On associative conformal algebras of linear growth II

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Abstract

We classify unital associative conformal algebras of linear growth and provide new examples of such.
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0. Introduction

Conformal algebras were introduced in [K1] to provide algebraic formalism for the singular part of the OPE in the theory of vertex algebras. Since then they turned out to be a useful instrument in the study of vertex algebras (see, e.g., [Ro]), infinite-dimensional Lie superalgebras [K3], and in a generalized form, hamiltonian structures in integrable systems [BDK].

Definition 0.1. A conformal algebra $C$ is a $\mathbb{R}[\partial]$-module endowed with bilinear operations $\odot : C \otimes C \to C$, $n \in \mathbb{Z}_{\geq 0}$, such that for any $a, b \in C$

1. (locality axiom) $a \odot b = 0$ for $n > N(a, b)$ ($N(a, b)$ is called the locality degree of $a$ and $b$);

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(2) $\partial(a \circledast b) = (\partial a) \circledast b + a \circledast (\partial b)$;
(3) $\partial a \circledast b = -na \circledast_{n-1} b$.

The number $n$ in $\circledast$ is called the order of multiplication $\circledast$.

In this paper $\mathbb{k}$ is an algebraically closed field of zero characteristic. When clear, we refer to objects that are finite as modules over $\mathbb{k}[\partial]$ simply as finite.

One of the basic questions in the study of conformal algebras is the theory of representations of finite modules. Besides being of independent interest, it is also related to the study of representations of algebras of differential operators [BKL2, Ze2]. Thus ones of the most important associative conformal algebras are the algebras $C_{\text{end}}$ which are the analogues of matrix algebras in “ordinary” theory. From an algebraic point of view, these are exactly simple unital associative conformal algebras of linear growth [Re1]. The next logical step is to describe all unital associative conformal algebras of linear growth. This paper contains such description.

**Theorem 0.2.** Let $C$ be a unital conformal algebra of linear growth. Then

- if $C$ is prime, then $C$ is isomorphic to either $C_{\text{end}}$ or a subalgebra of a current algebra over a prime algebra of linear growth;
- if $C$ is semisimple, then $C$ embeds into a direct sum of $C_{\text{end}}$ and a current algebra over a semiprime algebra of zero or linear growth;
- if $C$ contains a nilpotent ideal, then its coefficient algebra is not semiprime.

The paper is virtually self-contained. The first section contains all necessary definitions and statements and basic examples of associative conformal algebras.

Section 2 is devoted to classification results and contains the proof of Theorem 0.2 (see Theorems 2.12, 2.16, and Lemma 2.13). We deal with prime conformal algebras first. The proofs follow along the lines of those in [Re1]; however, since we work in a more general setup, we need to repeat some of the steps for consistency of presentation.

Section 3 is concerned with subalgebras of prime conformal algebras; in particular we exhibit a noncurrent unital subalgebra of a current algebra.

1. Preliminaries

A detailed exposition of conformal algebras can be found in [K1, Chapter 2] and in the survey papers [K2, Ze1]. The presentation here is guided by our needs in the next sections and is in no way complete. We also present several results on unital conformal algebras from [Re1, Re2].

Standard algebraic terminology easily carries over to the conformal case, thus an ideal $I$ of a conformal algebra $C$ is a conformal subalgebra such that $C \circledast I \subset I$, $I \circledast C \subset I$ for all $n$, a nilpotent ideal $I$ is such that the product of a fixed number of copies of $I$ (with multiplications of any order) is zero, a simple conformal algebra contains no ideals, a semisimple one contains no nilpotent ideals, etc.
1.1. Basic examples

Let $A$ be any algebra. We call elements of the extension $A[z, z^{-1}]$ formal distributions on $A$. Let $f(z) = \sum_{n \in \mathbb{Z}} f(n)z^{-n-1}$ and $g(z) = \sum_{n \in \mathbb{Z}} g(n)z^{-n-1}$ be formal distributions on $A$. Define the product $\circ_m$, $m \in \mathbb{Z}_{\geq 0}$, of $f(z)$ and $g(z)$ as

$$f(z) \circ_m g(z) = \text{Res}_{w=0} f(w)g(z)(w-z)^m$$

(by $\text{Res}_{w=0} h(w, z)$ we mean a formal distribution in $z$ that is a coefficient at $w^{-1}$ in $h(w, z)$ viewed as a formal distribution on the set $A[z, z^{-1}]$). Two formal distributions are called local if only a finite number of such products are nonzero.

A set of mutually local formal distributions that is closed with respect to products $\circ$ and the operator $\partial/\partial z$ is a conformal algebra. Clearly, as the application of $\partial/\partial z$ preserves locality (though changes it degree), the $[\partial/\partial z]$-span of a set of mutually local distributions closed with respect to products $\circ$ forms a conformal algebra. Moreover, we have

**Lemma 1.1** (Dong’s lemma [Li,K1]). Let $f$, $g$, and $h$ be pairwise mutually local formal distributions over either a Lie or associative algebra. Then for any $n \geq 0$, $f \circ g$ and $h$ are again pairwise mutually local.

