Available online at www.sciencedirect.com
ScienceDirect
Journal of Multivariate Analysis 98 (2007) 1583–1591
Journal of Multivariate Analysis
www.elsevier.com/locate/jmva

Extremes of conditioned elliptical random vectors

Enkelejd Hashorva[a, b, *]

[a] *Allianz Suisse Insurance Company, Laupenstrasse 27, CH-3001 Bern, Switzerland*
[b] *Department of Mathematical Statistics and Actuarial Science, University of Bern, Sidlerstrasse 5, CH-3012 Bern, Switzerland*

Received 1 October 2005
Available online 27 April 2007

Abstract

Let $\{\boldsymbol{X}_n, n \geqslant 1\}$ be iid elliptical random vectors in $\mathbb{R}^d$, $d \geqslant 2$ and let I, J be two non-empty disjoint index sets. Denote by $\boldsymbol{X}_{n,I}, \boldsymbol{X}_{n,J}$ the subvectors of $\boldsymbol{X}_n$ with indices in I, J, respectively. For any $\boldsymbol{a} \in \mathbb{R}^d$ such that $\boldsymbol{a}_J$ is in the support of $\boldsymbol{X}_{1,J}$ the conditional random sample $\boldsymbol{X}_{n,I} | \boldsymbol{X}_{n,J} = \boldsymbol{a}_J, n \geqslant 1$ consists of elliptically distributed random vectors. In this paper we investigate the relation between the asymptotic behaviour of the multivariate extremes of the conditional sample and the unconditional one. We show that the asymptotic behaviour of the multivariate extremes of both samples is the same, provided that the associated random radius of $\boldsymbol{X}_1$ has distribution function in the max-domain of attraction of a univariate extreme value distribution.

AMS 1991 subject classification: 60F05; 60G70

Keywords: Elliptical random vectors; Conditional distribution; Multivariate extremes; Max-domain of attraction; Weak convergence; Tail asymptotics

1. Introduction

Let $\boldsymbol{X} = (X_1, \ldots, X_d)^\top$ be an elliptical random vector in $\mathbb{R}^d$, $d \geqslant 2$ with stochastic representation $\boldsymbol{X} \stackrel{d}{=} RA^\top \boldsymbol{U}_d$, where R is an almost surely positive random variable independent of the random vector $\boldsymbol{U}_d$, which is uniformly distributed on the unit sphere of $\mathbb{R}^d$, and $A \in \mathbb{R}^{d \times d}$ is a non-singular matrix ($^\top$ and $\stackrel{d}{=}$ stand for the transpose sign and equality of distribution functions, respectively). For any vector $\boldsymbol{x} = (x_1, \ldots, x_d)^\top \in \mathbb{R}^d$ define $\boldsymbol{x}_I := (x_{i, i \in I})^\top, \boldsymbol{x}_J := (x_{i, i \in J})^\top$ with I, J two non-empty disjoint index sets such that $I \cup J = \{1, \ldots, d\}$.

* Corresponding author at: Department of Mathematical Statistics and Actuarial Science, University of Bern, Sidlerstrasse 5, CH-3012 Bern, Switzerland. Fax: +41 583584572.
E-mail address: enkelejd@stat.unibe.ch.

0047-259X/$ - see front matter
doi:10.1016/j.jmva.2007.04.008

The main distributional properties of elliptical random vectors are presented in [5,1,10,9,4,18,20,24] among several others.

As shown in [5] if $\boldsymbol{a}$ is a given vector in $\mathbb{R}^d$ such that $\boldsymbol{P}\{\boldsymbol{X}_J \leqslant \boldsymbol{a}_J\} \in (0,1)$, then the random vector $\boldsymbol{X}_I | \boldsymbol{X}_J = \boldsymbol{a}_J$ is elliptically distributed. This property is well known for Gaussian random vectors. Conditioning in that case changes only the mean and the covariance matrix of the Gaussian vector. In general for elliptical random vectors this is not the case; other moments may change too.

Asymptotic tail behaviour and extremes of spherical random vectors are discussed in details in [2–4,12,7]. From these results we know that the asymptotic tail behaviour of the associated random radius R determines the asymptotic behaviour of the sample extremes. This fact extends to the class of elliptical random vectors, see, e.g. [13].

Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n, n \geqslant 1$ be independent elliptical random vectors in $\mathbb{R}^d$ such that $\boldsymbol{X}_i \stackrel{d}{=} \boldsymbol{X}, 1 \leqslant i \leqslant n$. The conditional random sample $(\boldsymbol{X}_n)_I | (\boldsymbol{X}_n)_J = \boldsymbol{a}_J, n \geqslant 1$ consists of elliptical random vectors. In this paper we study the asymptotic behaviour of the sample maxima (defined componentwise) of the conditional and the unconditional samples.

