Accelerated Monotone Iterations for Numerical Solutions of Nonlinear Elliptic Boundary Value Problems

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Abstract—An accelerated monotone iterative scheme for numerical solutions of a class of nonlinear elliptic boundary value problems is presented. The mathematical analysis is devoted to a system of discretized equations of the elliptic boundary value problem by the finite-difference or finite-element method. It is shown that the sequence of iterations from a linear iteration process converges monotonically and quadratically to a unique solution in a sector between a pair of upper and lower solutions. This result is then used to show the quadratic convergence of the iterations to a maximal solution and a minimal solution when the nonlinear discrete system possesses multiple solutions. An application is given to a tabular reactor model from chemical engineering for numerical solutions, and the number of iterations are compared with that by the regular monotone iterative scheme. © 2003 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

In the treatment of numerical solutions of the nonlinear elliptic boundary value problem

\[-\nabla \cdot (Du) + \nu \cdot Vu = f(x, u), \quad x \in \Omega,\]
\[\alpha \frac{\partial u}{\partial \nu} + \beta u = g(x, u), \quad x \in \partial \Omega.\]

(1.1)

by the finite-difference (or finite-element) method, where \(\Omega\) is a bounded domain in \(\mathbb{R}^n\) with boundary \(\partial \Omega\), \(\nu \cdot \nabla u = \frac{\partial u}{\partial x_1} + \cdots + \frac{\partial u}{\partial x_n}\) denotes the outward normal derivative of \(u\) on \(\partial \Omega\), and \(f(x, u)\) and \(g(x, u)\) are, in general, nonlinear functions of \(u\), the discretized equations of the problem is a system of algebraic equations which may be written in the compact form

\[AU = F(U).\]

(1.2)

In the above system, \(U \equiv (u_1, \ldots, u_N)\) is a column vector representing the solution \(u_i \equiv u(x_i)\) at the mesh point \(x_i\) in \(\bar{\Omega} \equiv \Omega \cup \partial \Omega\) (\(i = 1, \ldots, N\)), \(A\) is an \(N\) by \(N\) block matrix that is associated

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with the diffusion-convection operator and the boundary condition in (1.1), and
\[
F(U) = (F_1(u_1), \ldots, F_N(u_N)),
\]
\[
F_i(u_i) = f_i(x_i, u(x_i)) + g_i(x_i, u(x_i)), \quad x_i \in \Omega, \quad x_i' \in \partial \Omega.
\] (1.3)

The function \( f_i(x, u(x)) \) appears at the interior mesh points \( \Omega \) while \( g(x'_i, u(x'_i)) \) appears on the boundary points and possibly neighboring boundary mesh points of \( \partial \Omega \). (See [1,2] for some detailed derivations.) It is assumed that \( D \equiv D(x) \) is positive on \( \Omega \), \( \alpha \) and \( \beta \) are nonnegative constants with \( \alpha + \beta > 0 \), and \( F_i(u_i) \) is a \( C^1 \)-function of \( u_i \). The assumption on \( \alpha, \beta \) includes the Dirichlet boundary condition \( (\alpha = 0, \beta = 1, g_i \equiv g(x_i)) \) and the Neumann-Robin boundary condition \( (\alpha = 1, \beta \geq 0) \).

Using the standard central difference approximation for the first and second derivatives (and an upwind differencing scheme if the convection coefficient \( v \equiv v(x) \) dominates the diffusion coefficient \( D(x) \)) the matrix \( A \) possesses the properties in Hypothesis (H) in Section 2 (cf. [2-4]). Under this condition and the existence of a pair of ordered upper and lower solutions, one can construct two monotone sequences which converge to a maximal solution and a minimal solution of (1.1), respectively (cf. [2,4-8]). The above monotone iterative method is well known and has been widely used for both continuous and discrete elliptic boundary value problems. Some of the works for numerical solutions by this method are given in [2,4,5-15] while those for the continuous problems can be found in [16] and the references therein. However, most of the discussions in the above works involve monotone iterative schemes whose rate of convergence are of linear order. In a recent article [3], some accelerated monotone iterative schemes for parabolic boundary value problems are given. The purpose of this paper is to extend the method used in [3] to obtain an accelerated monotone iterative scheme for the elliptic boundary value problem (1.1). An advantage of this method is that it leads to not only the existence and computational algorithm for maximal and minimal solutions but also quadratic convergence of the sequence of iterations for a certain class of nonlinear functions. Moreover, since the initial iteration in the monotone iterative scheme is either an upper solution or a lower solution which can be constructed from the equation without any knowledge of the solution, this method simplifies considerably the search for the initial guess as is often required in the Newton's method.

