag{2.5}$$

$${D_0}^2 x_1 + \omega^2 x_1 = -2D_0 D_1 x_0 - \alpha {x_0}^3, \tag{2.6}$$

where $D_i = \frac{\partial}{\partial T_i}$. The solution of (2.5) can be written as

$$x_0(T_0, T_1) = A(T_1)e^{i\omega T_0} + \overline{A}(T_1)e^{-i\omega T_0}. \tag{2.7}$$

Substituting (2.7) into (2.6), we obtain

$$D_0{}^2 x_1 + \omega^2 x_1 = -\left(2i\omega D_1 A + 3A^2\overline{A}\right) e^{i\omega T_0} + \left(2i\omega D_1 \overline{A} - 3\overline{A}^2 A\right) e^{-i\omega T_0}$$
$$-A^3 e^{3i\omega T_0} - \overline{A}^3 e^{-3\omega i T_0}. \tag{2.8}$$

To avoid the generation of secular terms in $x_1(t)$, the coefficients of $e^{i\omega T_0}$ and its complex conjugate must vanish; that is,

$$2i\omega D_1 A + 3A^2\overline{A} = 0. \tag{2.9}$$

Writing A in the polar form $A(T_1) = \frac{1}{2} a e^{i\beta}$ and separating (2.9) into its real and imaginary parts gives us

$$\frac{\partial a}{\partial T_1} = 0, \qquad a\frac{\partial \beta}{\partial T_1} - \frac{3}{8} a^3 = 0. \tag{2.10}$$

The solution to (2.4) is then given by

$$x(t) = a\cos(\omega t + \beta) + \cdots \tag{2.11}$$

where a and β are described by the so called evolution equations (2.10).

3. The Method of Normal Forms

The method of normal forms specifies a procedure to determine a near-identity coordinate transformation in which a given dynamical system takes the ”simplest” form. As presented by Wiggins (1990), a vector field of the type

$$\dot{W} = G(W), \qquad W \in \Re^n, \tag{3.1}$$

generates a vector field which can be approximated, in the neighborhood of a fixed point $W = W_0$, by its normal form, provided that G is C^r where r will be the order of the derivative, typically $r \geq 4$. The system (3.1) can be simplified by translating the fixed point to the origin using $v = W - W_0$, which gives

$$\dot{v} = G(v + W_0) \equiv H(v), \qquad v \in \Re^n. \tag{3.2}$$

The linear part of (3.2) can be taken out to get

$$\dot{v} = DH(0)v + \overline{H}(v), \tag{3.3}$$

where D is the linear part and, $\overline{H}(v) = H(v) - DH(0)v = O(|v|^2)$. Finally, $DH(0)$ can be put into Jordan canonical form with the transformation $v = Tx$, to get

$$\dot{x} = Jx + F(x), \qquad x \in \Re^n, \tag{3.4}$$

where $J \equiv T^{-1}DH(0)T$, and $F(x) \equiv T^{-1}\overline{H}(Tx)$. The above manipulation has simplified the linear part of (3.3) as much as possible. Taylor series can now be used to expand (3.4) into

$$\dot{x} = Jx + F_2(x) + F_3(x) + \cdots + F_{r-1}(x) + O(|x|^r) \tag{3.5}$$

where the $F_i(x)$ represents the term of order i in the Taylor series. Wiggins (1990) has shown that (3.5) can be further simplified and the essence of the simplification is stated in the following theorem.

THEOREM 3.1. *By a sequence of analytic coordinate changes (3.5) can be transformed into*

$$\dot{y} = Jy + F_2^r(y) + F_3^r(y) + \cdots + F_{r-1}^r(y) + O(|y|^r), \tag{3.6}$$

where $F_k^r(y) \in G_k$, $2 \leq k \leq r-1$, and G_k is a space complementary to the space generated by the Lie bracket operation on the vector space H_k and Jy, where H_k is the space of vector-valued monomials of degree k. Equation (3.6) is said to be in normal form.

