Stochastic Processes and their Applications 6 (1978) 223–240.

STRONG APPROXIMATION THEOREMS FOR DENSITY DEPENDENT MARKOV CHAINS

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Received 17 January 1977

AMS 1970 Subj. Class. Primary 60F15, 60H20, Secondary 60J10, 60J60, 80A30, 92A15

diffusion approximation — error estimates
embedding theorems — central limit theorem

A variety of continuous parameter Markov chains arising in applied probability (e.g. epidemic and chemical reaction models) can be obtained as solutions of equations of the form

$$X_N(t) = x_0 + \sum \frac{1}{N} lY_l\left(N\int_0^t f_l(X_N(s))\mathrm{d}s\right)$$

where $l \in \mathbf{Z}^d$, the Y_l are independent Poisson processes, and N is a parameter with a natural interpretation (e.g. total population size or volume of a reacting solution).

The corresponding deterministic model, satisfies

$$X(t) = x_0 + \int_0^t \sum lf_l(X(s))\mathrm{d}s$$

Under very general conditions $\lim_{N\to\infty} X_N(t) = X(t)$ a.s. The process $X_N(t)$ is compared to the diffusion processes given by

$$Z_N(t) = x_0 + \sum \frac{1}{N} lB_l\left(N\int_0^t f_l(Z_N(s))\mathrm{d}s\right)$$

and

$$V(t) = \sum l\int_0^t \sqrt{f_l(X(s))}\mathrm{d}\tilde{W}_l + \int_0^t \partial F(X(s))\cdot V(s)\mathrm{d}s.$$

Under conditions satisfied by most of the applied probability models, it is shown that X_N, Z_N and V can be constructed on the same sample space in such a way that

$$X_N(t) = Z_N(t) + \mathrm{O}\left(\frac{\log N}{N}\right)$$

and

$$\sqrt{N}(X_N(t) - X(t)) = V(t) + \mathrm{O}\left(\frac{\log N}{\sqrt{N}}\right)$$

1. Introduction

There is a large body of literature concerning diffusion and other approximations for Markov chains and similar processes (see [4, 11, 16, 17] for some examples and references). Most of these results involve convergence in distribution or weak convergence. In [11] we used a result of Kómlos, Major and Tusnady [8] to obtain diffusion processes and Markov chains built on the same sample space so that it is possible to give almost sure pathwise estimates for the error in the approximation of the Markov chain by the diffusion process.

In the present paper we extend this result to a somewhat larger class of chains and prove a similar strong convergence result for a central limit theorem.

The class of Markov chains we will consider is as follows:

Let $\mathbf{Z}^d$ be the d-dimensional integer lattice. For each $N>0$ $X_N(t)$ will be a Markov chain with state space $\{(1/N)k : k \in \mathbf{Z}^d\}$. We assume there exist non negative functions $f_l(x), f_l^N(x)$, $N = 1,2,\ldots$, $l \in \mathbf{Z}^d$, $x \in \mathbf{R}^d$ and constants ε_l, Γ such that

$$|f_l^N(x)| \leq \varepsilon_l(1+|x|) \tag{1.1}$$

and

$$|f_l^N(x)-f_l(x)| \leq \frac{\Gamma\varepsilon_l}{N}(1+|x|), \tag{1.2}$$

and that the infinitesimal parameter for a transition from k/N to $(k+l)/N$ is $Nf_l^N(k/N)$. Throughout we will assume $\sum \varepsilon_l|l|<\infty$. This implies that $X_N(t)$ is uniquely determined by the infinitesimal parameters. In fact $X_N(t)$ can be obtained as a solution of the stochastic equation

$$X_N(t) = X_N(0)+\sum \frac{1}{N}\, lY_l\left(N\int_0^t f_l^N(X_N(s))\mathrm{d}s\right) \tag{1.3}$$

where the Y_l are independent Poisson processes with $\mathbf{E}(Y_l(u)) = u$ and $X_N(0)$ is independent of the Y_l. We observe that

$$\begin{aligned}\mathbf{E}(X_N(t)) &= \mathbf{E}(X_N(0)) + \int_0^t \mathbf{E}\left(\sum lf_l^N(X_N(s))\right)\mathrm{d}s.\\ &\equiv \mathbf{E}(X_N(0)) + \int_0^t \mathbf{E}(F^N(X_N(s)))\mathrm{d}s.\end{aligned} \tag{1.4}$$

Most of the Markov chain models arising in applied probability (e.g. epidemic models and chemical reaction models) can be formulated in this way (see [9, 10, 11]). Equation (1.4) gives at least some motivation for considering

$$\begin{aligned}X(t) &= X(0)+\int_0^t \sum lf_l(X(s))\mathrm{d}s\\ &\equiv X(0)+\int_0^t F(X(s))\mathrm{d}s\end{aligned} \tag{1.5}$$

as the corresponding deterministic model.

The diffusion approximation corresponding to (1.3) is given by

$$Z_N(t) = Z_N(0) + \sum \frac{1}{N}\, lB_l\left(N\int_0^t f_l(Z_N(s))\mathrm{d}s\right) \tag{1.6}$$

where the B_l are independent Brownian motions with $E(B_l(t)) = \mathrm{Var}(B_l(t)) = t$. It is not obvious that (1.6) has a solution. However (1.6) is equivalent to the Ito equation

$$Z_N(t) = Z_N(0) + \sum \frac{1}{\sqrt{N}} l \int_0^t \sqrt{f_l(Z_N(s))}\,d\tilde{W}_l(s) + \int_0^t F(Z_N(s))\,ds. \quad (1.7)$$

This fact is used in [11] to prove

Lemma 1.1. *Suppose* $\sum |l|^2 \varepsilon_l < \infty$, $\sum_l |l|^2 |\sqrt{f_l(x)} - \sqrt{f_l(y)}|^2 \leq M|x-y|^2$, *and* $|F(x) - F(y)| \leq M|x-y|$ *for some M and all* $x, y \in \mathbf{R}^d$. *Then there exists a diffusion satisfying* (1.7) *which satisfies an identity of the form* (1.6).