Thus, mutually local formal distributions over a Lie or associative algebra generate a conformal algebra.

It can be shown [K1] (see also [Bo] for a vertex algebra version of the same construction) that every conformal algebra $C$ embeds into formal distributions on some algebra. Moreover, there exists a universal algebra Coeff $C$ (called the coefficient algebra of $C$) such that for any $A$, $C \to A[z, z^{-1}]$, there exists a unique homomorphism Coeff $C \to A$ such that the diagram

$$\text{Coeff } C \xrightarrow{\sim} A[z, z^{-1}]$$

commutes. In particular, Coeff $C$ “distinguishes” coefficients $f(n)$ of every element of $f \in C$ (or rather the corresponding formal distribution $f(z) \in \text{Coeff}$). The subalgebra of coefficients at $z^{-1}$ is denoted $(\text{Coeff } C)_0$.

A conformal algebra is called associative (respectively Lie) if the corresponding coefficient algebra is associative (respectively Lie). For every identity satisfied by Coeff $C$, one can write a corresponding conformal identity satisfied by $C$; however, for this exposition we do not require the explicit forms of conformal associativity, Jacobi identity, etc.

**Example 1.2.** Let $B$ be any algebra. For every $b \in B$ consider the following formal distribution on $B[t, t^{-1}]$:

$$\tilde{b} = \sum_n bt^n z^{-n-1}.$$
Clearly for any \( b_1, b_2 \in B \), the formal distributions \( \tilde{b}_1 \) and \( \tilde{b}_2 \) are mutually local: \( \tilde{b}_1 \otimes \tilde{b}_2 = \delta_{0,n} b_1 b_2 \). Thus by Dong’s Lemma 1.1, \( \tilde{b} \) generate a conformal algebra. It is called the current algebra over \( B \) and is denoted \( \text{Cur} \, B \).

Observe that the conformal algebra generated by formal distributions \( \sum b z^{-n-1} \) on \( B \) is isomorphic to \( \text{Cur} \, B \). However, for this conformal algebra \( B \) is not a coefficient algebra \( \text{Coeff} \, \text{Cur} \, B \), whereas \( B[t, t^{-1}] \) is.

**Example 1.3.** Denote by \( W \) the Weyl algebra \( \mathbb{k}[x, t \mid x t - t x = 1] \) and by \( W_t \) its localization at \( t \). We define the conformal algebra \( \text{Cend}_n \) as an algebra of formal distributions on \( W_t \) generated by distributions \( L^{k}_A = \sum A x^k t^n z^{-n-1} \), \( k \geq 0 \), \( A \in \text{End}_n(\mathbb{k}) \).

In particular, \( \text{Cend}_1 \) is generated by elements \( L^k = \sum x^k t^n z^{-n-1} \) for \( k = 0, 1 \). Their nonzero products are

\[
\begin{align*}
L^0 \otimes L^0 &= L^0, \\
L^0 \otimes L^1 &= L^1 \otimes L^0 = L, \\
L^0 \otimes L^1 &= L^1 \otimes L^0 = -L^0, \\
L^1 \otimes L^1 &= L^2, \\
L^1 \otimes L^1 &= -L^1.
\end{align*}
\]

(1.1)

It follows that \( \text{Cend}_1 \) (and, as a consequence, \( \text{Cend}_n \) for any \( n \)) is not finite over \( \mathbb{k}[\partial] \).

Observe also that \( L^0 \) generates a subalgebra isomorphic to \( \text{Cur} \, \mathbb{k} \) and that, in a broad sense, \( L^0 \) acts as (left) identity. This will be used later.

In the theory of representations of conformal algebras \( \text{Cend}_n \) plays the role \( \text{End}_n(\mathbb{k}) \) in ordinary theory, i.e., \( \text{Cend}_n \) is the conformal algebra of conformal linear maps on \( \mathbb{k}[\partial]^n \).

**Example 1.4.** Let \( B \) be an associative algebra with a locally nilpotent derivative \( \delta \). For every \( b \in B \) consider the following formal distribution in \( B[t, t^{-1}; \delta][z, z^{-1}] \):

\[
\tilde{b} = \sum_n b t^n z^{-n-1}
\]

(here \( B[t, t^{-1}; \delta] \) is the Ore extension of \( B \) localized at \( t \), i.e., for any \( b \in B \), \( bt - tb = \delta(b) \)). Such formal distributions are mutually local with each other, namely

\[
\tilde{b}_1 \otimes \tilde{b}_2 = (-1)^m b_1 \delta^m(\tilde{b}_2), \quad b_1, b_2 \in B,
\]

(1.2)

and locality follows from nilpotency of \( \delta \).

The family of formal distributions \( \{ \tilde{b} \mid b \in B \} \) spans the conformal algebra called the differential conformal algebra \( \text{Diff} \, B \). Obviously, \( (\text{Coeff} \, \text{Diff} \, B)_0 = B \).