Nayfeh (1993) has shown that the results obtained by the method of normal forms are equivalent to those obtained with other perturbation methods, such as the method of multiple scales. Therefore, it can be shown that the equations presented are the normal form of the simple oscillator with a cubic nonlinearity.

4. Algorithm for the Method of Multiple Scales

The following algorithm describes the main steps in developing a symbolic code to apply the MMS. The assumption is that the problem involves a weakly nonlinear ODE. The problem could be autonomous or nonautonomous, but if there are forcing terms, they are small. The algorithm can be modified to treat a variety of other cases, including systems of ODEs, PDEs, or forcing terms that are not small. The main steps in the algorithm are as follow:

Start Program

Step 1: An ODE of the type (1.1) is defined.
Step 2: The desired MMS is defined by specifying:
 (i) n, which sets $O(\epsilon^n)$, the order of the solution sought.
 (ii) polar or cartesian form for the solution sought.
Step 3: The operators derivative are replaced to include the time scales using $\frac{d}{dt} = \frac{d}{dT_0} + \epsilon\frac{d}{dT_1} + \cdots$, the main variable is perturbed using $x = x_0 + \epsilon x_1 + \cdots$, and t becomes T_0.
Step 4: The expression is expanded keeping only terms up to order n, and sinusoidals are
 expressed in complex exponential form.
Step 5: Terms in the ODE are separated by powers of ϵ, generating $n+1$ equations.
Step 6: The equation at order ϵ^0 is solved to get $x_0(T_0, T_1, \ldots)$ in terms of a complex amplitude, let's say $A(T_1, \ldots, T_n)$ and its conjugate.

Begin Loop $1 \leq r \leq n-1$

Step 7: $x_0, x_1, \ldots, x_{r-1}$ are replaced into the equation at order ϵ^r. Secular and nonsecular terms are separated. The secular terms are set to zero in the equation and stored in
equation S_r.

Step 8: Including only nonsecular terms, a solution is generated at order ϵ^r.

End Loop

Step 9: The solution can be written as $x = x_0 + \epsilon x_1 + \cdots + \epsilon^{n-1} x_{n-1} + O(\epsilon^n)$. The complex parameters A and $\overline{A}$ can be expressed in polar or cartesian form as desired. These parameters will be determined from the evolution equations (2.10), which describe A and $\overline{A}$, or a and β.

Begin Loop $1 \leq p \leq n$

Step 10: Solve equation S_p to determine $D_p A(T_1, \ldots, T_n)$ and its conjugate using $D_k A$, $k = 1, \ldots, p-1$, found in previous iterations, and their partial derivatives $D_j D_k A$.

End Loop

Step 11: The method of reconstitution (Nayfeh, 1981) is used to define the evolution equations. The method assumes $\frac{dA}{dt} = \epsilon D_1 A + \epsilon^2 D_2 A + \cdots$, using the $D_i A(T_1, \ldots, T_n)$ previously defined.

Step 12: The complex amplitude $A(T_1, \ldots, T_n)$ can be written in polar form, $A = ae^{(i\beta)}$, or cartesian form $A = \frac{1}{2}(p - iq)$.

Step 13: The reconstituted equation can be separated into real and imaginary parts, generating the modulation equations.

End Program

We implemented the above algorithm in Maple. All the code is available to perform all the functions that the previous section describes (sanchez@alamo.eng.utsa.edu); the following section shows some examples that illustrate the power of the procedure and how the algorithm is used.