Remark. This is a "weak" existence theorem for (1.6) in the sense that the $B_l(t)$ are defined in terms of the solution of (1.7). In particular

$$B_l(t) = \int_0^{\tau_l(t)} \sqrt{N f_l(Z_N(s))}\,d\tilde{W}_l(s) + t \quad (1.8)$$

where $\tau_l(t)$ is the solution of

$$t = N \int_0^{\tau_l(t)} f_l(Z_N(s))\,ds.$$

It follows that

$$B_l\left(N \int_0^t f_l(Z_N(s))\,ds\right) = \int_0^t \sqrt{N f_l(Z_N(s))}\,d\tilde{W}_l(s) + N \int_0^t f_l(Z_N(s))\,ds.$$

One of the goals of the paper is to give "analytic" proofs of the convergence and approximation theorems by using well known sample path properties of the Y_l and B_l in Equations (1.3) and (1.6).

In Section 2, we show that $X_N(t)$ converges almost surely to $X(t)$ (under the assumption that F is Lipschitz) by using the Law of Large Numbers for Y_l. In addition, under the assumption that $\sum |l|^2 \varepsilon_l < \infty$ we show that

$$X_N(t) = X(t) + O\left(\frac{1}{\sqrt{N}}\right).$$

This latter fact is, of course, a consequence of the known central limit theorem for $\sqrt{N}(X_N(t) - X(t))$ (Norman [15] and Kurtz [10]), but here we obtain the error estimate directly from equation (1.3) using properties of the Y_l.

In Section 3, we use the embedding theorem of Kómlos, Major and Tusnady [8] to construct solutions of (1.3) and (1.6) on the same sample space in such a way that

$$X_N(t) = Z_N(t) + O\left(\frac{\log N}{N}\right).$$

Once again the estimates are obtained analytically for almost every sample path using the error estimates in the embedding theorem and Levy's modulus of continuity.

Similarly, in Section 4 we show that

$$\sqrt{N}(X_N(t)-X(t)) = V(t) + O\left(\frac{\log N}{\sqrt{N}}\right)$$

where

$$V(t) = \sum l \int_0^t \sqrt{f_l(X(s))}\,d\tilde{W}_l + \int_0^t \partial F(X(s))\cdot V(s)\,ds. \tag{1.9}$$

(∂F denotes the gradient of F.)

Note that $V(t)$ is a Gaussian process.

Finally, we use this result to give Berry Esseen type estimates on the difference of the distributions of $\sqrt{N}(X_N(t)-X(t))$ and $V(t)$.

2. Deterministic limit

Theorem 2.1. *Let*

$$\alpha \equiv \sum \varepsilon_l |l| < \infty. \tag{2.1}$$

Then there exists a positive random variable K_N such that

$$|X_N(t)| \le K_N e^{\alpha t} - 1 \tag{2.2}$$

and

$$P\{K_N > x\} \le \frac{1}{x}(1+|X_N(0)|). \tag{2.3}$$

If $\sum \varepsilon_l |l|^\beta < \infty$ for some $\beta > 1$, then

$$\mathrm{E}(K_N^\beta) \le C_\beta (1+|X_N(0)|)^\beta \tag{2.4}$$

where C_β is independent of N.

If $\sum \varepsilon_l e^{\lambda|l|} < \infty$ for all $\lambda < \lambda_0$, then there is an N_λ and a D_λ independent of N such that

$$\mathrm{E}(e^{\lambda K_N}) \le [D_\lambda]^{(1+|X_N(0)|)} \quad \text{for all } \lambda < \lambda_0 \text{ and } N > N_\lambda. \tag{2.5}$$

Remark. Inequality (2.2) is essentially a stochastic version of Gronwall's Inequality.

Proof. Let $Q(t) = \sum |l|\, Y_l(\varepsilon_l t)$. Then

$$\begin{aligned} |X_N(t)| &\le |X_N(0)| + \sum \frac{1}{N}|l|\, Y_l\left(N\int_0^t f_l^N(X_N(s))\,ds\right) \\ &\le |X_N(0)| + \sum \frac{1}{N}|l|\, Y_l\left(N\varepsilon_l \int_0^t (1+|X_N(s)|)\,ds\right). \end{aligned} \tag{2.6}$$

From (2.6) it follows that

$$1+|X_N(t)| \leq M_N(t)$$

where

$$\begin{aligned} M_N(t) &= 1+|X_N(0)| + \sum \frac{1}{N}|l|\, Y_l\left(N\varepsilon_l \int_0^t M_N(s)\,\mathrm{d}s\right) \\ &= 1+|X_N(0)| + \frac{1}{N} Q\left(N\int_0^t M_N(s)\,\mathrm{d}s\right). \end{aligned} \tag{2.7}$$

Since Q is an increasing process with independent increments it follows that $M_N(t)\mathrm{e}^{-\alpha t}$ is a positive martingale with

$$\mathbf{E}(M_N(t)\mathrm{e}^{-\alpha t}) = 1+|X_N(0)|.$$

Therefore

$$K_N \equiv \sup M_N(t)\mathrm{e}^{-\alpha t} < \infty \quad \text{a.s.}$$

and (2.2) follows. The inequalities (2.3), (2.4) and (2.5) follow by martingale inequalities [3, page 217] and estimates on moments of $M_N(t)$ (see Appendix).

We observe that

$$\lim_{a\to\infty} \sup_{z\leq z_0} \left| \frac{1}{a} Q(az) - \alpha z \right| = 0 \quad \text{a.s.}$$

for each $z_0 > 0$, and hence, if $\lim_{N\to\infty} X_N(0) = X(0)$, then

$$\lim_{N\to\infty} \sup_{t\leq T} |M_N(t) - (1+|X(0)|)\mathrm{e}^{\alpha t}| = 0 \quad \text{a.s.}$$

for each $T > 0$.

Remark. $M_N(t)$ is a continuous state branching process (see Lamperti [13]).

For large values of N The Law of Large Numbers implies

$$\frac{Y_l(Nu)}{N} \approx u.$$

This suggests that

$$X_N(t) \approx X_N(0) + \int_0^t F_l^N(X_N(s))\,\mathrm{d}s$$

and in turn that $X_N(t) \approx X(t)$. This is made precise in the following theorem.

Theorem 2.2. *If*

$$|F(x)-F(y)| \leq M|x-y|, \quad \sum |l|\,\varepsilon_l < \infty \tag{2.8}$$

and $\lim_{N\to\infty} X_N(0) = X(0)$, *then*

$$\lim_{N\to\infty} \sup_{t\le T} |X_N(t) - X(t)| = 0 \quad a.s. \tag{2.9}$$

for every $T > 0$.