Observe that for a trivial \( \delta \), \( \text{Diff} \, B = \text{Cur} \, B \). Also, \( \text{Cend}_n = \text{Diff} \, \mathbb{k}[x] \) with the standard derivation.

It was shown in [Re2] that \( B \) and \( \text{Diff} \, B \) have equivalent categories of representations. The lattice of ideals of \( B \) is isomorphic to the lattice of \( \delta \)-stable ideals of \( \text{Diff} \, B \). Namely, to a \( \delta \)-stable ideal \( I \) of \( B \) there corresponds the ideal \( \tilde{I} \) of \( \text{Diff} \, B \) spanned by \( \{ \tilde{b} \mid b \in I \} \). Conversely, to an ideal \( J \) of \( \text{Diff} \, B \) there corresponds the ideal \( \tilde{J} = \{ \tilde{b} \mid \tilde{b} \in J \} \) of \( B \) and \( \tilde{J} = J, \tilde{I} = I \).
As a corollary we have the following

**Lemma 1.5.** Diff $A$ is simple if and only if $A$ is differentiably simple.

### 1.2. Unital conformal algebras

The study of ordinary associative algebras begins with the study of unital ones, i.e., those containing $k$. By analogy, in the conformal case one should start by considering associative conformal algebras that contain a subalgebra of rank 1 acting faithfully in some sense.

It can be shown that $\text{Cur}^k_k$ is the unique associative conformal algebra with nonzero multiplication that is free of rank 1. Moreover, it can be shown that every module $M$ over $\text{Cur}^k_k$ splits as $M = M_0 \oplus M_1$, where $\text{Cur}^k_k \otimes M_0 = 0$ for every $n$ and every element of $M_1$ is fixed by the action of $e \otimes$ for any generator $e$ of $\text{Cur}^k_k$ [Re2]. This motivates the following definition:

**Definition 1.6.** An associative conformal algebra $C$ is **unital** if $\text{Cur}^k_k$ embeds into $C$ and for the action of the image of this embedding, $C = C_1$.

A generator of $\text{Cur}^k_k \subset C$ is called a **conformal identity** and is denoted $e$. Observe that a conformal identity is not unique (see Lemma 2.8).

Unlike in the ordinary case, it is not clear how one can “adjoin identity” to a torsion-free conformal algebra (a unital conformal algebra is automatically torsion-free). However, a differential conformal algebra $\text{Diff} B$ always embeds into a unital conformal algebra: for this one needs only to adjoin identity to $B$. Thus we will always assume below that a differential conformal algebra is unital. Unless stated otherwise, we will also assume that $e = \tilde{1}$.

Modulo a technical condition, the converse of the above observation is also true. For a conformal algebra $C$ denote by $L(C)$ the set $\{a \mid a \otimes b = 0 \forall b \in C, n \in \mathbb{Z}_{\geq 0}\}$.

**Theorem 1.7.** Let $C$ be a unital conformal algebra such that $L(C) = 0$. Then $C = \text{Diff} B$ for a unital associative algebra $B$.

In particular, a semisimple unital conformal algebra is always differential.

### 1.3. Gelfand–Kirillov dimension

The Gelfand–Kirillov dimension of a finitely generated algebra (of any variety) $A$ is defined as

$$\text{GKdim } A = \limsup_{r \to \infty} \frac{\log \dim(V^1 + V^2 + \cdots + V^r)}{\log r},$$

where $V$ is a generating subspace of $A$ [KL]. This definition easily carries over to the conformal case.
Let $C$ be a finitely generated conformal algebra (over any variety). Define $C_r$ to be the $k[\partial]$-span of products of at most $r$ generators with any positioning of brackets and multiplications of any order.

Since the powers of $\partial$ can be gathered at the beginning of conformal monomials (with a probable change in the orders of multiplications), it is clear that $\bigcup_r C_r = C$. For a given ordered collection of generators and a given positioning of brackets, the number of nonzero monomials is finite because of locality. Therefore, $r_k C_r$ is finite.

Definition 1.8. Let $C$ be a finitely generated conformal algebra. Then

$$\text{GKdim } C = \limsup_{r \to \infty} \frac{\log r_k [\partial] C_r}{\log r}. \quad (1.3)$$

Conformal Gelfand–Kirillov dimension has the same basic properties as the ordinary one: it is invariant of the choice of the generating set, GKdim of a subalgebra or a quotient algebra does not exceed that of the algebra, etc.

For a conformal associative algebra $C$, $\text{GKdim Coeff } C \leq \text{GKdim } C + 1$ [Re1, Theorem 2.2] (the inequality is sometimes strict, e.g., when $C$ is torsion). One can show directly that for a differential conformal algebra, $\text{GKdim Diff } B = \text{GKdim } B$. In particular, $\text{GKdim Cend}_n = 1$. The main result of [Re1] is the following

Theorem 1.9. Let $C$ be a simple unital associative conformal algebra of Gelfand–Kirillov dimension 1. Then $C$ is isomorphic to $\text{Cend}_n$ for some $n$.

In the next section we present the generalization of this theorem.

2. Semisimple conformal algebras of linear growth

In this section we generalize Theorem 1.9 and achieve the complete description of unital conformal algebras of GKdim 1.