We show that the asymptotic tail behaviour of the associated random radius R determines the asymptotic behaviour of the maxima of the conditional sample. Further, we prove that the asymptotic tail behaviour of the associated random radius of the conditional sample determines the asymptotic tail behaviour of R.

Organisation of the paper: in Section 2 we present several notation and some key results for elliptical random vectors and multivariate extremes. The main results are given in Section 3 followed by the proofs in Section 4.

2. Preliminaries

We shall introduce first some notation, and then give a brief introduction to elliptical random vectors and multivariate extremes.

Let I, J be two non-empty disjoint index sets satisfying $I \cup J = \{1, \ldots, d\}, d \geqslant 2$. For any vector $\boldsymbol{x} = (x_1, \ldots, x_d)^\top \in \mathbb{R}^d$ the vector $\boldsymbol{x}_I := (x_{i, i \in I})^\top$ consists of the components of $\boldsymbol{x}$ with indices in I and similarly for a given $d \times d$ matrix $B \in \mathbb{R}^{d \times d}$, the matrix B_{IJ} is obtained by deleting the rows of B with indices in J and by deleting the columns of B with indices in I. Write $\boldsymbol{x}_I^\top$ instead of $(\boldsymbol{x}_I)^\top$. For two vectors $\boldsymbol{x}, \boldsymbol{y}$ in $\mathbb{R}^d$ the relations $\boldsymbol{x} \geqslant \boldsymbol{y}$, $\boldsymbol{x} > \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$ are understood componentwise. Further we write

$$\mathbf{0} := (0, \ldots, 0)^\top \in \mathbb{R}^d, \quad \mathbf{1} := (1, \ldots, 1)^\top \in \mathbb{R}^d,$$

$$\boldsymbol{a}\boldsymbol{x} := (a_1 x_1, \ldots, a_d x_d)^\top, \quad \boldsymbol{x}/\boldsymbol{a} := (x_1/a_1, \ldots, x_d/a_d)^\top, \quad c\boldsymbol{x} := (cx_1, \ldots, cx_d)^\top, \quad \boldsymbol{a} \in \mathbb{R}^d, \ c \in \mathbb{R}.$$

Let throughout the paper A be a $d \times d$ real non-singular matrix, and let $\boldsymbol{U}_k, k \geqslant 1$ denote a random vector uniformly distributed on the unit sphere of $\mathbb{R}^k$. In this paper we consider an elliptical random vector $\boldsymbol{X}$ in $\mathbb{R}^d$ with stochastic representation

$$\boldsymbol{X} \stackrel{d}{=} R A^\top \boldsymbol{U}_d, \tag{2.1}$$

where R is an almost surely positive random variable with distribution function F independent of $\boldsymbol{U}_d$.

A canonical example of elliptical random vectors is a Gaussian random vector with non-singular covariance matrix. In this case R^2 is chi-squared distributed with d degrees of freedom.

Let $\boldsymbol{a}$ be a given vector in $\mathbb{R}^d$ such that $F(q(\boldsymbol{a}_J)) \in (0,1)$ with $q(\boldsymbol{a}_J) := \boldsymbol{a}_J^\top (\Sigma_{JJ})^{-1}\boldsymbol{a}_J$, $\Sigma := A^\top A$. Cambanis et al. [5] showed that the random vector $\boldsymbol{X}_I|\boldsymbol{X}_J = \boldsymbol{a}_J$ is elliptically distributed with stochastic representation

$$(\boldsymbol{X}_I|\boldsymbol{X}_J = \boldsymbol{a}_J) \stackrel{d}{=} R_{I,\boldsymbol{a}_J} D^\top \boldsymbol{U}_m + \Sigma_{IJ}(\Sigma_{JJ})^{-1}\boldsymbol{a}_J, \tag{2.2}$$

where D is a square matrix such that $D^\top D = \Sigma_{II} - \Sigma_{IJ}(\Sigma_{JJ})^{-1}\Sigma_{JI}$ and $R_{I,\boldsymbol{a}_J}$ is a positive random radius independent of $\boldsymbol{U}_m$, $m := |I|$. The distribution function $F_{I,\boldsymbol{a}_J}$ of $R_{I,\boldsymbol{a}_J}$ is given by

$$F_{I,\boldsymbol{a}_J}(x) = c(\boldsymbol{a}_J)\int_{\sqrt{q(\boldsymbol{a}_J)}}^{(q(\boldsymbol{a}_J)+x^2)^{1/2}} (r^2 - q(\boldsymbol{a}_J))^{m/2-1} r^{-(d-2)}\, dF(r), \quad x > 0, \tag{2.3}$$

with

$$c(\boldsymbol{a}_J) := \left(\int_{\sqrt{q(\boldsymbol{a}_J)}}^{\infty} (r^2 - q(\boldsymbol{a}_J))^{m/2-1} r^{-(d-2)}\, dF(r)\right)^{-1}.$$