If $\sum |l|^\beta \varepsilon_l < \infty$, $\beta \ge 2$, *then there is a random variable* $L_N(T)$ *such that*

$$\sup_{t\le T} |X_N(t) - X(t)| \le \left(|X_N(0) - X(0)| + \frac{L_N(T)}{\sqrt{N}} + \frac{\Gamma K_N}{N}\right) \mathrm{e}^{MT} \tag{2.10}$$

and

$$\sup_N \mathbf{E}(L_N(T)^\beta) < \infty. \tag{2.11}$$

If $\sum \mathrm{e}^{\lambda |l|} \varepsilon_l < \infty$, *then*

$$\sup_N \mathbf{E}(\mathrm{e}^{\lambda L_N(T)}) < \infty. \tag{2.12}$$

Proof. We have

$$\begin{aligned} |X_N(t) - X(t)| \le{} & |X_N(0) - X(0)| \\ & + \left|\sum l \left(\frac{1}{N} Y_l\left(N\int_0^t f_l^N(X_N(s))\mathrm{d}s\right) - \int_0^t f_l^N(X_N(s))\mathrm{d}s\right)\right| \\ & + \left|\int_0^t (F^N(X_N(s)) - F(X_N(s)))\mathrm{d}s\right| \\ & + \left|\int_0^t (F(X_N(s)) - F(X(s)))\mathrm{d}s\right| \\ \le{} & |X_N(0) - X(0)| \\ & + \left|\sum l \left(\frac{1}{N} Y_l\left(N\int_0^t f_l^N(X_N(s))\mathrm{d}s\right) - \int_0^t f_l^N(X_N(s))\mathrm{d}s\right)\right| \\ & + \frac{\Gamma}{N} \alpha \int_0^t M_N(s)\mathrm{d}s \\ & + \int_0^t M |X_N(s) - X(s)| \mathrm{d}s \end{aligned} \tag{2.13}$$

where $M_N(s)$ is given by (2.7). Since $M_N(s) \le K_N \mathrm{e}^{\alpha s}$, Gronwall's Inequality gives (2.10) with

$$L_N(T) = \sup_{t\le T} \left| \sqrt{N} \sum l \left(\frac{1}{N} Y_l\left(N\int_0^t f_l^N(X_N(s))\mathrm{d}s\right) - \int_0^t f_l^N(X_N(s))\mathrm{d}s\right)\right|. \tag{2.14}$$

Define $\tau_T^N = \int_0^T M_N(s)\mathrm{d}s$. By the observation at the end of the proof of Theorem (2.1)

$$\lim_{N\to\infty} \tau_T^N = (1 + |X(0)|) \int_0^T \mathrm{e}^{\alpha s}\,\mathrm{d}s \quad \text{a.s.} \tag{2.15}$$

From (1.1) and $1+|X_N(t)|\leq M_N(t)$

$$\begin{aligned}\frac{1}{\sqrt{N}}L_N(T)&\leq\sum|l|\sup_{u\leq\tau_T^N}\left|\frac{1}{N}Y_l(\varepsilon_l Nu)-\varepsilon_l u\right|\\&\leq\sum|l|\left(\frac{1}{N}Y_l(\varepsilon_l N\tau_T^N)+\varepsilon_l\tau_T^N\right)\\&=\frac{1}{N}Q(N\tau_T^N).\end{aligned}\tag{2.16}$$

Note that the second inequality holds term by term and that each term in the first series converges to zero a.s. The Law of Large Numbers shows that the limit as $N\to\infty$ and summation are interchangeable for the second series, and hence for the first series. This implies

$$\lim_{N\to\infty}\frac{1}{\sqrt{N}}L_N(T)=0\quad\text{a.s.}$$

and (2.9) follows.

To estimate the moments of $L_N(T)$ we observe that

$$Z_N(t)=\sqrt{N}\sum l\left(\frac{1}{N}Y_l\left(N\int_0^t f_l^N(X_N(s))\mathrm{d}s\right)-\int_0^t f_l^N(X_N(s))\mathrm{d}s\right)\tag{2.17}$$

is a martinguale ($\mathbf{R}^d$ valued). Furthermore the optional sampling theorem for martingales indexed by partially ordered sets [12] implies

$$\mathbf{E}(\varphi(Z_N(T)))\leq\mathbf{E}\left(\varphi\left(\sqrt{N}\sum l\left(\frac{1}{N}Y_l(N\varepsilon_l\tau_T^N)-\varepsilon_l\tau_T^N\right)\right)\right)\tag{2.18}$$

for any convex function φ.

Each component of

$$U(t)=\sqrt{N}\sum l\left(\frac{1}{N}Y(N\varepsilon_l t)-\varepsilon_l t\right)$$

is a process with independent increments and τ_T^N is a stopping time.

If $\sum|l|^\beta\varepsilon_l<\infty$, $\beta\geq 2$ then Theorem 7 of [1] implies

$$\mathbf{E}(|U(\tau_T^N)|^\beta)\leq C\max\{\mathbf{E}(\tau_T^N),\mathbf{E}((\tau_T^N)^{\beta/2})\}\tag{2.19}$$

and an inspection of the proof of that result shows that C may be selected independently of N. The uniform boundedness of the right hand side of (2.19) follows from the estimates in the Appendix.

3. Diffusion approximation

We will now consider estimates on the difference $|X_N(t)-Z_N(t)|$ between solutions of (1.3) and (1.6) with $Z_N(0)=X_N(0)$. To simplify the calculations we will assume

$$f_l^N(x) \leq \varepsilon_l, \tag{3.1}$$

$$|f_l(x) - f_l^N(x)| \leq \frac{\Gamma\varepsilon_l}{N}. \tag{3.2}$$

Given the estimates on the growth of $|X_N(t)|$ in Section 2 and the fact that similar estimates can be given for the growth of $|Z_N(t)|$, (3.1) and (3.2) are not really much more restrictive than the previous assumptions.

The basis for our diffusion approximation is the following lemma which is an immediate consequence of a Theorem of Kómlos, Major and Tusnady [8].