The plan of the paper is as follows. In Section 2, we present an accelerated monotone iterative scheme for the computation of maximal and minimal sequences, including the case where \( A \) is a singular matrix. Section 3 is devoted to the quadratic convergence of the maximal and minimal sequences for a certain class of nonlinear functions \( F(U) \). In Section 4, we give an application of the accelerated monotone iterative scheme to a tabular reactor model in chemical engineering where the number of iterations between the regular and the accelerated monotone iterations are compared.

### 2. MONOTONE ITERATIVE SCHEMES

To develop monotone iterative schemes for (1.2), we need a pair of ordered upper and lower solutions \( \bar{U}, \hat{U} \) which are required to satisfy the relation \( \bar{U} \geq \hat{U} \) and the inequalities
\[
A\bar{U} \geq F(\bar{U}) \quad \text{and} \quad A\hat{U} \leq F(\hat{U}),
\] (2.1)

where the inequality between vectors is always in the componentwise sense. For a given pair of ordered upper and lower solutions \( \bar{U} \equiv (\bar{u}_1, \ldots, \bar{u}_N), \hat{U} \equiv (\hat{u}_1, \ldots, \hat{u}_N) \), we set
\[
\langle \bar{U}, \bar{U} \rangle \equiv \{ U \in \mathbb{R}^N; \bar{U} \leq U \leq \bar{U} \},
\]
\[
\langle \hat{u}_i, \hat{u}_i \rangle \equiv \{ u_i \in \mathbb{R}; \hat{u}_i \leq u_i \leq \hat{u}_i \}, \quad i = 1, \ldots, N
\]

and make the following hypothesis on \( A \).
(H) The matrix $A \equiv (a_{ij})$ is irreducible, and $a_{ii} > 0$, $a_{ij} \leq 0$ for $j \neq i$, and
\[
\sum_{j=1}^{N} a_{ij} \geq 0, \quad \text{with strict inequality holds for at least one } i. \tag{2.2}
\]

Hypothesis (H) implies that $A$ is an $M$-matrix, and for any nonnegative diagonal matrix $C$, including $C = 0$, the inverse $(A + C)^{-1}$ exists and is a positive matrix (cf. [17, p. 85]). Moreover, the smallest eigenvalue of $A$, denoted by $\mu_0$, is real and positive, and if condition (2.2) is replaced by the condition
\[
\sum_{j=1}^{N} a_{ij} = 0, \quad \text{for } i = 1, \ldots, N, \tag{2.3}
\]
then $\mu_0 = 0$ and $A$ is singular. Condition (2.3) corresponds to the case of the Neumann boundary condition (i.e., $\beta = 0$) and will be treated in the following discussion.

Consider the linear iteration process
\[
(A + yI)U^{(m+1)} = yU^{(m)} + F(U^{(m)}), \quad m = 0, 1, 2, \ldots, \tag{2.4}
\]
where $y$ is any nonnegative constant satisfying
\[
y \geq y_0 \equiv \max \left\{ -\frac{\partial F_i}{\partial u_i}(u_i); \ u_i \leq U_i \leq \bar{u}_i \right\}.
\]

It is well known that if the initial iteration in (2.4) is taken as $U^{(0)} = 0$ (respectively, $U^{(0)} = \bar{U}$) then the corresponding sequence $\{U^{(m)}\}$ converges monotonically to a maximal solution $\bar{U}$ (respectively, a minimal solution $U$) of (1.2) (cf. [2-6,8,13,14]). To increase the rate of convergence while maintaining the monotone property of the sequence, we modify the iteration process (2.4) by choosing some suitable nonnegative diagonal matrices
\[
C^{(m)} = \text{diag} \left( c_1^{(m)}, \ldots, c_N^{(m)} \right), \quad m = 0, 1, 2, \ldots, \tag{2.5}
\]
and construct a sequence from the iterative scheme
\[
(A + C^{(m)})U^{(m+1)} = C^{(m)}U^{(m)} + F(U^{(m)}), \quad m = 0, 1, 2, \ldots, \tag{2.6}
\]
It is clear from Hypothesis (H) and the nonnegative property of $C^{(m)}$ that the inverse $(A+C^{(m)})^{-1}$ exists and is a positive matrix for every $m$ (cf. [17]). This implies that the sequence governed by (2.6) is well defined whenever $C^{(m)}$ exists and is nonnegative. Denote the sequence by $\{\bar{U}^{(m)}\}$ if $U^{(0)} = \bar{U}$ and by $\{\underline{U}^{(m)}\}$ if $U^{(0)} = \underline{U}$, and refer to them as the maximal and minimal sequence, respectively.

