Note
The toughness of split graphs

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Abstract

In this short note we argue that the toughness of split graphs can be computed in polynomial time. This solves an open problem from a recent paper by Kratsch et al. (Discrete Math. 150 (1996) 231–245). © 1998 Elsevier Science B.V. All rights reserved

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1. Introduction

1.1. Definitions

In this note we only consider finite undirected graphs \( G = (V, E) \) without loops or multiple edges. A graph \( G = (V, E) \) is a split graph (cf. [4, 5]) if its vertex set can be partitioned into an independent set \( I \) and a clique \( C \). Usually, a split graph is denoted by \( G = (C, I, E) \). For \( v \in V \), the set of vertices adjacent to \( v \) is denoted by \( \Gamma(v) \). For \( U \subseteq V \), the set of vertices adjacent to vertices in \( U \) is denoted by \( \Gamma(U) \). The number of connected components of a graph \( G \) is denoted by \( c(G) \). For \( S \subseteq V \), denote by \( G - S \) the vertex-induced subgraph of \( G \) on the vertex set \( V - S \). A set \( S \subseteq V \) is a cutset of \( G \) if \( c(G - S) > 1 \).

Chvátal [2] introduced the toughness \( t(G) \) of a graph as follows. The toughness of a complete graph is infinite, \( t(K_n) = \infty \). If \( G \) is not complete then

\[
t(G) = \min \left\{ \frac{|S|}{c(G - S)} : c(G - S) > 1 \right\}
\]

(1)

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A graph $G$ is said to be $t$-tough if $\tau(G) \geq t$ holds, i.e. $|S| \geq \tau(G - S)$ for every cutset $S$ of $G$.

1.2. Previous work

Chvátal [2] conjectures that there exists a $t_0$ such that every $t_0$-tough graph is Hamiltonian. This conjecture is still unresolved. Bauer et al. [1] prove that computing the toughness of a graph $G$, in general, is an NP-hard problem. Chvátal’s paper [2] exhibits $(\frac{1}{2} - \epsilon)$-tough split graphs that are not Hamiltonian for every $\epsilon > 0$. As a complementary result, Kratsch et al. [7] prove that every $\frac{1}{2}$-tough split graph is Hamiltonian. Moreover, they give a polynomial-time algorithm for deciding whether a given split graph is $1$-tough. The computational complexity of deciding whether a given split graph is $t$-tough for arbitrary rational $t$ remained open.

1.3. Result of this note

We observe that the toughness of a split graph can be computed in polynomial time. This is done by rewriting the problem as minimization of a submodular function and by applying well-known results from mathematical programming.

2. The polynomial-time algorithm

Throughout this section, let $t$ be some fixed positive rational number and let $G = (C, I, E)$ be a split graph. To avoid trivialities, assume that $G$ is connected, that $C \neq \emptyset$ and that $I \neq \emptyset$ holds. Our first goal is to decide whether there exists a cutset $S^*$ of $G$ for which $|S^*| < \tau(G - S^*)$ holds. In case such a cutset $S^*$ does exist, one may assume without loss of generality, that $S^* \subseteq C$: otherwise, replace $S^*$ by $S^* \cap C$. This does not increase $|S^*|$ and cannot decrease $\tau(G - S^*)$.

If $S^* = C$, then $\tau(G - S^*) = |I|$ holds, and this case is trivial to check. Otherwise, $S^* \subset C$ holds. Then $\tau(G - S^*)$ equals the number of vertices $v \in I$ with $\Gamma(v) \subseteq S^*$ plus one, where the ‘plus one’ accounts for the component that contains $C - S^*$. Hence, the problem boils down to deciding whether there exists a proper subset $X$ of $I$ such that

$$|\Gamma(X)| < t(|X| + 1),$$

or in other words, to decide whether there exists a proper subset $X \subset I$ for which

$$f(X) := |\Gamma(X)| - t|X| < t.$$  \hfill (3)

Observe that $f(X)$ is a submodular function $2^I \to \mathbb{R}$, i.e. that $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$ holds for all $A, B \subseteq I$ (cf. e.g. [6]). Therefore, the minimum of $f(X)$ can be determined in polynomial time by the ellipsoid method [6] or by Cunningham’s combinatorial algorithm [3] (although the running time of Cunningham’s algorithm is
only pseudopolynomial in the values of $f(X)$, it is strictly polynomial in our case: the values of $f(X)$ are polynomially bounded in $|V|$. Consequently, one can decide in polynomial time whether there exists an $X$ that fulfills inequality (3).

Finally, observe that the toughness of a graph $G = (V, E)$ is a positive rational number with nominator and denominator bounded by $|V|$. One can enumerate all these numbers $t$ in polynomial time and check whether the split graph $G$ is $t$-tough.

**Theorem 1.** The toughness of a split graph $G = (C, I, E)$ can be computed in polynomial time.

**References**