Write in the following $\omega := \sup\{x : F(x) < 1\} > 0$ for the upper endpoint of F. Clearly, $\omega_{\boldsymbol{a}_J} := \sqrt{\omega^2 - q(\boldsymbol{a}_J)}$ is the upper endpoint of the distribution function $F_{I,\boldsymbol{a}_J}$, and $\omega = \infty$ implies $\omega_{\boldsymbol{a}_J} = \infty$. Cambanis et al. [5] showed further that $F_{I,\boldsymbol{a}_J}$ defines F in $[q(\boldsymbol{a}_J), \omega)$ by the following relation:

$$1 - F(r) = \frac{1}{c(\boldsymbol{a}_J)}\int_{\sqrt{r^2 - q(\boldsymbol{a}_J)}}^{\omega_{\boldsymbol{a}_J}} (x^2 + q(\boldsymbol{a}_J))^{d/2-1} x^{-(m-2)}\, dF_{I,\boldsymbol{a}_J}(x) \tag{2.4}$$

for $r \in [q(\boldsymbol{a}_J), \omega)$.

Next, we give few details for multivariate sample extremes.

Let G, H be two distribution functions on $\mathbb{R}^d$. If there exist functions $a_i > 0, b_i, i = 1, \ldots, d$ such that

$$\lim_{t\to\infty} \sup_{\boldsymbol{x}=(x_1,\ldots,x_d)\in\mathbb{R}^d} \left|G^t(\boldsymbol{a}(t)\boldsymbol{x} + \boldsymbol{b}(t)) - H(\boldsymbol{x})\right| = 0, \tag{2.5}$$

with $\boldsymbol{a}(t) := (a_1(t), \ldots, a_d(t))^\top$, $\boldsymbol{b}(t) := (b_1(t), \ldots, b_d(t))^\top$, then we say that G is in the max-domain of attraction of H. This is abbreviated as $G \in MDA(H)$.

It is well known that H is a max-stable distribution function, see, e.g. [23,22] or [8]. If $G_i, H_i, 1 \leqslant i \leqslant d$ are the marginal distributions of G, H, then (2.5) implies

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} \left|G_i^t(a_i(t)x + b_i(t)) - H_i(x)\right| = 0,$$

which is again abbreviated as $G_i \in MDA(H_i)$. The univariate extreme value distribution function $H_i, 1 \leqslant i \leqslant d$ is either the Gumbel distribution $\Lambda(x) = \exp(-\exp(-x)), x \in \mathbb{R}$, or the Weibull distribution $\Psi_\gamma(x) = \exp(-|x|^\gamma), \gamma > 0, x < 0$, or the Fréchet distribution $\Phi_\gamma(x) = \exp(-x^{-\gamma}), \gamma > 0, x > 0$.

We present next a result for the asymptotic behaviour of the sample maxima considering multivariate elliptical random vectors with representation (2.1) and A such that $\Sigma = A^\top A$ is a positive definite correlation matrix with diagonal entries equal 1. We omit the proof, since it follows easily using the results of Berman [4] or Hashorva [13].

Proposition 2.1. *Let X be an elliptical random vector in* $\mathbb{R}^d$, $d \geqslant 2$ *with distribution function G and stochastic representation* (2.1). *Assume that the random radius R has distribution function F in the max-domain of attraction of* Λ *or* $\Psi_\gamma, \gamma > 0$. *Then* $G \in MDA(Q)$ *with* $Q(\boldsymbol{x}) = \prod_{i=1}^d \Lambda(x_i)$ *or* $Q(\boldsymbol{x}) = \prod_{i=1}^d \Psi_{\gamma+(d-1)/2}(x_i), \boldsymbol{x} \in \mathbb{R}^d$, *respectively. If* $F \in MDA(\Phi_\gamma), \gamma > 0$ *then* $G \in MDA(Q)$ *with Q a max-stable distribution function which has dependent Fréchet marginal distributions with index* γ.

3. Main results

In this section we deal with multivariate elliptical random vectors with stochastic representation (2.1), and consider I, J, $\boldsymbol{a}$, $R_{I,\boldsymbol{a}_J}$, $c(\boldsymbol{a}_J)$ as in the previous section.

We discuss separately the three possible cases for the max-domain of attractions of the distribution function F, $F_{I,\boldsymbol{a}_J}$ of the associated random radius R, $R_{I,\boldsymbol{a}_J}$, respectively. Proposition 2.1 shows that the asymptotic behaviour of the extremes of elliptical random vectors is known if the asymptotic tail behaviour of the associated random radius is known. Therefore, we treat in details the asymptotic relation between R and $R_{I,\boldsymbol{a}_J}$. We begin first with the Gumbel case, followed by the Fréchet and Weibull one. The extreme value results mentioned in the following can be found in [6,21,11,23,22,8,19] therefore we omit some details.