Lemma 3.1. *Let $B(t)$ be Brownian motion with*

$$\mathrm{E}(B(t)) = \mathrm{Var}(B(t)) = t. \tag{3.3}$$

Then a Poisson process $Y(t)$ can be constructed on the same sample space as $B(t)$ so that

$$\sup_{t\geq 0} \frac{|Y(t) - B(t)|}{\log t \vee 2} \equiv K < \infty \quad a.s. \tag{3.4}$$

and $\mathrm{E}(\exp\{\lambda K\}) < \infty$ *for some* $\lambda > 0$.

Remark. By Lemma 3.1 we have

$$\sup_{s\leq t} |Y(s) - B(s)| \leq \begin{cases} \sup\limits_{s\leq t} |Y(s) - s| + \sup\limits_{s\leq t} |B(s) - s| & \text{for } t < 2, \\ K \log t \quad \text{for } t \geq 2. \end{cases} \tag{3.5}$$

We write $B_l(t) = W_l(t) + t$ where $W_l(t)$ is Brownian motion with mean zero and variance t. After a small amount of manipulation we have

$$\begin{aligned} X_N(t) - Z_N(t) = {} & \sum \frac{1}{N} l \left[W_l\left(N\int_0^t f_l(X_N(s))\,\mathrm{d}s\right) - W_l\left(N\int_0^t f_l(Z_N(s))\,\mathrm{d}s\right)\right] \\ & + \int_0^t [F(X_N(s)) - F(Z_N(s))]\,\mathrm{d}s \\ & + \sum \frac{1}{N} l \left[Y_l\left(N\int_0^t f_l^N(X_N(s))\,\mathrm{d}s\right) - B_l\left(N\int_0^t f_l^N(X_N(s))\,\mathrm{d}s\right)\right] \\ & + \sum \frac{1}{N} l \left[B_l\left(N\int_0^t f_l^N(X_N(s))\,\mathrm{d}s\right) - B_l\left(N\int_0^t f_l(X_N(s))\,\mathrm{d}s\right)\right]. \end{aligned} \tag{3.6}$$

The second term on the right can be handled using Gronwall's inequality, the third term can be estimated using Lemma 3.1, and the fourth term is estimated using the modulus of continuity for Brownian motion.

In order to handle the first term on the right we prove the following lemma.

Lemma 3.2. *Suppose*

$$|f_l(x)-f_l(y)| \leq \varepsilon_l M|x-y|, \qquad \sum |l|\sqrt{\varepsilon_l} < \infty \tag{3.7}$$

and

$$|X_N(t)-Z_N(t)| \leq e^{Mt}\delta_N + e^{Mt}\sum \frac{|l|}{N}\sup_{u\leq t}\left| W_l\left(N\int_0^u f_l(X_N(s))ds\right) - W_l\left(N\int_0^u f_l(Z_N(s))ds\right)\right|. \tag{3.8}$$

Define

$$M_l \equiv \sup_{u,v\leq N\varepsilon_l T} \frac{|W_l(u)-W_l(v)|}{\sqrt{|u-v|(1+\log(N\varepsilon_l T/|u-v|))}},$$

Then

$$\sup_{t\leq T}|X_N(t)-Z_N(t)| \leq \leq \max\left\{\frac{1}{N},\ 2e^{MT}\delta_N + e^{2MT}\frac{MT(1+\log N/M)}{N}\left(\sum |l|\sqrt{\varepsilon_l}M_l\right)^2\right\}. \tag{3.10}$$

Remark. M_l is finite by a result of Levy (McKean [14, page 14]) and a result of Fernique [5] implies $\mathbf{E}(\exp\lambda M_l^2)<\infty$ for some $\lambda>0$. This in turn implies $\mathbf{E}(\exp\lambda(\sum|l|\sqrt{\varepsilon_l}M_l)^2)<\infty$ for some $\lambda>0$, since the M_l are independent. The M_l are also identically distibuted and the distribution of $\sum|l|\sqrt{\varepsilon_l}M_l$ does not depend on N.

Proof. Define $\gamma(t)=N|X_N(t)-Z_N(t)|$ and $\gamma=\sup_{t\leq T}\gamma(t)$. We have

$$N\left|\int_0^t f_l(X_N(s))ds - \int_0^t f_l(Z_N(s))ds\right| \leq \varepsilon_l M\int_0^t \gamma(s)ds. \tag{3.11}$$

Noting that

$$\sqrt{x(1+\log(N\varepsilon_l T/x))} \tag{3.12}$$

is increasing in x we have

$$\gamma(t) \leq e^{MT}N\delta_N + e^{MT}\sum |l|\sqrt{\varepsilon_l}M_l\sqrt{M\int_0^t \gamma(s)ds\left(1+\log\left(NT/M\int_0^t\gamma(s)ds\right)\right)} \tag{3.13}$$

and hence

$$\gamma \leq e^{MT}N\delta_N + e^{MT}\sum |l|\sqrt{\varepsilon_l}M_l\sqrt{MT(1+\log N/M\gamma)}\sqrt{\gamma}. \tag{3.14}$$

By (3.14) either $\gamma\leq 1$ or

$$\gamma \leq e^{MT}N\delta_N + e^{MT}\sum |l|\sqrt{\varepsilon_l}M_l\sqrt{MT(1+\log N/M)}\sqrt{\gamma}. \tag{3.15}$$

Inequality (3.15) implies

$$\gamma \leq 2e^{MT}N\delta_N + e^{2MT}MT(1+\log N/M)\left(\sum |l|\sqrt{\varepsilon_l}M_l\right)^2.$$

and (3.10) follows.

Returning to (3.6), if we assume $|F(x)-F(y)| \leq M|x-y|$ then Gronwall's Inequality implies (3.8) with δ_N equal to the sum of

$$\sum \frac{1}{N}|l| \sup_{u \leq N\varepsilon_l T} |Y_l(u) - B_l(u)|, \tag{3.16}$$

and

$$\sum \frac{1}{N}|l|[M_l\sqrt{\varepsilon_l \Gamma T(1+\log(N/\Gamma)} + \varepsilon_l \Gamma T]. \tag{3.17}$$

If $\sum |l|\sqrt{\varepsilon_l} < \infty$ and $N > 1$, then (3.17) is bounded by

$$\frac{\log N}{N}\left(c_1 \sum |l|\sqrt{\varepsilon_l}M_l + c_2\right) \tag{3.18}$$

where c_1 and c_2 are constants independent of N.

From Lemma 3.2 and (3.18) it is clear that the rate of convergence is determined by (3.16).

Using various assumptions on ε_l we make various estimates on (3.16) to obtain the following theorem.