To obtain the monotone property of these sequences, we choose the elements $c_i^{(m)}$ of $C^{(m)}$ by
\[
c_i^{(m)} = \begin{cases} 
\gamma_i^{(m)}, & \text{if } \gamma_i^{(m)} \geq 0, \\
0, & \text{if } \gamma_i^{(m)} < 0,
\end{cases} \quad m = 0, 1, 2, \ldots, \tag{2.7}
\]
where $\gamma_i^{(m)}$ is given by
\[
\gamma_i^{(m)} \equiv \max \left\{ -\frac{\partial F_i}{\partial u_i}(u_i); \ u_i \leq U_i^{(m)} \leq \bar{U}_i^{(m)} \right\}, \quad m = 0, 1, 2, \ldots, \tag{2.8}
\]
and $U_i^{(m)}$ and $\bar{U}_i^{(m)}$ are the respective components of $U^{(m)}$ and $\bar{U}^{(m)}$. The choice of $c_i^{(m)}$ in (2.7) implies that $C^{(m)}$ is nonnegative and
\[
C^{(m)}U + F(U) \geq C^{(m)}V + F(V), \quad \text{whenever } \bar{U}^{(m)} \geq U \geq U^{(m)}. \tag{2.9}
\]
To ensure that $\gamma_i^{(m)}$ exists, we must show $\underline{U}_i^{(m)} \leq U_i^{(m)}$ for every $m$. This relation and the monotone property of the maximal and minimal sequences are given by the following lemma.
Lemma 2.1. The sequences \( \{U^{(m)}\}, \{U'(m)\} \) governed by (2.6) with \( U^{(0)} = \bar{U} \) and \( U'(0) = \bar{U} \) are well defined and possess the monotone property

\[
\dot{U} \leq U^{(m)} \leq U^{(m+1)} \leq \bar{U}^{(m+1)} \leq \bar{U}^{(m)} \leq \bar{U}, \quad m = 1, 2, \ldots \quad (2.10)
\]

Moreover, for each \( m = 1, 2, \ldots, \bar{U}^{(m)} \) and \( U'(m) \) are ordered upper and lower solutions of (1.1).

Proof. It is clear from (2.7) and (2.8) that \( C^{(0)} \) is well defined. By (2.6) and (2.1), the vector \( \bar{V}^{(0)} \equiv \bar{U}^{(0)} - U^{(1)} = \bar{U} - \bar{U}^{(1)} \) satisfies the relation

\[
\left( A + C^{(0)} \right) \bar{V}^{(0)} = \left( A + C^{(0)} \right) \bar{U} - \left[ C^{(0)} \bar{U}^{(0)} + F \left( \bar{U}^{(0)} \right) \right] = A\bar{U} - F \left( \bar{U} \right) \geq 0.
\]

Since \( (A + C^{(0)})^{-1} \) exists and is positive, it follows that \( \bar{V}^{(0)} \geq 0 \). This proves \( U^{(1)} \leq U^{(0)} \). A similar argument using the property of a lower solution gives \( U^{(2)} \geq U^{(0)} \). Let \( W^{(1)} = U^{(1)} - U^{(1)} \).

Then by (2.6) and (2.9) (with \( m = 0 \)),

\[
\left( A + C^{(0)} \right) W^{(1)} = \left[ C^{(0)} U^{(0)} + F \left( U^{(0)} \right) \right] - \left[ C^{(0)} U^{(0)} + F \left( U^{(0)} \right) \right] \geq 0.
\]

This leads to \( U^{(1)} \geq U^{(1)} \). The above conclusions show that \( U^{(0)} \leq U^{(1)} \leq U^{(1)} \leq U^{(0)} \). This relation and (2.8) imply that \( C^{(1)} \) is well defined which ensures that \( U^{(2)} \) and \( U^{(2)} \) exist and can be computed from (2.6) with \( U^{(1)} = U^{(1)} \) and \( U^{(1)} = U^{(1)} \), respectively. Assume, by induction, that \( U^{(m)} \) and \( U^{(m)} \) exist and satisfy the relation \( U^{(m-1)} \leq U^{(m)} \leq U^{(m)} \leq U^{(m-1)} \) for some \( m > 1 \). Then by (2.7) and (2.8), \( C^{(m)} \) is well defined and is nonnegative. Therefore, \( (A + C^{(m)})^{-1} \) is positive which implies that \( U^{(m+1)} \) and \( U^{(m+1)} \) exist. Moreover, by (2.6), (2.9), and the induction hypothesis \( U^{(m-1)} \geq U^{(m)} \), we have

\[
\left( A + C^{(m)} \right) \left( \bar{U}^{(m)} - U^{(m+1)} \right) = \left( C^{(m)} - C^{(m-1)} \right) \bar{U}^{(m)}
\]

\[
+ \left( A + C^{(m-1)} \right) \bar{U}^{(m)} - \left( A + C^{(m-1)} \right) \bar{U}^{(m+1)}
\]

\[
= \left( C^{(m)} - C^{(m-1)} \right) \bar{U}^{(m)} + \left[ C^{(m-1)} \bar{U}^{(m-1)} + F \left( \bar{U}^{(m-1)} \right) \right]
\]

\[
- \left[ C^{(m)} \bar{U}^{(m)} + F \left( \bar{U}^{(m)} \right) \right]
\]

\[
\geq 0.
\]

This yields \( U^{(m)} \geq U^{(m+1)} \). A similar argument shows that \( U^{(m+1)} \geq U^{(m)} \) and \( U^{(m+1)} \geq U^{(m+1)} \). The existence of the sequences \( \{\bar{U}^{(m)}\}, \{U^{(m)}\} \) and the monotone property (2.10) follow from the principle of induction.