3.1. Gumbel case

Let F be a distribution function on $\mathbb{R}$ with upper endpoint ω. A necessary and sufficient condition for $F \in MDA(\Lambda)$ is the existence of a positive scaling function w such that

$$\lim_{t\uparrow\omega} \frac{1-F(t+x/w(t))}{1-F(t)} = \exp(-x) \quad \forall x \in \mathbb{R}. \tag{3.1}$$

In the next proposition we derive the asymptotic tail behaviour of $R_{I,\boldsymbol{a}_J}$ assuming that the distribution function F of R is in the Gumbel max-domain of attraction. For any $\boldsymbol{x} \in \mathbb{R}^d$ define $\omega_{\boldsymbol{x}_J} := (\omega^2 - q(\boldsymbol{x}_J))^{1/2}$ whenever $q(\boldsymbol{x}_J) \in [0, \sqrt{\omega})$, $\omega > 0$ and denote by $G_{I,\boldsymbol{x}_J}$ the distribution function of $\boldsymbol{X}_I|\boldsymbol{X}_J = \boldsymbol{x}_J$. (Recall $q(\boldsymbol{x}_J) := \boldsymbol{x}_J^\top (\Sigma_{JJ})^{-1}\boldsymbol{x}_J, \boldsymbol{x} \in \mathbb{R}^d$).

Proposition 3.1. *Let A be a* $d \times d$ *real non-singular matrix, and let R be an almost surely positive random variable with distribution function F independent of* $\boldsymbol{U}_d$. *Define the elliptical random vector* $\boldsymbol{X} \stackrel{d}{=} RA^\top \boldsymbol{U}_d$ *with distribution function G and let I, J be two non-empty disjoint index sets such that* $I \cup J = \{1, \ldots, d\}$. *If* (3.1) *holds, then for any* $\boldsymbol{a} \in \mathbb{R}^d$ *such that* $\omega_{\boldsymbol{a}_J} \in (0, \omega)$ *we have* $F_{I,\boldsymbol{a}_J} \in MDA(\Lambda)$.

Furthermore

$$1 - F_{I,\boldsymbol{a}}(u) = (1+o(1))c(\boldsymbol{a}_J)\frac{{\omega_{\boldsymbol{a}_J}}^{m-2}}{\omega^{d-2}}\left[1 - F\left(\sqrt{q(\boldsymbol{a}_J)+u^2}\right)\right], \quad u \uparrow \omega_{\boldsymbol{a}_J} \tag{3.2}$$

holds with $m := |I| \geqslant 1$ *where we set* $\omega_{\boldsymbol{a}_J}/\omega := u$ *if* $\omega = \infty$.

Corollary 3.2. *Under the assumptions of Proposition* 3.1 *the multivariate distribution function* $G_{I,\boldsymbol{a}_J}$ *is in the max-domain of attraction of a product distribution with unit Gumbel marginal distributions.*

In the next proposition the converse of Proposition 3.1 is stated.

Proposition 3.3. *Let $F, G, I, J, \boldsymbol{X}$ be as in Proposition* 3.1 *and let $\boldsymbol{b}$ be a given vector in $\mathbb{R}^d$ with $\omega_{\boldsymbol{b}_J} \in (0, \omega)$. If $F_{I,\boldsymbol{b}_J} \in MDA(\Lambda)$ then $F \in MDA(\Lambda)$ and $G \in MDA(Q)$ with Q a product distribution with unit Gumbel marginal distributions.*

We give next an example.

Example 1 (*Kotz type I*). Let $\boldsymbol{X}$ be an elliptical random vector as in Proposition 3.1 and let $\boldsymbol{I}, \boldsymbol{J}$ be partitions of $\{1, \ldots, d\}$. We consider the special case R has asymptotic tail behaviour given by

$$\boldsymbol{P}\{R > x\} = (1 + o(1)) \exp(-Kx^{\delta}), \quad K > 0, \delta > 0, \ x \to \infty.$$

It follows easily that (3.1) is satisfied with scaling function $w(u) = K\delta u^{\delta-1}$. Hence (3.2) implies for any $\boldsymbol{a} \in \mathbb{R}^d, \boldsymbol{a}_J \neq \boldsymbol{0}_J$

$$1 - F_{I,\boldsymbol{a}_J}(u) = (1 + o(1))c(\boldsymbol{a}_J)u^{m-d} \exp(-K(q(\boldsymbol{a}_J) + u^2)^{\delta/2}), \quad u \to \infty.$$

Note in passing that if $K\delta = 1$ and $\delta = 2$ then $w(u) = u$ which is in particular the case if $\boldsymbol{X}$ is a standard Gaussian random vector with positive definite covariance matrix Σ.