As follows from Lemma 1.5, $C$ is simple if and only if $\text{Coeff } C$ is differentiably simple. Following this correspondence, we define a larger subclass of unital associative conformal algebras: we call $C$ prime whenever $(\text{Coeff } C)_0$ is prime (since being prime is, in some sense, equivalent to being differentiably prime). Also, recall that $C$ is semisimple if it does not contain nonzero nilpotent ideals. The latter condition is equivalent to having a semiprime coefficient algebra (see Lemma 2.13).

2.1. Classification of associative algebras of linear growth

The following theorem was proven in [SSW] (see also [SW]):

Theorem 2.1. Let $A$ be a finitely generated algebra of linear growth. Then

(i) The nilradical $N(A)$ of $A$ is nilpotent.
(ii) If $A$ is semiprime (i.e., if $N(A) = 0$), then it is a finite module over its center $Z(A)$ which is also finitely generated.

Several facts from the original proof of this theorem will be used below as well.

### 2.2. Prime unital conformal algebras of linear growth

We are going to classify prime unital conformal algebras of linear growth. It is well known that a differentially simple algebra is necessarily prime (see, e.g., [Po]), hence Theorem 1.9 also follows from such a classification.

Let $A$ be a finitely generated prime algebra of linear growth. Then by Theorem 2.1, it is a finite module over its center $Z(A)$. Moreover, it is easy to see that for any derivation $\delta$ of $A$, $Z(A)$ is $\delta$-stable: for $a \in Z(A)$, $0 = \delta([a, b]) = [\delta(a), b] + [a, \delta(b)] = [\delta(a), b]$ for any $b \in A$.

Thus, we begin by considering the case of a prime commutative finitely generated algebra.

**Lemma 2.2.** Let $A$ be a finitely generated prime commutative algebra with a locally nilpotent derivation $\delta$, $\text{GKdim} A = 1$. Then either $A \cong k[x]$, $\delta = \partial/\partial x$ or $\delta = 0$.

**Proof.** Since $A$ is prime, a nonzero algebraic element of $A$ must be invertible, hence all its algebraic elements lie in $k$.

Consider two sets of transcendental elements of $A$:

$$S_1 = \{ x \in A \mid \text{all nonzero } \delta^n(x) \text{ are transcendental} \},$$

$$S_2 = \{ x \in A \mid \text{for some } n, \delta^n(x) \neq 0 \text{ and is algebraic} \}.$$

Clearly, both sets are $\delta$-stable. Assume both are nonempty. Without loss of generality we can pick $x_1 \in S_1$, $x_2 \in S_2$ such that $\delta(x_1) = 0$, $\delta(x_2) = 1$. As $\text{tr.deg} A = 1$, $x_1$ and $x_2$ are algebraically dependent. In any statement of dependence of $x_2$ over $k[x_1]$ the degree in $x_2$ can be lowered by application of $\delta$. Therefore, one of the $S_i$’s is empty.

Consider now the case $A = k + S_1$. Assume that there exists an element with a nonzero derivation. Without loss of generality we can consider $x$ and $y$ such that $\delta(x) = y$, $\delta(y) = 0$. Just as above, consider a statement of dependence of $x$ over $k[y]$. Application of $\delta$ lowers the degree in $x$ (though it increases the degree in $y$), a contradiction. Therefore, $\delta$ kills all transcendental elements.

The remaining case is $A = k + S_2$. Choose $x$ such that $\delta(x) = 1$. Let $y$ be an arbitrary element with $\delta(y) \in k$. Then $x - y(\delta(y))^{-1} \in k$ and $y \in k[x]$. For an arbitrary $y \in S_2$, by induction on the minimal $n$ such that $\delta^n(y) \in k$, we also obtain $y \in k[x]$. $\square$

**Remark 2.3.** The final part of the proof of the above lemma can be also deduced from a result in [Wr].

**Corollary 2.4.** Let $A$ be a finitely generated differentiably simple commutative algebra of growth 1 with a locally nilpotent derivation. Then $A \cong k[x]$. 
Proof. Indeed, if $A \not\cong k[x]$, it must be simple. Therefore, $A$ is a field of transcendental degree 1 and cannot be finitely generated. □

Lemma 2.5. Let $A$ be a finitely generated prime algebra with a locally nilpotent derivation $\delta$, $\GKdim A = 1$. Then $A$ is either isomorphic to $\End_n(k[x])$, $\delta = \partial/\partial x$, or $A$ can be embedded into an algebra $B$ such that $\delta$ extends to an inner derivation of $B$ determined by a nilpotent element.

Proof. As mentioned above, $A$ is a finite module over its center $Z(A)$ which is finitely generated and has linear growth. Clearly, $Z(A)$ is prime and $\delta$-stable.

Case 1. $Z(A) = k[x]$ and $\delta|Z(A) = \partial/\partial x$.