Finally, by (2.6), (2.9), and (2.10),

\[
A\bar{U}^{(m)} = C^{(m-1)} \left( \bar{U}^{(m-1)} - U^{(m-1)} \right) + F \left( \bar{U}^{(m-1)} \right) \geq F \left( \bar{U}^{(m)} \right),
\]

\[
A U^{(m)} = -C^{(m-1)} \left( U^{(m)} - U^{(m-1)} \right) + F \left( U^{(m-1)} \right) \leq F \left( U^{(m)} \right),
\]

for every \( m \). This relation and (2.10) show that \( \bar{U}^{(m)} \) and \( U^{(m)} \) are ordered upper and lower solutions of (1.2).
In view of the monotone property (2.10), we have the following existence-comparison theorem.

**Theorem 2.1.** Let \( \overline{U}, \underline{U} \) be a pair of ordered upper and lower solutions of (1.2), and let Hypothesis (H) hold. Then the maximal sequence \( \{\overline{U}^{(m)}\} \) converges monotonically from above to a maximal solution \( \overline{U} \) of (1.1), and the minimal sequence \( \{\underline{U}^{(m)}\} \) converges monotonically from below to a minimal solution \( \underline{U} \). Moreover,

\[
\overline{U} \leq U^{(m)} \leq U^{(m+1)} \leq \overline{U} \leq U \leq U^{(m+1)} \leq \overline{U}^{(m)} \leq \overline{U}, \quad m = 1, 2, \ldots \tag{2.11}
\]

**Proof.** In view of the monotone property (2.10), the limits

\[
\lim_{m \to \infty} U^{(m)} = \overline{U} \quad \text{and} \quad \lim_{m \to \infty} \underline{U}^{(m)} = \underline{U}
\]

exist and satisfy relation (2.11). Since by (2.10), \( \gamma^{(m)} \) is a nonincreasing function of \( m \) and is bounded from below by \( \gamma_i \), where

\[
\gamma_i = \max \left\{ -\frac{\partial F_i}{\partial u_i}(u); \; \underline{u}_i \leq u_i \leq \overline{u}_i \right\},
\]

we see from (2.7), (2.8), and (2.5) that the sequence \( \{C^{(m)}\} \) converges to a matrix \( C \) as \( m \to \infty \). It follows by letting \( m \to \infty \) in (2.6) that \( \overline{U} \) and \( \underline{U} \) are solutions of the equation \( (A + C)\overline{U} = CU + F(U) \) which is equivalent to (1.2). The maximal and minimal property of \( \overline{U} \) and \( \underline{U} \) follows from the fact that every solution of (1.2) is an upper solution as well as a lower solution (see also, [4,16]).

It is seen from the proofs of Lemma 2.1 and Theorem 2.1 that the monotone convergence of the maximal and minimal sequences is based on the positive property of the matrix \( (A + C^{(m)})^{-1} \) and the monotone property (2.9). These properties are ensured by the condition on \( A \) in (H) and the choice of \( c_i^{(m)} \) in (2.7), including the case \( c_i^{(m)} = 0 \) for all \( i \). When the elements \( a_{ij} \) satisfy condition (2.3), which corresponds to Neumann boundary condition in (1.2), the matrix \( A \) is singular (and \( \mu_0 = 0 \)). In this situation, the inverse matrix \( (A + C^{(m)})^{-1} \) exists and remains positive if \( c_i^{(m)} > 0 \) for at least one \( i \) (cf. [17]). This is clearly the case if \( \gamma_i^{(m)} > 0 \) for some \( i \). In the case of \( \gamma_i^{(m)} \leq 0 \) for all \( i \) (that is, \( F_i(u_i) \) is a nondecreasing function of \( u_i \)), we choose \( c_i^{(m)} > 0 \) for at least one \( i \). In each of the above cases, the inverse \( (A + C^{(m)})^{-1} \) is positive and condition (2.9) holds. By the argument in the proof of Theorem 2.1 we have the following conclusion for the singular matrix \( A \).