3.2. Fréchet case

If the distribution function F of the associated random radius is in the max-domain of attraction of $\Phi_\gamma, \gamma > 0$, then the upper endpoint ω equals ∞. A typical example $F \in MDA(\Phi_\gamma)$ is when F has the following tail asymptotics:

$$1 - F(u) = (1 + o(1))cu^{-\gamma}, \quad c > 0, \ \gamma > 0, \ u \to \infty.$$

Denote by $G^{(j)}_{I,\boldsymbol{a}_J}, 1 \leqslant j \leqslant m$ the marginal distributions of the distribution function $G_{I,\boldsymbol{a}_J}, \boldsymbol{a} \in \mathbb{R}^d$.

Proposition 3.4. *Let $F, G, I, J, m, \boldsymbol{X}$ be as in Proposition* 3.1. *If $F \in MDA(\Phi_\gamma), \gamma > 0$ then we have for any vector $\boldsymbol{a} \in \mathbb{R}^d, \boldsymbol{a}_J \neq \boldsymbol{0}_J$*

$$1 - F_{I,\boldsymbol{a}_J}(u) = \frac{(1 + o(1))c(\boldsymbol{a}_J)}{1 + (d - m)/\gamma} u^{m-d}[1 - F(u)], \quad u \to \infty, \tag{3.3}$$

and both $F_{I,\boldsymbol{a}_J}$ and $G^{(j)}_{I,\boldsymbol{a}_J}, 1 \leqslant j \leqslant m$ are in the Fréchet max-domain of attraction with index $\gamma + d - m > 0$.

Note that $F \in MDA(\Phi_\gamma)$ is equivalent with X_1 has distribution function in the Fréchet max-domain of attraction (see [15,17]). Assuming that the conditional radius $R_{I,\boldsymbol{a}_J}$ has distribution function in the Fréchet max-domain of attraction we obtain the converse results.

Proposition 3.5. *Let $F, G, I, J, m, \boldsymbol{X}$ be as in the above proposition. Assume that for a vector $\boldsymbol{a} \in \mathbb{R}^d, \boldsymbol{a} \neq \boldsymbol{0}_J$ we have $F_{I,\boldsymbol{a}_J} \in MDA(\Phi_\gamma), \gamma > d - m$. Then $F \in MDA(\Phi_{\gamma+m-d})$ and* (3.3) *hold. Furthermore, $G \in MDA(Q)$ with Q a max-stable distribution function on $(0, \infty)^d$ which is not a product distribution. The marginal distributions of Q are Fréchet with positive parameter $\gamma + m - d$.*

We remark that if $F_{I,\boldsymbol{a}_J} \in MDA(\Phi_\gamma)$, $\gamma \leqslant d-m$ the above proposition does not provide the tail asymptotic behaviour of F.

3.3. Weibull case

We consider now the last case $F \in MDA(\Psi_\gamma)$, with $\gamma > 0$. Both the upper endpoints $\omega, \omega_{\boldsymbol{b}_J}$ of F and $F_{I,\boldsymbol{a}_J}$ are necessarily finite. We assume for simplicity that $\omega = 1$. The other case $\omega \in (0,\infty) \setminus \{1\}$ follows easily.

Proposition 3.6. *Let $F, G, I, J, m, \boldsymbol{X}$ be as in Proposition* 3.1. *If $F \in MDA(\Psi_\gamma), \gamma > 0$ with upper endpoint $\omega = 1$, then we have for any vector $\boldsymbol{a} \in \mathbb{R}^d$ such that $\omega_{\boldsymbol{a}_J} \in (0,1)$*

$$1 - F_{I,\boldsymbol{a}_J}(\omega_{\boldsymbol{a}_J} - 1/u) = (1+o(1))c(\boldsymbol{a}_J)\omega_{\boldsymbol{a}_J}^{m-2}\left[1 - F\left(1 - \frac{\omega_{\boldsymbol{a}_J}}{u}\right)\right], \quad u \to \infty. \tag{3.4}$$

Furthermore $F_{I,\boldsymbol{a}_J} \in MDA(\Psi_\gamma)$ and $G_{I,\boldsymbol{a}_J} \in MDA(Q)$ with $Q(\boldsymbol{x}_I) = \prod_{i\in I}\Psi_{\gamma+(m-1)/2}(x_i), \boldsymbol{x} \in (-\infty,0)^d$.

Conversely, if for a given $\boldsymbol{a} \in \mathbb{R}^d$ such that $\omega_{\boldsymbol{a}_J} \in (0,1)$ we have $F_{I,\boldsymbol{a}_J} \in MDA(\Psi_\gamma), \gamma > 0$, then $G \in MDA(Q)$ with $Q(\boldsymbol{x}) = \prod_{i=1}^d \Psi_{\gamma+(d-1)/2}(x_i), \boldsymbol{x} \in \mathbb{R}^d$ and (3.4) *holds.*

Next we give a simple example.