5. Examples

5.1. Parametrically excited Duffing oscillator

The Duffing oscillator with softening nonlinearity of the type (2.4) has been extensively studied in the context of a large variety of physical systems, from the oscillation of a pendulum to charge variations in superionic conductors. A parametrically excited Duffing oscillator is a particular case that has many physical applications (e.g., Sanchez et al., 1990a), some of which we will illustrate. The basic oscillator is of the form

$$\ddot{x} + x = -\epsilon \left(2\mu \dot{x} - \alpha x^3 + gx \cos(\Omega t) \right), \tag{5.1}$$

where the natural frequency has been scaled to unity. It can be shown that this oscillator exhibits resonances for $\Omega \approx 1,\ 2,$ and 4. The MMS can be used to approximate the response near any resonance. For example, to obtain the response near $\Omega \approx 1$ we use $\Omega^2 = 1 + \epsilon\sigma$, where σ is a detuning parameter. Substituting this relation in (5.1), we obtain

$$\ddot{x} + \Omega^2 x = \epsilon \left(\sigma x - 2\mu\dot{x} + \alpha x^3 - g x \cos(\Omega t)\right). \tag{5.2}$$

A Maple V Release 2 of the MMS algorithm was used to determine a second-order perturbation solution for $\Omega \approx 1$. The solution found was

$$\begin{aligned} x(t) = \quad & \left(\left(\frac{g\cos(\beta(t) + 2\,OMG\,t)}{6\,OMG^2} - \frac{g\cos(\beta(t))}{2\,OMG^2}\right)\ a(t) \right. \\ & \left. - \frac{a(t)^3\,alph\ \cos(3\,\beta(t) + 3\,OMG\,t)}{32\ \ OMG^2}\right) w + a(t)\cos(\beta(t) + OMG\,t) \end{aligned} \tag{5.3}$$

where a and β are found from the evolution equations

$$\frac{da(t)}{dt} = \left(-\frac{3\,MU\ alph\,a(t)\ ^3}{8\,OMG^2} - \frac{g^2\sin(2\,\beta(t))a(t)}{8\quad OMG^3}\right) w^2 - wMU\,a(t) \tag{5.4}$$

$$\begin{aligned} a(t)\frac{d\beta(t)}{dt} = \quad & \left(-\frac{15\,a(t\ )^5\,alph^2}{256\,OMG^3} - \frac{3\,SIG\ alph\,a(t)^3}{16\,OMG^3} + \left(-\frac{MU^2}{2\ \ OMG} \right.\right. \\ & \left.\left. - \ \frac{g^2}{12\,OMG^3} - \frac{SIG^2}{8\,OMG^3} - \frac{g^2\cos(2\,\beta(t))}{8\,OMG^3}\ \right) a(t)\right) w^2 \\ & + \left(-\frac{3\,a(t)^3\,alph}{8OMG} - \frac{SIG\,a(t)}{2\,OMG}\right) w \end{aligned} \tag{5.5}$$

and where $w = \epsilon$, OMG=Ω, SIG=σ , $Alph = \alpha$, and MU= μ. As it was mentioned before, the above differential equations are the normal forms of (5.1).

5.2. Rolling of a ship in longitudinal seas

Sanchez and Nayfeh (1990b) have shown that the behavior of a ship rolling in longitudinal waves can be described by an equation of the type

$$\ddot{x} + x + \epsilon \left(2\mu\dot{x} + \mu_3\dot{x}^3 + \alpha_3 x^3 + \alpha_5 x^5 + h x \cos(\Omega t)\right) = 0. \tag{5.6}$$

It can be shown that resonances could be excited when $\Omega \approx 1,\ 2,\ 4,\ and\ 6$. To find a solution of order $O(\epsilon^2)$ for $\Omega \approx 1$, we can use a detuning

$$\Omega^2 = 1 + \epsilon\sigma. \tag{5.7}$$

Implementing (5.6) and (5.7) in the Maple procedure, paralleling the previous case, leads to the solution