**Theorem 2.2.** Let the conditions in Theorem 2.1 be satisfied except that condition (2.2) is replaced by (2.3). For each \( m = 0, 1, 2, \ldots \), define \( c_i^{(m)} \) by (2.7) if \( \gamma_i^{(m)} > 0 \) for at least one \( i \), and by \( c_i^{(m)} = \delta > 0 \) for some \( i \) if \( \gamma_i^{(m)} \leq 0 \) for all \( i \). Then all the conclusions in Theorem 2.1 hold true.

It is seen from Theorems 2.1 and 2.2 that the main conditions for the monotone convergence of the maximal and minimal sequences are Hypothesis (H) and the existence of a pair of ordered upper and lower solutions. The latter requirement depends mainly on the nonlinear function \( F(U) \). Various methods and techniques for the construction of upper and solutions have been discussed in [4,16]. In particular, if there exist constant vectors \( a = (a_1, \ldots, a_N) \) and \( b = (b_1, \ldots, b_N) \) with \( a \leq o \leq b \) such that

\[
F_i(a_i) \geq 0 \geq F_i(b_i), \quad \text{for} \; i = 1, \ldots, N, \tag{2.14}
\]

then the pair \( \overline{U} = b \) and \( \underline{U} = a \) are ordered upper solutions. This pair will be used for our numerical example in the final section.
3. QUADRATIC RATE OF CONVERGENCE

In this section we show, under some conditions on $F_i(u_i)$, that the sequences $\{U^{(m)}\}, \{U_i^{(m)}\}$ converge quadratically to a solution of (1.2). It is assumed that for each $i = 1, \ldots, N$, $F_i(u_i)$ is a $C^2$-function of $u_i$ for $u_i \in (\hat{u}_i, \hat{u}_i)$. We first consider the case where $F_i(u_i)$ possesses the nonincreasing property

$$\frac{\partial F_i}{\partial u_i}(u_i) \leq 0, \quad \text{for all } u_i \in (\hat{u}_i, \hat{u}_i), \quad i = 1, \ldots, N. \quad (3.1)$$

It is known that under the above condition, $U = U^*$ ($\equiv U^*$) and $U^*$ is the unique solution of (1.1) (cf. [2,4]). Define

$$F_u(U) \equiv \text{diag} \left( \frac{\partial F_1}{\partial u_1}(u_1), \ldots, \frac{\partial F_N}{\partial u_N}(u_N) \right),$$

$$M \equiv \max \left\{ \left| \frac{\partial^2 F_i}{\partial^2 u_i^2}(u_i) \right| ; u_i \in (\hat{u}_i, \hat{u}_i), \quad i = 1, \ldots, N \right\},$$

$$\rho_\epsilon \equiv \rho \left[ (A + C)^{-1} \right] + \epsilon,$$

where $C \equiv \text{diag}(c_1, \ldots, c_N)$ with $c_i = \max\{0, \gamma_i\}$, $\epsilon > 0$ is an arbitrary small constant, and $\rho[B]$ denotes the spectral radius of $B$. In the following theorem, we show the quadratic convergence of the maximal and minimal sequences to $U^*$.

**Theorem 3.1.** Let the hypotheses in Theorem 2.1 and condition (3.1) be satisfied, and let $U^*$ be the unique solution of (1.2) in $(\hat{U}, \hat{U})$. Then for any $\epsilon > 0$, there exists a norm in $\mathbb{R}^n$ such that

$$\|U^{(m+1)} - U^*\| \leq \rho_\epsilon M \|U^{(m)} - U^*\|^2, \quad m = 0, 1, 2, \ldots, \quad (3.3)$$

if $\left( \frac{\partial^2 F_i}{\partial^2 u_i^2} \right)(u_i) \leq 0$ for $u_i \in (\hat{u}_i, \hat{u}_i)$, and

$$\|U^* - U^{(m+1)}\| \leq \rho_\epsilon M \|U^* - U^{(m)}\|^2, \quad m = 0, 1, 2, \ldots, \quad (3.4)$$

if $\left( \frac{\partial^2 F_i}{\partial^2 u_i^2} \right)(u_i) \geq 0$ for $u_i \in (\hat{u}_i, \hat{u}_i)$.