Consider subalgebra $A_0 = \ker \delta$. We begin by demonstrating that $A_0$ generates $A$ as a module over $Z(A)$. More precisely, every $a \in A$ is of the form $\sum x^i a_i$, $a_i \in A_0$, where $n$ is such that $\delta^n(a) = 0$, $\delta^{n-1}(a) \neq 0$. Indeed, with the inductive assumption $\delta(a) = \sum x^i b_i$, $b_i \in A_0$, consider $a_0 = a - \sum x^i \frac{x^{i+1}}{(i+1)!} b_i$. Since $\delta(a_0) = 0$, we see that $a$ is also a polynomial in $x$ over $A_0$. Moreover, this polynomial form is unique for any $a \in A$. Indeed, if $\sum x^i a_i = 0$, $a_i \in A$, applying a necessary number of derivations shows that the coefficient at the highest power is 0. In particular, this implies that $A = k[x] \otimes A_0$.

Fix a subset $\{a_i\}$ of $A_0$ that generates $A$ as a module over $Z(A)$. Any product of elements of $A_0$ is a linear combination $\sum p_i(x) a_i$ over $Z(A)$ with the derivation $\sum (\partial p_i(x)/\partial x) a_i = 0$. This implies $A_0 = \Span_k(a_i)$ is finite dimensional.

Clearly any ideal of $A_0$ can be lifted to $A$, thus $A_0$ is prime as well and therefore simple over $k$ [Rw, 2.1.15]. Hence, $A = \End_n(k[x])$ and $\delta = \partial/\partial x$.

Case 2. $\delta|Z(A) = 0$.

Let $F$ be the field of fractions of $Z(A)$ and consider the finite-dimensional simple $F$-algebra $B = F \otimes Z(A) A$. Clearly, $\delta$ extends to a derivation of $B$ and is, therefore, an inner nilpotent derivation, $\delta = \text{ad}_a [Ja]$. We may take $a$ to be nilpotent (it is enough to pick any $a$ such that $\delta = \text{ad}_a$ and take its nilpotent part, since the semisimple part must commute with all elements of $B$). □

Remark 2.6. [Re1] If we strengthen the condition of Lemma 2.5 to $A$ being differentiably simple, by Corollary 2.4 we will have to consider Case 1 of the above lemma only. This implies Theorem 1.9.

Remark 2.7. It follows that there exist no finitely generated simple associative current conformal algebras of linear growth. However, this (quite unexpected) result can be deduced directly from Theorem 2.1. Indeed, if $\text{Cur} A$ is such an algebra, then $A$ must be simple and have linear growth. So should its center, hence it is a field of transcendental degree 1.

In the more general framework of prime conformal algebras, the second case of Lemma 2.5 merits further consideration. We notice first that by a change of conformal
identity, one can discount the twisting on the coefficient algebra introduced by an inner derivation:

**Lemma 2.8.** Let $C$ be a differential conformal algebra over algebra $A$ with an inner derivation determined by a nilpotent element. Then $C \cong \text{Cur } A$.

**Proof.** We have $\delta = \text{ad } r$ for a nilpotent $r$. Clearly, $A[t, t^{-1}; \delta]$ is isomorphic to the polynomial algebra $A[s, s^{-1}]$ via the mapping $t \rightarrow s$. To prove that corresponding differential algebras $C$ and $\text{Cur } A$ are isomorphic as well, we first show that the formal distribution $e' = \sum (t - r)^n z^{-n-1}$ belongs to $C$. Indeed, in this case, the conformal subalgebra of $C$ generated by $e'$ and $\tilde{a} \otimes e'$, $a \in A$, is isomorphic to $\text{Cur } A$.

Let $m$ be the degree of nilpotency of $r$. Since $\delta(r) = 0$, $t$ and $r$ commute; therefore,

$$e' = \sum_{k=0}^{m-1} \frac{1}{k!} \partial^k \left( \sum r^k t^n z^{-n-1} \right) \in \text{Diff } A.$$

Conversely, $(\tilde{r} \otimes e') \otimes e' = \tilde{r}$ lies in the above subalgebra, hence, so does $e$. Thus, this subalgebra coincides with $C$. □

**Corollary 2.9.** Let $C$ be a simple unital associative conformal algebra that is finite over $k[\partial]$. Then $C \cong \text{Cur } \text{End}_n(k)$.

**Proof.** Let $C = \text{Diff } A$ where $A$ is differentiably simple. Since $A$ is finite, it must be simple [Bl]; thus $A \cong \text{End}_n(k)$ and all its derivations are inner. The rest follows from Lemma 2.8. □

**Remark 2.10.** The above result also follows from the classification of simple Lie conformal algebras that are finite over $k[\partial]$ in [DK]. In fact, in this line of proof one does not require unitality; though, we still get only $\text{Cur } \text{End}_n(k)$ as an answer [K2]. This shows that every simple associative conformal algebra that is finite over $k[\partial]$ is unital.

Of course, in general not every associative conformal algebra is unital and one cannot simply “adjoin” a conformal identity as in the ordinary case. However, in every known case, a conformal algebra can be embedded into a unital one.

**Conjecture 2.11.** Every $\partial$-torsion free associative conformal algebra can be embedded into a unital conformal algebra.