Example 2. Let $\boldsymbol{X}$ be a random vector as in the above theorem with associated random radius R with distribution function F. We suppose in this example that F is the Beta distribution with positive parameters α, β. It follows easily that

$$1 - F(1-1/u) = (1+o(1))\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta+1)}u^{-\beta}, \quad u \to \infty$$

holds with $\Gamma(\cdot)$ the Gamma function implying that F is in the Weibull max-domain of attraction with index β. Consequently, the above theorem yields for $\boldsymbol{a} \in \mathbb{R}^d$ such that $q(\boldsymbol{a}_J) \in (0,1)$ with I, J partitions of $\{1,\ldots,d\}$

$$1-F_{I,\boldsymbol{a}_J}\left(\sqrt{1-q(\boldsymbol{a}_J)}-1/u\right) = (1+o(1))c(\boldsymbol{a}_J)\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta+1)}(1-q(\boldsymbol{a}_J))^{m/2-1+\beta/2}u^{-\beta}$$

as $u \to \infty$.

4. Proofs

Proof of Proposition 3.1. Define $u_* := (q(\boldsymbol{a}_J) + u^2)^{1/2}$, $u \in (0, \omega_{\boldsymbol{b}_J})$. By (2.3) we have

$$1 - F_{I,\boldsymbol{a}_J}(u) = c(\boldsymbol{a}_J)\int_{u_*}^{\omega}(x^2 - q(\boldsymbol{a}_J))^{m/2-1}x^{-(d-2)}\,dF(x).$$

We prove first (3.2). Assume therefore that $\omega \in (0,\infty)$. Clearly, if $u \uparrow \omega_{\boldsymbol{b}_J}$ then $u_* \to \omega$. Since the integrand is positive and continuous at ω it follows easily that

$$1 - F_{I,\boldsymbol{a}_J}(u) = (1+o(1))c(\boldsymbol{a}_J)\omega^{2-d}\omega_{\boldsymbol{b}_J}^{m-2}\left[1 - F\left(\sqrt{q(\boldsymbol{a}_J)+u^2}\right)\right], \quad u \uparrow \omega_{\boldsymbol{b}_J}$$

holds, thus (3.2) is satisfied. By the properties of the scaling function w we have (see, e.g. [23,4]) $\lim_{u_*\uparrow\infty} w(u_*) = \infty$, hence we obtain for any $x \in \mathbb{R}$

$$\lim_{u\uparrow\omega_{\boldsymbol{b}_J}} \frac{1 - F_{I,\boldsymbol{a}_J}(u + x/w(u_*))}{1 - F_{I,\boldsymbol{a}_J}(u)} = \exp(-x\omega_{\boldsymbol{b}_J}/\omega),$$

consequently $F_{I,\boldsymbol{a}_J} \in MDA(\Lambda)$. Utilising [14, Lemmas 4.3, 4.4] we obtain for $\omega = \infty$

$$\begin{aligned}1 - F_{I,\boldsymbol{a}_J}(u) &= c(\boldsymbol{a}_J)\int_{u_*}^{\omega} (x^2 - q(\boldsymbol{a}_J))^{m/2-1} x^{-(d-2)}\, dF(x)\\ &= (1 + o(1))c(\boldsymbol{a}_J)u^{m-d}\left[1 - F\left(\sqrt{q(\boldsymbol{a}_J) + u^2}\right)\right], \quad u \to \infty.\end{aligned}$$

Since further $\lim_{u\uparrow\infty} uw(u) = \infty$ and $\lim_{u\to\infty} u_*/u = 1$ we have

$$\begin{aligned}\lim_{u\to\infty} \frac{1 - F_{I,\boldsymbol{a}_J}(u + x/w(u_*))}{1 - F_{I,\boldsymbol{a}_J}(u)} &= \lim_{u\to\infty} \frac{1 - F\left(\sqrt{q(\boldsymbol{a}_J) + (u + x/w(u_*))^2}\right)}{1 - F\left(\sqrt{q(\boldsymbol{a}_J) + u^2}\right)}\\ &= \exp(-x),\end{aligned}$$

hence $F_{I,\boldsymbol{a}_J} \in MDA(\Lambda)$. Thus the proof is complete. □

Proof of Corollary 3.2. By Proposition 3.1 using further Proposition 2.1 for the Gumbel case we have $G_{I,\boldsymbol{a}_J} \in MDA(Q)$ with Q a product distribution with unit Gumbel marginal distributions. □