Theorem 3.3. *Assume that* (3.1) *and* (3.2) *hold, and suppose*

$$\sum \sqrt{\varepsilon_l}|l| < \infty, \qquad |f_l(x) - f_l(y)| \leq \varepsilon_l M|x-y| \tag{3.19}$$

and

$$|F(x) - F(y)| \leq M|x-y| \quad \text{for some } M > 0.$$

Assume that Y_l and B_l satisfy the conclusion of Lemma 3.1.

(a) *If $\varepsilon_l = 0$ for all but finitely many l, then for $N \geq 2$ there is a random variable β_N^T with distribution independent of N and* $\mathbf{E}(\exp\{\lambda\beta_N^T\}) < \infty$ *for some $\lambda > 0$ such that*

$$\sup_{t \leq T} |X_N(t) - Z_N(t)| \leq \beta_N^T \frac{\log N}{N}. \tag{3.20}$$

(b) *If $\sum e^{\mu|l|}\varepsilon_l < \infty$ for some $\mu > 0$, then for $N \geq 2$ there is a random variable β_N^T with* $\sup_N \mathbf{E}(\exp\{\lambda\beta_N^T\}) < \infty$ *for some $\lambda > 0$ such that*

$$\sup_{t \leq T} |X_N(t) - Z_N(t)| \leq \beta_N^T \frac{(\log N)^{d+2}}{N} \tag{3.21}$$

(c) *If $\varepsilon_l \leq |l|^{-\alpha}$ for $\alpha > 2d+2$, then for $N \geq 2$ there is a random variable β_N^T with* $\sup_N \mathbf{E}((\beta_N^T)^\eta) < \infty$ *for $\eta < \alpha - d$ such that*

$$\sup_{t \leq T} |X_N(t) - Z_N(t)| \leq \beta_N^T(\log N)N^{(d+1)/\alpha - 1}. \tag{3.22}$$

Proof. Part (a) follows from Lemma 3.1 and Lemma 3.2. β_N^T may be written as a function of M_l and

$$K_l \equiv \sup_{t\geq 0} \frac{|Y_l(t)-B_l(t)|}{\log t \vee 2} \tag{3.23}$$

To obtain parts (b) and (c) we write (3.16) as

$$\begin{aligned}
&\sum_{|l|>C_N} \frac{1}{N}|l| \sup_{u\leq N\varepsilon_l T} |Y_l(u)-B_l(u)| \\
&\quad + \sum_{|l|\leq C_N} \frac{1}{N}|l| \sup_{u\leq N\varepsilon_l T} |Y_l(u)-B_l(u)| \\
&\leq \sum_{|l|>C_N} \frac{1}{N}|l| \Big[\sup_{u\leq N\varepsilon_l T} |Y_l(u)-u| + \sup_{u\leq N\varepsilon_l T} |B_l(u)-u|\Big] \\
&\quad + \sum_{|l|\leq C_N} \frac{1}{N}|l| K_l \log(N\varepsilon_l T \vee 2).
\end{aligned} \tag{3.24}$$

The second term on the right can be bounded by a constant times

$$\begin{aligned}
&\frac{\log N}{N}\Big(\sum_{|l|\leq C_N}|l|\Big) \frac{\sum_{|l|\leq C_N}|l|K_l}{\sum_{|l|\leq C_N}|l|} = \\
&= \frac{\log N}{N}\Big(\sum_{|l|\leq C_N}|l|\Big)\beta_1(N,T).
\end{aligned} \tag{3.25}$$

The random variable $\beta_1(N,T)$ is a convex combination of independent identically distributed random variables satisfying $\mathrm{E}(\exp\{\lambda K_l\})<\infty$ for some $\lambda>0$. Consequently

$$\mathrm{E}(\exp\{\lambda\beta_1(N,T)\}) \leq \mathrm{E}(\exp\{\lambda K_l\}).$$

We split the first term on the right of (3.24) and observe that

$$\begin{aligned}
&\sum_{|l|>C_N} \frac{1}{N}|l| \sup_{u\leq N\varepsilon_l T} |B_l(u)-u| \leq \\
&\leq \sum_{|l|>C_N} \frac{1}{N}|l| \Big[\sqrt{N\varepsilon_l T} \sup_{u\leq N\varepsilon_l T}\Big|\frac{B_l(u)-u}{\sqrt{N\varepsilon_l T}}\Big|\Big].
\end{aligned} \tag{3.26}$$

Since $\sup_{u\leq N\varepsilon_l T}|(B_l(u)-u)/\sqrt{N\varepsilon_l T}|$ is equal in distribution to $\bar{W} = \sup_{u\leq 1}|W(u)|$ where $W(u)$ is a standard Brownian motion, (3.26) can be bounded by

$$\sqrt{\frac{T}{N}}\Big(\sum_{|l|>C_N}|l|\sqrt{\varepsilon_l}\Big)\beta_2(N,T) \tag{3.27}$$

where $\mathrm{E}(\exp\{\lambda\beta_2(N,T)\}) \leq \mathrm{E}(\exp\{\lambda\bar{W}\})$.

Finally we have

$$\sum_{|l|>C_N} \frac{1}{N}|l| \sup_{u\le N\varepsilon_l T} |Y_l(u)-u| \le \le \sum_{|l|>C_N} \frac{1}{N}|l|\, Y_l(N\varepsilon_l T) + \sum_{|l|>C_N} |l|\,\varepsilon_l T. \tag{3.28}$$

We consider these terms separately as

$$\frac{\alpha(N)}{N}\left[\frac{1}{\alpha(N)}\sum_{|l|>C_N}|l|\, Y_l(N\varepsilon_l T)\right] =: \frac{\alpha(N)}{N}\beta_3(N,T) \tag{3.29}$$

and

$$\sum_{|l|>C_N}|l|\,\varepsilon_l T. \tag{3.30}$$

To complete the proofs we must select C_N and $\alpha(N)$, estimate the sums in (3.25), (3.27) and (3.30) and the moments of $\beta_3(N,T)$.

To obtain part (b) note that $\varepsilon_l \le a\mathrm{e}^{-\mu|l|}$ for some a. Set $C_N = k\log N$ where k is a sufficiently large constant and $\alpha(N)=(\log N)^{d+2}$.