Another question is: if such embeddings $C \rightarrow C'$ exist for a given conformal algebra $C$, what is the lower bound on $\text{GKdim } C'$?

Clearly, when $C$ has a faithful representation that is finite over $k[\partial]$, $C$ embeds into $\text{Cend}_n$. Moreover, when $C$ is finite itself it can be embedded into a unital conformal algebra (this follows from the classification and so far no direct proof is known). So, in such cases it is always possible to find $C'$ with $\text{GKdim } C' = \text{GKdim } C$. 
We can now translate the statement of Lemma 2.5 into the language of conformal algebras:

**Theorem 2.12.** Let $C$ be a prime unital finitely generated associative conformal algebra with $\text{GKdim } C = 1$. Then $C$ is isomorphic to either $C_{\text{end}} n$ or a subalgebra of a current algebra over a finitely generated prime algebra.

**Proof.** By Lemma 2.5, $(\text{Coeff } C)_0$ is either $\text{End}_R(k) \otimes k[x]$ with a nilpotent derivation given by $\partial/\partial x$ or a subalgebra of a prime algebra with an inner derivation determined by a nilpotent element. In the first case, $C$ is isomorphic to $C_{\text{end}} n$ and in the second case it is a subalgebra of a current algebra by Lemma 2.8. In general, this current algebra might be infinitely generated but by construction one may choose an appropriate prime current subalgebra.  

Essentially, this classification is the best one can hope for; this is explained in Section 3.

### 2.3. Classification of semisimple conformal algebras of linear growth

We begin by translating semisimplicity of conformal algebras of linear growth into a condition for its coefficient algebra.

**Lemma 2.13.** Let $\text{Diff } A$ be an associative conformal algebra of zero or linear growth. Then $\text{Diff } A$ is semisimple if and only if $A$ is semiprime.

**Proof.** By construction from Lemma 1.5, an ideal $J$ of $\text{Diff } A$ is nilpotent if and only if the corresponding ideal $\bar{J}$ of $A$ is nilpotent. Indeed, if for any $a_0, \ldots, a_n$, $(\cdots(\tilde{a}_0 \oplus \cdots) \oplus \tilde{a}_n) = 0$, then $a_0 \cdot \ldots \cdot a_n = 0$ and, conversely, if $(\bar{J})^n = 0$, then $(\cdots(\tilde{a}_0 \oplus \cdots) \oplus \tilde{a}_n) = 0$ as $\bar{J}$ is $\delta$-stable.

Thus, if $\text{Diff } A$ contains a nilpotent ideal, so does $A$. Conversely, if $A$ is not semiprime, its nilradical $N(A)$ is $\delta$-stable [Rw, 2.6.28] and nilpotent (Theorem 2.1(i) for the case of linear growth). Hence, $\bar{N}(A)$ is nilpotent.

Thus, we are able to classify semisimple conformal algebras of linear and zero growth.

We need an easy (and probably known) technical lemma first.

**Lemma 2.14.** Let $A = \bigoplus_i A_i$ be a finite direct sum of unital associative algebras $A_i$. Then a derivation $\delta$ on $A$ restricts to each $A_i$.

**Proof.** Let $e_i$ be the identity in $A_i$. Since $\delta(e_i e_j) = e_i \delta(e_j) + \delta(e_i) e_j = 0$ and the summands lie in $A_i$ and $A_j$ respectively, we have $\delta(e_i e_j) = 0$ for any $j \neq i$. Thus, $\delta(e_i) \in A_i$. Now, as $\delta(e_i) = \delta(e_i^2) = 2\delta(e_i)$, it follows that $\delta(e_i) = 0$. Consequently, $\delta(A_i) = \delta(A_i e_i) = \delta(A_i) e_i \subset A_i$.  

The classification of semisimple unital conformal algebras of $\text{GKdim } \leq 1$ follows:
Lemma 2.15. [K2] Let $C$ be a semisimple unital finitely generated associative conformal algebra that is finite over $\mathbb{k}[\partial]$. Then $C \cong \bigoplus_{i=1}^{k} \text{Cur} \text{End}_{n_i}(\mathbb{k})$.

**Proof.** The proof is the same as for Corollary 2.9. Let $C = \text{Diff} A$. By Lemma 2.13, we have $A \cong \bigoplus_{i=1}^{k} \text{End}_{n_i}(\mathbb{k})$. The nilpotent derivation that leads to $C$ is inner, hence its effects can be removed by a change of conformal identity as in Lemma 2.8.

**Theorem 2.16.** Let $C$ be a semisimple unital finitely generated associative conformal algebra, $\text{GKdim} C = 1$. Then $C$ embeds into a direct sum of a current algebra over a semiprime algebra of zero or linear growth and $\text{Cend}_n$ for some $n$.

**Proof.** Let $C = \text{Diff} A$ be determined by a nilpotent derivation $\delta$. By Lemma 2.13, $A$ is semiprime. It follows from various lemmas in [SSW] that $A$ splits as $A = B \oplus F$ where $B$ is semiprime Goldie and $F$ finite dimensional.