Proof of Proposition 3.3. Set $b_*^2 := \boldsymbol{b}_J^\top(\Sigma_{JJ})^{-1}\boldsymbol{b}_J$ and $u_* := \sqrt{u^2 - b_*^2}$. In the case ω is finite (2.4) implies

$$\begin{aligned}1 - F(u) &= \frac{1}{c(\boldsymbol{b}_J)}\int_{\sqrt{u^2-b_*^2}}^{\omega_{\boldsymbol{b}_J}} (x + b_*^2)^{d/2-1} x^{-(m-2)}\, dF_{I,\boldsymbol{b}_J}(x)\\ &= \frac{1}{c(\boldsymbol{b}_J)}\int_{u_*}^{\omega_{\boldsymbol{b}_J}} (x + b_*^2)^{d/2-1} x^{-(m-2)}\, dF_{I,\boldsymbol{b}_J}(x)\\ &= \frac{1}{c(\boldsymbol{b}_J)}\omega^{d-2}\omega_{\boldsymbol{b}_J}^{2-m}\left[1 - F_{I,\boldsymbol{b}_J}\left(\sqrt{u^2 - b_*^2}\right)\right], \quad u \uparrow \omega.\end{aligned}$$

The above asymptotic relation yields thus for any $x \in \mathbb{R}$

$$\begin{aligned}\lim_{u\uparrow\omega} \frac{1 - F(u + x/w(u_*))}{1 - F(u)} &= \lim_{u\uparrow\omega} \frac{1 - F_{I,\boldsymbol{b}_J}\left(\sqrt{(u + x/w(u_*))^2 - b_*^2}\right)}{1 - F_{I,\boldsymbol{b}_J}\left(\sqrt{u^2 - b_*^2}\right)}\\ &= \exp(-x\omega/\omega_{\boldsymbol{b}_J}),\end{aligned}$$

consequently $F \in MDA(\Lambda)$. Hence the proof for this case follows easily using further Propositions 2.1 and 3.3. If $\omega = \infty$ we obtain utilising again [14, Lemmas 4.3, 4.4]

$$\begin{aligned}1 - F(u) &= \frac{1}{c(\boldsymbol{b}_J)}\int_{\sqrt{u^2-b_*^2}}^{\infty} (x + b_*^2)^{d/2-1} x^{-(m-2)}\, dF_{I,\boldsymbol{b}_J}(x)\\ &= \frac{(1 + o(1))}{c(\boldsymbol{b}_J)}u^{d-m}\left[1 - F_{I,\boldsymbol{b}_J}\left(\sqrt{u^2 - b_*^2}\right)\right], \quad u \to \infty.\end{aligned}$$

Since further $\lim_{u\to\infty} u/u_* = 1$ we have

$$\lim_{u\to\infty}\frac{1-F(u+x/w(u_*))}{1-F(u)} = \lim_{u\to\infty}\frac{1-F_{I,\boldsymbol{b}_J}(u_*+ux/(u_*w(u_*))(1+o(1)))}{1-F_{I,\boldsymbol{b}_J}(u_*)}$$
$$= \lim_{u\to\infty}\frac{1-F_{I,\boldsymbol{b}_J}(u+x/w(u))}{1-F_{I,\boldsymbol{b}_J}(u)}$$
$$= \exp(-x),$$

hence the proof follows using further Propositions 2.1 and 3.1. □

Proof of Proposition 3.4. $F \in MDA(\Phi_\gamma)$ implies

$$\lim_{u\to\infty}\frac{1-F(xu)}{1-F(u)} = x^{-\gamma} \quad \forall x > 0,$$

hence utilising [6, Theorem 1.2.1] and (2.3) we obtain

$$1-F_{I,\boldsymbol{a}_J}(u) = c(\boldsymbol{a}_J)\int_{\sqrt{q(\boldsymbol{a}_J)+u^2}}^{\infty}(x^2-q(\boldsymbol{a}_J))^{m/2-1}x^{-(d-2)}\,dF(x)$$
$$= \frac{1+o(1)}{1+(d-m)/\gamma}u^{m-d}[1-F(u)], \quad u\to\infty.$$

The rest of the proof follows by Proposition 4.2 of Hashorva (2005). □

Proof of Proposition 3.5. The proof follows easily by (2.4), Propositions 2.1 and 3.4. □

Proof of Proposition 3.6. By the definition, the distribution function $F_{I,\boldsymbol{a}_J}$ has upper endpoint $\omega_{\boldsymbol{a}_J} := \sqrt{1-q(\boldsymbol{a}_J)}$. Next, using (2.3) we have as $u\to\infty$

$$1-F_{I,\boldsymbol{a}_J}(\omega_{\boldsymbol{a}_J}-1/u) = c(\boldsymbol{a}_J)\int_{\sqrt{q(\boldsymbol{a}_J)+(\omega_{\boldsymbol{a}_J}-1/u)^2}}^{\omega}(x^2-q(\boldsymbol{a}_J))^{m/2-1}x^{-(d-2)}\,dF(x)$$
$$= (1+o(1))c(\boldsymbol{a}_J)\omega_{\boldsymbol{a}_J}^{m-2}[1-F(1-\omega_{\boldsymbol{a}_J}/u)],$$

where we used the fact that the integrand is continuous at 1. Consequently, $F_{I,\boldsymbol{a}_J} \in MDA(\Psi_\gamma)$ and the proof follows applying Proposition 2.1.