**Proof.** Consider the maximal sequence $\{U^{(m)}\}$. Since $U^*$ satisfies the equation

$$(A + C^{(m)})U^* = C^{(m)}U^* + F(U^*), \quad (3.5)$$

for every $m$, a subtraction of the above equation from (2.6) (with $U^{(m)} = U^{(m)}$) leads to

$$(A + C^{(m)}) \left( U^{(m+1)} - U^* \right) = C^{(m)} \left( U^{(m)} - U^* \right) + F \left( U^{(m)} \right) - F(U^*). \quad (3.6)$$

Since by (2.8), (3.1), and the hypothesis $\frac{\partial^2 F_i}{\partial^2 u_i} \leq 0$,

$$\gamma_i^{(m)} = -\frac{\partial F_i}{\partial u_i}(u_i^{(m)}) \geq 0,$$

we see from (2.5), (2.7), and (3.2) that $C^{(m)} = -F_u(U^{(m)})$. Using this relation and the mean-value theorem in (3.6) yields

$$(A + C^{(m)}) \left( U^{(m+1)} - U^* \right) = \left[ -F_u \left( U^{(m)} \right) + F_u \left( \xi^{(m)} \right) \right] \left( U^{(m)} - U^* \right),$$
where \( \xi^{(m)} \equiv (\xi_1^{(m)}, \ldots, \xi_N^{(m)}) \) is an intermediate value between \( \overline{U}^{(m)} \) and \( U^* \). Define

\[
V^{(m)} = - \left[ F_u \left( \overline{U}^{(m)} \right) - F_u \left( \xi^{(m)} \right) \right] \left( \overline{U}^{(m)} - U^* \right). \tag{3.7}
\]

Then the above relation may be written as

\[
\overline{U}^{(m+1)} - U^* = \left( A + C^{(m)} \right)^{-1} V^{(m)}. \tag{3.8}
\]

Since by the mean-value theorem, the elements of the diagonal matrix \( F_u(\overline{U}^{(m)}) - F_u(\xi^{(m)}) \) are given by

\[
\frac{\partial F_i}{\partial u_i} (\overline{u}_i^{(m)}) - \frac{\partial F_i}{\partial u_i} (\xi_i^{(m)}) = \left( \frac{\partial^2 F_i}{\partial u_i^2} \left( \overline{u}_i^{(m)} \right) \right) \left( \overline{u}_i^{(m)} - \xi_i^{(m)} \right), \quad i = 1, \ldots, N,
\]

where \( \eta_i^{(m)} \) is an intermediate value between \( u_i^{(m)} \) and \( \xi_i^{(m)} \), we see that the components \( v_i^{(m)} \) of \( V^{(m)} \) are given by

\[
v_i^{(m)} = - \left( \frac{\partial^2 F_i}{\partial u_i^2} \left( \overline{u}_i^{(m)} \right) \right) \left( \overline{u}_i^{(m)} - \xi_i^{(m)} \right) \left( \overline{u}_i^{(m)} - u_i^* \right), \quad i = 1, \ldots, N.
\]

Now, by (2.11) and \( \overline{U} = \overline{U} \equiv U^* \), we have

\[
\overline{U}^{(m)} \geq \eta \geq \xi \geq U^*.
\]

It follows from (3.2) and (2.11) that

\[
\left| v_i^{(m)} \right| \leq M \left( \overline{u}_i^{(m)} - \xi_i^{(m)} \right) \left( \overline{u}_i^{(m)} - u_i^* \right) \leq M \left( \overline{u}_i^{(m)} - u_i^* \right)^2. \tag{3.9}
\]

Moreover, by (2.5), (2.8), and (2.11), \( C^{(m)} \geq C \) for every \( m \). This implies that (cf. [17, p. 301]).

\[
\rho \left[ \left( A + C^{(m)} \right)^{-1} \right] \leq \rho \left[ \left( A + C \right)^{-1} \right], \quad m = 1, 2, \ldots, \tag{3.10}
\]

It is well known that given any \( \epsilon > 0 \) and any matrix \( B \), there exists a matrix norm and a vector norm in \( \mathbb{R}^N \) such that \( \|B\| \leq \rho \|B\| + \epsilon \) and \( \|B V\| \leq \|B\| \|V\| \) for every \( V \in \mathbb{R}^N \) (cf. [18]). By applying these norms in (3.8) with \( B = (A + C^{(m)})^{-1} \) and using the relations in (3.7), (3.9), and (3.10), we obtain

\[
\left\| \overline{U}^{(m+1)} - U^* \right\| \leq (\rho \left[ \left( A + C \right)^{-1} \right] + \epsilon) \cdot M \left\| \overline{U}^{(m)} - U^* \right\|^2.
\]

This proves relation (3.3).

To show relation (3.4), we observe from (2.6) and (3.5) (with \( \overline{U}^{(m)} = U^{(m)} \)) that

\[
\left( A + C^{(m)} \right) \left( U^* - \overline{U}^{(m+1)} \right) = C^{(m)} \left( U^* - U^{(m)} \right) + F \left( U^* \right) - F \left( U^{(m)} \right).
\]

By (2.8), (3.1), and the hypothesis \( \frac{\partial^2 F_i}{\partial u_i^2} \geq 0 \),

\[
\gamma_i^{(m)} = - \frac{\partial F_i}{\partial u_i} (\overline{u}_i^{(m)}) \geq 0.
\]

This leads to the relation

\[
\left( A + C^{(m)} \right) \left( U^* - U^{(m+1)} \right) = \left[ - F_u \left( U^{(m)} \right) + F_u \left( \xi^{(m)} \right) \right] \left( U^* - \overline{U}^{(m)} \right).
\]
for some intermediate value $\xi^{(m)}$ between $U^*$ and $U^{(m)}$. The conclusion in (3.4) follows from the same argument as that for (3.3). This proves the theorem.