The sum in (3.25) can be bounded by a constant times

$$\int_{|x|\le C_N} |x|\,dx = a_d C_N^{d+1} = \mathrm{O}((\log N)^{d+1}). \tag{3.31}$$

Similarly in (3.27) the sum is of the order of

$$\int_{C_N}^{\infty} r\,\mathrm{e}^{-\frac{1}{2}\mu r} r^{d-1}dr = \mathrm{O}(C_N^d \mathrm{e}^{-\frac{1}{2}\mu C_N}) = \mathrm{O}((\log N)^d N^{-\frac{1}{2}\mu k}), \tag{3.32}$$

and in (3.30) the sum is

$$\mathrm{O}((\log N)^d N^{-\mu k}). \tag{3.33}$$

Finally

$$\mathbf{E}(\exp\{\lambda\beta_3(N,T)\}) = \prod_{|l|>C_N}\mathbf{E}\left(\exp\left\{\frac{\lambda}{\alpha(N)}|l|\,Y_l(N\varepsilon_l T)\right\}\right) = \exp\left\{\sum_{|l|>C_N} N\varepsilon_l T(\mathrm{e}^{\lambda|l|/\alpha(N)}-1)\right\}. \tag{3.34}$$

The sum in the exponent is

$$\mathrm{O}\left(N\int_{C_N}^{\infty}\mathrm{e}^{-\mu r}(\mathrm{e}^{\lambda r/\alpha(N)}-1)r^{d-1}\mathrm{d}r\right). \tag{3.35}$$

If $(\mu-\lambda/\alpha(N))k \ge 1$, this is

$$\mathrm{O}\left(\frac{N}{\alpha(N)}\int_{C_N}^{\infty}\mathrm{e}^{-(\mu-\lambda/\alpha(N))r}r^d\,\mathrm{d}r\right) = \mathrm{O}\left(\frac{N}{\alpha(N)}C_N^d\mathrm{e}^{-(\mu-\lambda/\alpha(N))C_N}\right) = \mathrm{O}(1). \tag{3.36}$$

Part (b) now follows.

To obtain part (c) we have similar estimates on the sums. In (3.25)

$$\sum_{|l|\le C_N} |l| = O(C_N^{d+1}); \tag{3.37}$$

in (3.27)

$$\sum_{|l|>C_N} |l|\sqrt{\varepsilon_l} \le \sum_{|l|>C_N} |l|^{-(\alpha/2-1)} = O(C_N^{d+1-\alpha/2}); \tag{3.38}$$

in (3.30)

$$\sum_{|l|>C_N} |l|\,\varepsilon_l = O(C_N^{d+1-\alpha}). \tag{3.39}$$

To obtain the moment estimate in (3.29) define

$$U(t) = \sum_{|l|>C_N} |l|\, Y_l(N\varepsilon_l t).$$

Then for $1<\eta<\alpha-d$

$$\begin{aligned}\mathbf{E}(U(T)^\eta) &= \mathbf{E}\int_0^T \sum_{|l|>C_N} N\varepsilon_l[(U(s)+|l|)^\eta - U(s)^\eta]\,ds\\ &\le C\mathbf{E}\int_0^T \sum_{|l|>C_N} N\varepsilon_l(|l|U(s)^{\eta-1}+|l|^\eta)\,ds \qquad (3.40)\\ &\le CTN\left[\left(\sum_{|l|>C_N}\varepsilon_l|l|\right)\mathbf{E}(U(T)^{\eta-1}) + \sum_{|l|>C_N}\varepsilon_l|l|^\eta\right]\end{aligned}$$

where C is a constant independent of N. (The fact that $\mathbf{E}(U(T)^\eta)<\infty$ is a consequence of Theorem 5 of [1].)

To complete the proof, set $C_N = N^{1/\alpha}$ and $\alpha(N) = N^{(d+1)/\alpha}$. The estimates involving the sums are immediate. Inequality (3.40) implies

$$\mathbf{E}(U(T)^\eta) \le C'[N^{(d+1)/\alpha}\mathbf{E}(U(T)^{\eta-1}) + N^{(d+\eta)/\alpha}], \tag{3.41}$$

and hence

$$\mathbf{E}(\beta_3(N,T)^\eta) \le C'[\mathbf{E}(\beta_3(N,T)^{\eta-1})+1] \tag{3.42}$$

which in turn implies $\mathbf{E}(\beta_3(N,T)^\eta)$ is uniformly bounded in N.

The various estimates can now be combined to give part (c).

4. Central Limit Theorem

Throughout this section we will assume (3.1), (3.2)

$$\sum |l|^2|\sqrt{f_l(x)} - \sqrt{f_l(y)}|^2 \le M|x-y|^2,$$

$$\left|\frac{\partial}{\partial x_i} F(x)\right| \le M$$

and

$$\left|\frac{\partial^2}{\partial x_i \partial x_j} F(x)\right| \leq M.$$

We begin by comparing the solution of (1.7) to $\sqrt{N}(Z_N(t)-X(t))$, assuming $Z_N(0)=X(0)$.

Using (1.5) this gives

$$\begin{aligned}
&\sqrt{N}(Z_N(t)-X(t))-V(t)= \\
&=\sum l\int_0^t \left[\sqrt{f_l(Z_N(s))}-\sqrt{f_l(X(s))}\right]\mathrm{d}\tilde{W} \\
&\quad+\int_0^t [\sqrt{N}(F(Z_N(s))-F(X(s)))-\partial F(X(s))\cdot V(s)]\,\mathrm{d}s \\
&=\sum l\int_0^t [\sqrt{f_l(Z_N(s))}-\sqrt{f_l(X(s))}]\,\mathrm{d}\tilde{W}_l \\
&\quad+\int_0^t \partial F(X(s))\cdot[\sqrt{N}(Z_N(s)-X(s))-V(s)]\,\mathrm{d}s \\
&\quad+\int_0^t \sqrt{N}(Z_N(s)-X(s))\cdot\partial^2 F(\theta(s))(Z_N(s)-X(s))\,\mathrm{d}s.
\end{aligned} \tag{4.1}$$

We need the following consequence of Ito's Formula (see Friedman [6, page 87]).