Utilising (2.4) and the assumption $F_{I,\boldsymbol{a}_J} \in MDA(\Psi_\gamma)$ we obtain with similar arguments for $u\to\infty$

$$1-F(1-1/u) = \frac{1}{c(\boldsymbol{a}_J)}\int_{\sqrt{(1-1/u)^2-q(\boldsymbol{a}_J)}}^{\omega_{a_J}}(x^2+q(\boldsymbol{a}_J))^{d/2-1}x^{-(m-2)}\,dF_{I,\boldsymbol{a}_J}(x)$$
$$= \frac{(1+o(1))}{c(\boldsymbol{a}_J)}\omega_{\boldsymbol{a}_J}^{2-m}[1-F_{I,\boldsymbol{a}_J}(\omega_{\boldsymbol{a}_J}-1/u)]$$

implying $F \in MDA(\Psi_\gamma)$. The proof follows now by Propositions 2.1 and 3.6. □

Acknowledgment

I would like to thank Professor Kano for kindly providing a copy of [18] and Professor Szabłowski for sending several research articles.

References

[1] T.W. Anderson, K.T. Fang, On the theory of multivariate elliptically contoured distributions and their applications, in: K.T. Fang, T.W. Anderson (Eds.), Statistical Inference in Elliptically Contoured and Related Distributions, Allerton Press, New York, 1990, pp. 1–23.

[2] M.S. Berman, Sojourns and extremes of stationary processes, Ann. Probab. 10 (1982) 1–46.

[3] M.S. Berman, Sojourns and extremes of Fourier sums and series with random coefficients, Stochastic Process. Appl. 15 (1983) 213–238.

[4] M.S. Berman, Sojourns and Extremes of Stochastic Processes, Wadsworth & Brooks, Cole, Belmont, CA, 1992.

[5] S. Cambanis, S. Huang, G. Simons, On the theory of elliptically contured distributions, J. Multivariate Anal. 11 (1981) 368–385.

[6] L. de Haan, On Regular Variation and its Applications to the Weak Convergence of Sample Extremes, Mathematisch Centrum Amsterdam, The Netherlands, 1970.

[7] W.F. Eddy, J.D. Gale, The convex hull of a spherically symmetric sample, Adv. Appl. Probab. 13 (1981) 751–763.

[8] M. Falk, J. Hüsler, R.-D. Reiss, Laws of Small Numbers: Extremes and Rare Events, DMV Seminar, vol. 23, second ed., Birkhäuser, Basel, 2004.

[9] K. Fang, Y. Zhang, Generalized Multivariate Analysis, Springer, Berlin, 1990.

[10] K.-T. Fang, S. Kotz, K.-W. Ng, Symmetric Multivariate and Related Distributions, Chapman & Hall, London, 1990.

[11] J. Galambos, The Asymptotic Theory of Extreme Order Statistics, second ed., Krieger, Malabar, 1987.

[12] J.D. Gale, The asymptotic distribution of the convex hull of a random sample, Ph.D. Thesis, Carnegie-Mellon University, 1980.

[13] E. Hashorva, Extremes of asymptotically spherical and elliptical random vectors, Insurance Math. Econom. 36 (3) (2005) 285–302.

[14] E. Hashorva, Gaussian approximation of conditional elliptical random vectors, Stochastic Models 22 (2006) 441–457.

[15] E. Hashorva, On the regular variation of elliptical random vectors, Statist. Probab. Lett. 76 (14) (2006) 1427–1434.

[17] E. Hashorva, S. Kotz, A. Kume, L_p-norm symmetrised Dirichlet distributions, Albanian J. Math. 1 (2007) 31–56.

[18] Y. Kano, Consistency property of elliptical probability density functions, J. Multivariate Anal. 51 (1994) 139–147.

[19] S. Kotz, S. Nadarajah, Extreme Value Distributions, Theory and Applications, Imperial College Press, London, United Kingdom, 2005 (second printing).

[20] S. Kotz, I.V. Ostrovskii, Characteristic functions of a class of elliptical distributions, J. Multivariate Anal. 49 (1) (1994) 164–178.

[21] M.R. Leadbetter, G. Lindgren, H. Rootzén, Extremes and Related Properties of Random Sequences and Processes, Springer, New York, 1983.

[22] R.-D. Reiss, Approximate Distributions of Order Statistics: With Applications to Nonparametric Statistics, Springer, New York, 1989.

[23] S.I. Resnick, Extreme Values, Regular Variation and Point Processes, Springer, New York, 1987.

[24] P.J. Szabłowski, Uniform distributions on spheres in finite dimensional L_α and their generalizations, J. Multivariate Anal. 64 (1998) 103–117.