$$\begin{aligned} x(t) = \quad & \left(\left(\frac{a\_5\ \cos(5\,\beta(t) + 5\,OMG\,t)}{384\,OMG^2} + \frac{5\,a\_5\ \cos(3\,\beta(t) + 3\,OMG\,t)}{256\,OMG^2} \right.\right. \\ & \left. + \ \frac{5\,a\_5\ \cos(5\,\beta(t) + 3\,OMG\,t)}{256\,OMG^2}\right) a(t)^5 + \left(\frac{a\_3\ \cos(3\,\beta(t) + 3\,OMG\,t)}{32\,OMG^2} \right. \\ & \left. + \ \frac{OMG\ MU3\ \sin(3\,\beta(t) + 3\,OMG\,t)}{32}\right) a(t)^3 + \left(\frac{h\cos(\beta(t) + 2\,OMG\,t)}{6\,OMG^2} \right. \\ & \left.\left. - \ \frac{h\cos(\beta(t))}{2\,OMG^2}\ \right) a(t)\right) w + a(t)\cos(\beta(t) + OMG\,t) \end{aligned} \tag{5.8}$$

where $w = \epsilon$, OMG= Ω, SIG= σ, Alph= α, and MU= μ have been defined as before. The evolution equations found in this case are :

$$
\begin{aligned}
\frac{da(t)}{dt} = & \left(\frac{25\,a(t)^9 a\_5^2 \sin(2\ \beta(t))}{4096\,OMG^3} + \left(\frac{5\,MU\ a\_5}{8\,OMG^2} - \frac{3\,a\_3\ MU3}{32}\right)\ a(t)^5 \right. \\
& + \left(\frac{3\,MU\ a\_3}{8\,OMG^2} - \frac{3\,MU3}{8} + \frac{3\,MU3\ SIG}{8}\right) a(t)^3 + \frac{15\,MU3\ a(t)^7 a\_5}{256} \\
& \left. - \frac{h^{\ 2}\sin(2\,\beta(t))a(t)}{4\,OMG^3}\right) w^2 + \left(-\ MU\ a(t) - \frac{3\,OMG^2\,MU3\ a(t)^3}{8}\right) w
\end{aligned} \tag{5.9}
$$

$$
\begin{aligned}
a(t)\frac{d\beta(t)}{dt} = & \left(\left(\frac{9\,OMG^3\,MU3^2}{256} + \frac{5\,SIG\ a\_5}{32\,OMG^3} - \frac{5\,a\_\ 5}{32\,OMG^3} - \frac{15\,a\_3^2}{256\,OMG^3}\right) a(t)^5 \right. \\
& - \frac{5\,a\_3\ a(t)^7 a\_5}{64\ \ OMG^3} + \left(\frac{25\,a\_5^2\cos(2\,\beta(t))}{4096\,OMG^3} - \frac{295\,a\_5^2}{12288\,OMG^3}\right) a(t)^9 \\
& + \left(\frac{3\,a\_3\ SIG}{16\,OMG^3} - \frac{3\,a\_3}{16\,OMG^3}\right) a(t)^3 + \left(\frac{SIG}{4\,OMG^3} - \frac{MU^2}{2\,OMG} \right. \\
& \left.\left. - \frac{1}{8\,OMG^3} + \frac{h^2}{24\,OMG^3} - \frac{SIG^2}{8\,OMG^3} - \frac{h^2\cos(2\,\beta(t))}{4\,OMG^3}\right) a(t)\right) w^2 \\
& + \left(\frac{3\,a\_3\ a(t)^3}{8\,OMG} + \frac{5\,a\_5\ a(t)^5}{16\,OMG} + \left(\frac{1}{2\,OMG} - \frac{SIG}{2\,OMG}\right) a(t)\right) w
\end{aligned} \tag{5.10}
$$

In the same form, a detuning like $\left(\frac{\Omega}{k}\right)^2 = 1 + \epsilon\sigma$ can be used to determine a solution near $\Omega \approx k$.

6. Conclusions

The algorithm presented to implement the method of multiple scales (MMS) is very effective for computing approximate solutions of nonlinear oscillatory problems and for transforming a differential equation into its normal form. The scheme shown can be modified to handle systems of ODE's and PDE's. The examples shown illustrated that a significant number of nonlinear problems can be approached with the procedure described.

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