Lemma 4.1. *Let $g_l(s)$ be non-anticipating functions. Then for each $n \geq 1$ there exists a constant K_n such that*

$$\mathbf{E}\left(\sup_{t\leq T}\left|\sum l\int_0^t g_l(s)\,\mathrm{d}\tilde{W}_l\right|^{2n}\right)\leq K_n T^{n-1}\int_0^T \mathbf{E}\left(\left(\sum |l|^2 g_l^2(s)\right)^n\right)\mathrm{d}s. \tag{4.2}$$

If

$$\mathbf{E}\left(\exp\left\{\mu\int_0^T \sum |l|^2 g_l^2(s)\,\mathrm{d}s\right\}\right)<\infty \quad \textit{for some } \mu>0, \tag{4.3}$$

then for each $1\leq i\leq d$,

$$\exp\left\{\lambda\int_0^t \sum l_i g_l(s)\,\mathrm{d}\tilde{W}_l-\lambda^2\int_0^t \sum l_i^2 g_l^2(s)\,\mathrm{d}s\right\} \tag{4.4}$$

is a mean one martingale.

As a corollary we have:

Lemma 4.2. *Suppose $\sum |l|^2 \varepsilon_l<\infty$. Then*

$$\mathbf{E}\left(\sup_{t\leq T}|\sqrt{N}(Z_N(t)-X(t))|^{2n}\right)\leq K_n T^n \mathrm{e}^{2nMT}\sum |l|^2\varepsilon_l \tag{4.5}$$

and

$$\sup_N \mathbf{E}\left(\exp\left\{\lambda \sup_{t\le T} \sqrt{N}|Z_N(t)-X(t)|\right\}\right)<\infty \tag{4.6}$$

for every $\lambda>0$.

Proposition 4.3. *There is a constant* C_n *such that*

$$\mathbf{E}\left(\sup_{t\le T}(\sqrt{N}|\sqrt{N}(Z_N(t)-X(t))-V(t)|)^{2n}\right)\le$$

$$\le C_n\,\mathrm{e}^{2nMT}\mathbf{E}\left(T^{n-1}\int_0^T(\sqrt{N}|Z_N(s)-X(s)|)^n\,\mathrm{d}s \right. \tag{4.7}$$

$$\left.+\int_0^T(\sqrt{N}|Z_N(s)-X(s)|)^{4n}\,\mathrm{d}s\right)$$

and

$$\sup_N \mathbf{E}\left(\exp\left\{\lambda \sup_{t\le T}\sqrt{N}|\sqrt{N}(Z_N(t)-X(t))-V(t)|\right\}\right)<\infty \tag{4.8}$$

for every $\lambda>0$.

Remark. This proposition follows from (4.1) using the estimates in Lemma 4.1. If $\sqrt{f_l(x)}\equiv g_l(x)$ is continuously differentiable then $\sqrt{N}(\sqrt{N}(Z_N(t)-X(t))-V(t))$ converges in probability to the solution of

$$\hat{V}(t)=\sum l\int_0^t \partial g_l(X(s))\cdot V(s)\,\mathrm{d}\tilde{W}_l+\int_0^t \partial F(X(s))\cdot\hat{V}(s)\,\mathrm{d}s$$

$$+\int_0^t V(s)\cdot\partial^2F(X(s))V(s)\,\mathrm{d}s. \tag{4.9}$$

Observing that

$$\sqrt{N}(X_N(t)-X(t))-V(t)$$
$$=\sqrt{N}(X_N(t)-Z_N(t))+\sqrt{N}(Z_N(t)-X(t))-V(t). \tag{4.10}$$

Proposition 4.3 and Theorem 3.3. imply the following Theorem.

Theorem 4.4. *Assume that* $X_N(0)=X(0)$.

Suppose $\varepsilon_l=0$ *for all but finitely many* l, *then for* $N\ge 2$ *there are random variables* γ_N^T *with* $\sup_N \mathbf{E}(\exp\{\lambda\gamma_N^T\})<\infty$ *for some* $\lambda>0$ *such that*

$$\sup_{t\le T}|\sqrt{N}(X_N(t)-X(t))-V(t)|\le \gamma_N^T\frac{\log N}{\sqrt{N}}. \tag{4.11}$$

Remark. Similar Theorems corresponding to parts (b) and (c) of Theorem 3.3. can, of course, also be stated.

Corollary 4.5. *Under the conditions of Theorem* 4.4

$$|\mathbf{P}\{\sqrt{N}(X_N(t)-X(t))\in\Gamma\}-\mathbf{P}\{V(t)\in\Gamma\}|\leq K(\Gamma,t)\frac{(\log N)^2}{\sqrt{N}} \tag{4.12}$$

for every open set Γ in the subspace $\mathbf{E}$ *spanned by the l for which $\varepsilon_l>0$, such that $\partial\Gamma$ has finite surface area.*

Remark. Barbour [2] gives a rate of convergence of $O(\log N/\sqrt{N})$ for a somewhat complicated functional of $\sqrt{N}(X_N(t)-X(t))$.

Proof. Let $A=\bigcup_{x\in\partial\Gamma}S(x,\gamma_N^T\log N/\sqrt{N})$ where $S(x,a)$ is the sphere of radius a centered at x. Then

$$\mathbf{P}\{V(t)\in\Gamma-A\}\leq\mathbf{P}\{\sqrt{N}(X_N(t)-X(t))\in\Gamma\}\leq\mathbf{P}\{V(t)\in\Gamma\cup A\}. \tag{4.13}$$

Let $A'=\bigcup_{x\in\partial\Gamma}S(x,k\,(\log N)^2/\sqrt{N})$. Then

$$\begin{aligned}\mathbf{P}\{V(t)\in\Gamma\cup A\}-\mathbf{P}\{V(t)\in\Gamma-A\}&=\mathbf{P}\{V(t)\in A\}\\&\leq\mathbf{P}\{V(t)\in A'\}+\mathbf{P}\{\gamma_N^T>k\log N\}\\&\leq\mathbf{P}\{V(t)\in A'\}+C\exp\{-\lambda k\log N\}.\end{aligned}$$

Since $V(t)$ has a bounded density with respect to Lebesque measure on E it follows that

$$\mathbf{P}\{V(t)\in A'\}=O\left(k\frac{(\log N)^2}{\sqrt{N}}\right).$$

The Corollary follows by taking $k=1/\lambda$.