We have $Q = Q(B) = \bigoplus_i Q_i$, a semisimple Artinian quotient algebra of $B$. By Lemma 2.14, $\delta$ restricts to $B$. The standard construction of $Q(B)$ implies that $\delta$ can be extended to $Q$ and we can again restrict it to $Q_i$ (though it is no longer locally nilpotent at this stage). We obtain a system of prime ideals $P_i = B \cap \bigoplus_{i \neq j} Q_j$. Clearly, $P_i$ is $\delta$-stable, hence so is $B/P_i$. As $B \leftrightarrow \bigoplus_i B/P_i$ with the action of $\delta$ preserved, we have $\text{Diff} B \leftrightarrow \bigoplus_i \text{Diff} B/P_i$. Also, $\text{Diff} A = \text{Diff} B \oplus \text{Diff} F$.

Thus, $\text{Diff} A$ embeds into a direct sum of prime conformal algebras of linear growth and a $\mathbb{k}[[\partial]]$-finite semisimple conformal algebra. We have two types of components in this sum: some come from subalgebras of $\text{Cend}_m$, hence their sum can be viewed as a subalgebra of $\text{Cend}_n$ for some $n$. Others are subalgebras of prime current algebras of growth not exceeding 1. Thus, their sum is a subalgebra of a semiprime current algebra.

3. Examples of subalgebras of prime unital conformal algebras

In this section we discuss what kinds of conformal algebras can live inside typical examples of prime conformal algebras.

The most innocently looking one is a current algebra over a prime algebra.

In the proof of Lemma 2.8 we relied on the simple fact that to each locally nilpotent derivation of $\text{Coeff} C$ corresponds a conformal identity and a canonical basis of $C$, hence the effect of the derivation could be “untwisted.” This is not necessarily true for all subalgebras of $C$.

**Remark 3.1.** Let $C'$ be a unital conformal subalgebra of a unital current algebra $C$ with the same conformal identity. Then $C'$ itself is current. Indeed, let $a = \sum_{n=0}^{n} \partial^{k} \tilde{a}_{k} \in C'$. Then $(-1)^{n!} \tilde{a}_{n} = a \otimes e \in C'$, hence, by induction all $\tilde{a}_{k} \in C$.

However, when conformal identities of the conformal algebra and its subalgebra are different, the approach in the above remark cannot be used. Such a situation arises in the setting of Theorem 2.12: let $C'$ be a subalgebra of $C = \text{Diff} A$, where the derivation on $A$ is inner, $\delta = \text{ad} a$. If $\tilde{a} \notin C'$, the change of conformal identity prescribed by Lemma 2.8
cannot be performed inside $C'$, thus $C'$ becomes a possibly noncurrent subalgebra of a current algebra $C$. Consider the following

**Example 3.2.** Let $A = \text{End}_2(k[x])$ and $\delta = \text{ad} e_1 e_2$ be a locally nilpotent derivation on $A$. Remark that $\text{Diff} A$ is current by Lemma 2.8. Let $B = \text{End}_2(x^0 k[x])$ with the identity adjoined. $B$ is $\delta$-stable, so $\text{Diff} B \subset \text{Diff} A$; however, we will show below that $\text{Diff} B$ is not current for any choice of conformal identity.

More generally, it turns out that whenever the derivation is external with respect to $C'$, $C'$ cannot be current. The following statement is, in some sense, the converse of Lemma 2.8.

**Lemma 3.3.** Let $C$ be an associative conformal algebra such that for different choices of conformal identities, $C \cong \text{Cur} A$ and $C \cong \text{Diff} B$ for a nilpotent derivation $\delta$. Then $\delta$ can be made inner on $A$, i.e., there exists $a \in A$ such that the conformal algebra $\text{Diff} A$ determined by $\text{ad} a$ is isomorphic to $\text{Diff} B$ with the isomorphism preserving the conformal identity. Moreover, $a$ is nilpotent.

**Proof.** We fix the following notations: $\tilde{a}, a \in A$, is the canonical basis of $\text{Cur} A$ and $\tilde{e}$ is its conformal identity. For $\text{Diff} B$, $\tilde{b}$ is the canonical basis and $\tilde{e}$ the conformal identity.

We also identify the elements of $\text{Cur} A$ and $\text{Diff} B$ via the given isomorphism. Thus, we have $\tilde{e} = \sum a^i \tilde{e}_i$ for some $e_i \in A$. For an arbitrary $b \in B$, $\tilde{b} = \sum a^i \tilde{b}_i$. Since $\tilde{b} = \tilde{b} \circ \tilde{e}$, we have $\tilde{b} = (\sum a^i \tilde{b}_i) \circ (\sum a^i \tilde{e}_i) = \sum a^i \tilde{b}_0 \tilde{e}_i$. Thus, $b_i = b_0 e_i$.

In particular this implies that if $b \neq 0$, then $b_0 \neq 0$. Moreover, $(bb')_0 = b_0 b'_0$ as $\tilde{b} \circ \tilde{b}' = \tilde{b} \circ \tilde{b}'$. This establishes a map $B \rightarrow A$, $b \mapsto b_0$. Now we have to show that it extends to a map of given conformal algebras.