In Theorem 3.1, it is assumed that condition (3.1) is satisfied for all $u_i \in (\hat{u}_i, \bar{u}_i)$. This condition implies that a unique solution $U^*$ to (1.1) exists and both the maximal and minimal sequences converge to $U^*$. In the general case without this condition, Theorem 2.1 ensures the convergence of $\{U^{(m)}\}$ to a maximal solution $\bar{U}$, and $\{U^{(m)}\}$ to a minimal solution $\underline{U}$. To obtain the quadratic convergence of these sequences, we assume the existence of a constant $\delta > 0$ such that either

$$\frac{\partial F_i}{\partial u_i}(u_i) \leq 0 \quad \text{and} \quad \frac{\partial^2 F_i}{\partial u_i^2}(u_i) \leq 0,$$

(3.11)

$$\frac{\partial F_i}{\partial u_i}(u_i) \leq 0 \quad \text{and} \quad \frac{\partial^2 F_i}{\partial u_i^2}(u_i) \geq 0,$$

(3.12)

where $\bar{u}_i$ and $\underline{u}_i$ are the respective components of $\bar{U}$ and $\underline{U}$. The above conditions require that $F_i(u_i)$ be nonincreasing and has a concavity property in a neighborhood of the maximal solution or the minimal solution. Under these conditions, we have the following quadratic convergence of the maximal and minimal sequences.

**Theorem 3.2.** Let the hypotheses in Theorem 2.1 be satisfied, and let $\bar{U}$ and $\underline{U}$ be the maximal and minimal solutions of (1.2) in $(\bar{U}, \underline{U})$. Then there exists an integer $m^* \geq 0$ and a constant $K$, independent of $m$, such that

$$\|\bar{U}^{(m+1)} - \bar{U}\| \leq K \|\bar{U}^{(m)} - U^*\|^2,$$

(3.13)

if condition (3.11) holds, and

$$\|\underline{U} - \underline{U}^{(m+1)}\| \leq K \|\underline{U} - \underline{U}^{(m)}\|^2,$$

(3.14)

if condition (3.12) holds.

**Proof.** By (2.1), every solution of (1.2) is an upper solution as well as a lower solution, and by Lemma 2.1, $\bar{U}^{(m)}$ and $\underline{U}^{(m)}$ are upper and lower solutions for each $m$. This implies that the pair $(\bar{U}^{(m)}, \bar{U})$ and the pair $(\underline{U}, \underline{U}^{(m)})$ are both ordered upper and lower solutions. Consider the maximal sequences $\{\bar{U}^{(m)}\}$. By condition (3.11) and the convergence of $\{\bar{U}^{(m)}\}$ to $\bar{U}$, there exists an integer $m^* \geq 0$ such that for every $m \geq m^*$,

$$\frac{\partial F_i}{\partial u_i}(u_i) \leq 0 \quad \text{and} \quad \frac{\partial^2 F_i}{\partial u_i^2}(u_i) \leq 0,$$

where $\bar{u}_i$ and $\bar{u}_i^{(m)}$ are the respective components of $\bar{U}$ and $\bar{U}^{(m)}$. This condition and the maximal property of the solution $\bar{U}$ in $(\bar{U}, \bar{U})$ implies that $\bar{U}$ is the unique solution between $\bar{U}$ and $\bar{U}^{(m)}$. Hence, by considering $\bar{U} = \bar{U}^{(m^*)}$ and $\hat{U} = \bar{U}$ in Theorem 3.1, we obtain the relation

$$\|\bar{U}^{(m+1)} - \bar{U}\| \leq \rho' M' \|\bar{U}^{(m)} - \bar{U}\|^2,$$

(3.13)

where $\rho'\prime$ and $M'$ are given by (3.2) with respect to the sector $\langle \bar{u}_i, \bar{u}_i^{(m^*)} \rangle$. This proves relation (3.13) with $K = \rho' M'$. Similarly, if condition (3.12) holds then there exists $m^* \geq 0$ such that for $m \geq m^*$,

$$\frac{\partial F_i}{\partial u_i}(u_i) \leq 0 \quad \text{and} \quad \frac{\partial^2 F_i}{\partial u_i^2}(u_i) \geq 0,$$