Appendix

Let φ be twice continuously differentiable, and $M_N(t)$ be given by (2.7). Then (at least formally)

$$\begin{aligned}&\mathbf{E}(\varphi(M_N(t)e^{-\alpha t}))=\\&=\varphi(1+|X_N(0)|)+\int_0^t\mathbf{E}\left(\left\{\sum\left[\varphi\left(\left(M_N(s)+\frac{1}{N}|l|\right)e^{-\alpha s}\right)-\varphi(M_N(s)e^{-\alpha s})\right]N\varepsilon_l\right.\right.\\&\qquad\left.\left.-\alpha e^{-\alpha s}\varphi'(M_N(s)e^{-\alpha s})\right\}M_N(s)\right)\mathrm{d}s\\&=\varphi(1+|X_N(0)|)\\&\quad+\int_0^t\mathbf{E}\left(M_N(s)\sum N\varepsilon_l\int_0^{1/N|l|e^{-\alpha s}}\left(\frac{1}{N}|l|e^{-\alpha s}-u\right)\varphi''(M_N(s)e^{-\alpha s}+u)\mathrm{d}u\right)\mathrm{d}s.\end{aligned} \tag{A.1}$$

If we have $\varphi''(x+y) \leq K(\varphi''(x)+\varphi''(y))$, then

$$\begin{aligned} \mathbf{E}(\varphi(M_N(t)\mathrm{e}^{-\alpha t})) &\leq \\ &\leq \varphi(1+|X_N(0)|) \\ &\quad + K\int_0^t \mathbf{E}\left(\varphi''(M_N(s)\mathrm{e}^{-\alpha s})M_N(s)\mathrm{e}^{-2\alpha s}(2N)^{-1}\sum |l|^2\varepsilon_l\right)\mathrm{d}s \\ &\quad + K\int_0^t \mathbf{E}\left(M_N(s)N\sum \varepsilon_l\varphi\left(\frac{1}{N}|l|\mathrm{e}^{-\alpha s}\right)\right)\mathrm{d}s. \end{aligned} \tag{A.2}$$

Let $R_N(t) = M_N(t)\mathrm{e}^{-\alpha t}$, and recall

$$\mathbf{E}(R_N(t)) = 1+|X_N(0)| = R_N(0).$$

If $\sum |l|^\beta\varepsilon_l < \infty$, $\beta \geq 2$, then

$$\begin{aligned} \mathbf{E}((R_N(t))^\beta) \leq{} & (1+|X_N(0)|)^\beta \\ & + K'N^{-1}\int_0^t \mathbf{E}((R_N(s))^{\beta-1})\mathrm{e}^{-\alpha s}\,\mathrm{d}s \\ & + K'N^{-(\beta-1)}\int_0^t (1+|X_N(0)|)\mathrm{e}^{-(\beta-1)s}\,\mathrm{d}s, \end{aligned} \tag{A.3}$$

where K' is independent of N, t and $|X_N(0)|$.

The fact that

$$\sup_{N,t,|X_N(0)|} \mathbf{E}((R_N(t)))/(1+|X_N(0)|)^\beta < \infty$$

follows by iteration.

If $\sum \varepsilon_l\,\mathrm{e}^{\lambda_0|l|} < \infty$, then for $\lambda < \lambda_0$

$$\begin{aligned} \mathbf{E}(\exp\{\lambda R_N(t)\}) ={} & \exp\{\lambda(1+|X_N(0)|)\} \\ & + \int_0^t \mathbf{E}(\exp\{\lambda R_N(s)\}R_N(s))\mathrm{e}^{\alpha s}\sum N\varepsilon_l[\exp\{\lambda|l|\mathrm{e}^{-\alpha s}/N\} - 1 - \lambda|l|\mathrm{e}^{-\alpha s}/N]\,\mathrm{d}s \\ \leq{} & \exp\{\lambda(1+|X_N(0)|\} + CN^{-1}\int_0^t \mathbf{E}(\exp\{\lambda R_N(s)\}R_N(s))\mathrm{e}^{-\alpha s}\,\mathrm{d}s, \end{aligned} \tag{A.4}$$

where C is independent of N and can be taken independent of $\lambda \leq \lambda_0 - \delta$ for fixed $\delta > 0$.

Since the last inequality holds point wise in s, for $u(t,\lambda) = \mathbf{E}(\exp\{\lambda R_N(t)\})$ we have

$$\frac{\partial}{\partial t}u(t,\lambda) \leq C_\delta N^{-1}\frac{\partial}{\partial\lambda}u(t,\lambda)\mathrm{e}^{-\alpha t} \quad \text{for } t>0, \quad \lambda < \lambda_0 - \delta.$$

Consequently

$$\frac{\mathrm{d}}{\mathrm{d}t}u\left(t, a+\frac{1}{\alpha}C_\delta N^{-1}\mathrm{e}^{-\alpha t}\right) \leq 0 \quad \text{for } t>0 \quad \text{and } a+\frac{1}{\alpha}C_\delta N^{-1}\mathrm{e}^{-\alpha t} < \lambda_0 - \delta.$$

Therefore

$$u\left(t, a + \frac{1}{\alpha} C_\delta N^{-1} e^{-\alpha t}\right) \leq u\left(0, a + \frac{1}{\alpha} C_\delta N^{-1}\right)$$

$$= \exp\left\{\left(a + \frac{1}{\alpha} C_\delta N^{-1}\right)(1 + |X_N(0)|)\right\}$$

for all t and $a < \lambda_0 - \delta - (1/\alpha)C_\delta N^{-1}$.

Set $a = \lambda - (1/\alpha)C_\delta N^{-1}e^{-\alpha t}$.

If $\lambda < \lambda_0 - \delta - (1/\alpha)C_\delta N^{-1}(1 - e^{-\alpha t})$

$$u(t, \lambda) \leq \exp\{(\lambda + \tfrac{1}{2}C_\delta N^{-1}(1 - e^{-\alpha t}))(1 + |X_N(0)|)\}.$$

Therefore, if $\lambda < \lambda_0 - \delta - \frac{1}{2}C_\delta N^{-1}$

$$\sup_t \mathrm{E}(\exp\{\lambda R_N(t)\}) \leq \exp\{(\lambda + \tfrac{1}{2}CN^{-1})(1 + |X_N(0)|)\}.$$

Since we may select δ small and N_λ large (2.5) follows.

In general we cannot take $N_\lambda = 1$ even for the Yule process (see Karlin [7, page 180]) for which $\mathrm{E}(\exp\{\lambda R_N(t)\})$ can be explicitly computed.

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