where $\underline{u}_i$ and $\underline{u}_i^{(m)}$ are given by (3.2) with respect to the sector $\langle \underline{u}_i, \underline{u}_i^{(m^*)} \rangle$. This proves relation (3.14) holds. This proves the theorem.
To demonstrate the rate of convergence of the monotone iterative schemes (2.4) and (2.6) numerically, we consider a tabular reactor model in chemical engineering which is given by (cf. [19-21])

\[
\begin{align*}
\frac{u_r + 1}{r} = \sigma (1 - u) \exp \left( -\frac{\gamma}{1 + u} \right), \quad 0 < r < 1,
\end{align*}
\]

\[
\begin{align*}
u_r(0) = 0, \quad u_r(1) + \beta u(1) = 0,
\end{align*}
\]

where \( u_r = \frac{du}{dr} \) and \( \sigma, \gamma, \) and \( \beta \) are positive constants. Using the central difference approximation for \( u_r \) and \( u_{rr} \), an elementary derivation gives a finite-difference system of (4.1) in the form (1.2), where the coefficient matrix \( A \) is given by

\[
A = h^{-2} \begin{bmatrix}
4 & -4 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & -b_i & 2 & -c_i & 0 \\
0 & \cdots & -2 & d_N & -& \\
\end{bmatrix}, \quad i = 1, 2, \ldots, N - 1,
\]

with \( b_i = (1 - 1/2i) \), \( c_i = (1 + 1/2i) \), and \( d_N = 2 + 2\beta(1 + 1/2N) \) (cf. [1-3]). The function \( F(U) \) is given by (1.3) with

\[
F_i (u_i) = \sigma (1 - u_i) \exp \left( -\frac{\gamma}{1 + u_i} \right), \quad i = 0, 1, \ldots, N.
\]

It is clear that the matrix \( A \) in (4.2) possesses the properties in (H), including the strict inequality in (2.2) at \( i = N \) because \( \beta > 0 \). Moreover, \( F_i(0) \geq 0 \geq F_i(1) \) so that condition (2.14) is satisfied with \( a_i = 0 \) and \( b_i = 1 \). This implies that the constant vector \( b \equiv (1, \ldots, 1) \) and \( a \equiv (0, \ldots, 0) \) are ordered upper and lower solutions. With these constant vectors as the initial iterations, we can find the matrix \( C^{(m)} \) from (2.7),(2.8) and compute the maximal and minimal sequences \( \{\bar{U}^{(m)}\}, \{\underline{U}^{(m)}\} \) from either (2.4) or (2.6).

It is known that for fixed values of \( \gamma \) and \( \beta \), problem (4.1) has a unique solution if \( \sigma \) is either small or large, and it has a maximal solution \( \bar{u}(r) \) and a minimal solution \( \underline{u}(r) \) if \( \sigma \) is in a certain finite interval (cf. [16, p. 135]). According to the literature in chemical engineering, the value of \( \sigma \) can vary from 1 to \( 10^7 \) (cf. [20,21]). In the case of \( \gamma = 15 \), \( \beta = 1 \), our numerical experiment demonstrates that problem (4.1) has a unique positive solution if \( \sigma \leq \sigma \) or \( \sigma \geq \sigma \) and it has multiple positive solutions if \( \sigma < \sigma < \sigma \), where \( \sigma \approx 26,700 \) and \( \sigma \approx 149,000 \). Indeed, by increasing the value of \( \sigma \) from 26,700 slightly the unique solution bifurcates to multiple solutions, and it remains so until \( \sigma \) is increased beyond 149,000. An interesting observation in our numerical computations is that in the iteration process (2.4) the rate of convergence of the monotone iterations is moderate (within 100 iterations) if \( \sigma \) is small, and it becomes very slow if \( \sigma \) is either near \( \sigma \) or near \( \sigma \). On the other hand, if the iteration process (2.6) is used then the rate of convergence is much accelerated. Some numerical results of the maximal and minimal solutions \( \bar{u}(r_i), \underline{u}(r_i) \) at the quarter points of the interval \( (0,1) \) together with the number of iterations in the iteration processes (2.4) and (2.6), called regular and accelerated iteration, respectively, are given in Table 1.

In the above iteration processes, the value of \( \sigma \) and the mesh size \( h \) are taken as \( \sigma = 27,000 \) and \( h = 0.05 \). Also, a tolerance of \( \epsilon = 10^{-8} \) is used in the convergence criterion

\[
\max \left\{ \frac{|u_i^{(m+1)} - u_i^{(m)}|}{|u_i^{(m)}|}, \quad i = 0, 1, \ldots, N \right\} < \epsilon,
\]
for the sequence $\{u_i^{(m)}\}$ and $\{u_i^{(m)}\}$. This convergence criterion leads to $u_i^{(m+1)} = u_i^{(m)}$ up to the first six digits which are listed in the table. More numerical results, including some results of the corresponding time-dependent problem, can be found in [3,4].

**REFERENCES**