We also remark that as $\tilde{b} = \tilde{e} \circ \tilde{b}$, we see that $e_0 b_0 = b_0$.

According to (1.2), $\delta(b) = -\tilde{e} \circ \tilde{b}$ for any $b \in B$. We will now calculate $\tilde{e} \circ \tilde{b}$ in $\text{Cur} A$:

\[
\tilde{e} \circ \tilde{b} = (\sum a^i \tilde{e}_i) \circ (\sum a^i b_0 \tilde{e}_i) = \tilde{e}_0 \circ (\sum a^i b_0 \tilde{e}_i) - \tilde{e}_1 \circ (\sum a^i b_0 \tilde{e}_i) = \sum a^i (\tilde{e}_0 \circ b_0 \tilde{e}_i) + \sum i a^{i-1} (\tilde{e}_0 \circ b_0 \tilde{e}_i) - \sum a^i \tilde{e}_1 \circ b_0 \tilde{e}_i = \sum i a^{i-1} \tilde{e}_0 b_0 \tilde{e}_i - \sum a^i \tilde{e}_1 b_0 \tilde{e}_i
\]

(3.1)

(The last equality is valid, as we are working with a current algebra: products of positive orders of basis elements are 0).

On the other hand,

\[
\delta(b) = \sum a^i \delta(b)_0 \tilde{e}_i. \quad (3.2)
\]
Comparing the coefficients at $\partial^0$ in (3.1) and (3.2), we see that $\delta(b) e_0 = -e_0 b_0 e_1 + e_1 b_0 e_0$. Since $\delta(b) e_0 = \delta(b)$, we see that $\delta(b) = \text{ad}(e_1) b_0$. In particular, this implies that $\text{ad}(e_1)$ is nilpotent.

Consider now a differential algebra $\text{Diff} A$ over $A$ determined by $\text{ad}(e_1)$. By the above, we obtained an injective map $(B, \delta) \to (A, \text{ad}(e_1))$ of differential algebras; therefore, $\text{Diff} B$ embeds into $\text{Diff} A$ with the embedding given by $b \mapsto \tilde{b}$.

It remains to show that such an embedding is surjective. Since $\text{Cur} A \cong \text{Diff} A$, it is possible to express $\tilde{a} = \sum_j \partial^j \tilde{a}_j$, where $a_j \in B$. Hence, $\tilde{a} = \sum_{i,j} \partial^i \partial^j (\tilde{a}_j)_i$. As $C$ is free over $k[a]$, $a = (\tilde{a}_0)_0$. This shows that the constructed map is an isomorphism.

For our purposes, we need also to show that $e_1$ is nilpotent. Substitute $\tilde{b} = e_1$ in (3.1). The coefficient at $\partial^j$ in the last line is $e_{j+1} - e_1 e_j$. Since $\tilde{e} \text{ad} e = 0$, we obtain by induction that $e_j = (e_1)^j$ for $j \geq 1$. As the expression for $\tilde{e}$, $\sum \partial^i \tilde{e}_i$, is a finite sum, we see that $e_1$ is nilpotent. This completes the proof.

**Corollary 3.4.** Let $A'$ be a subalgebra of $A$ and $a \in A$ a nilpotent element. Then the conformal algebra $\text{Diff} A'$ determined by $\text{ad} a$ is current if and only if there exists $a' \in A'$, $\text{ad} a = \text{ad} a'$.

**Proof.** If such $a'$ exists, the statement follows from the proof of Lemma 2.8.

Otherwise, assume that $\text{Diff} A'$ is current for some choice of conformal identity. By Lemma 3.3, $\text{Diff} A'$ is determined by an inner derivation (for the same choice of conformal identity). Hence, there exists such $a'$. 

This shows that the subalgebra in Example 3.2 is indeed never current.

**Remark 3.5.** Such examples can be constructed with ease. In particular, this means that the classification in Theorem 2.12 is the best possible for the case of prime conformal algebras.

Just as subalgebras of current algebras, subalgebras of $\text{Cend}_n$ also appear naturally. However, this happens either when such a subalgebra acts on a given finite module or in the context of Theorem 2.16. In either case the conformal identities of $\text{Cend}_n$ and its subalgebra coincide. Thus, we essentially speak of subalgebras of $\text{End}_n(k) \otimes k[x]$ (with the same identity).

Such subalgebras are too general to describe (cf. Theorem 2.1), even in the prime case [SW]. The only known result is that on differentiably simple subalgebras, i.e., Theorem 1.9 and Corollary 2.9, and here the resulting conformal subalgebras are isomorphic to either $\text{Cend}_m$ or $\text{Cur End}_m(k)$.

The case of simple subalgebras is obviously important for conformal representation theory. Kac’s conjecture [K2] describes all subalgebras of $\text{Cend}_n$ that act irreducibly on the standard module $k[\partial]^n$. The unital conformal algebras from the list are the ones described above; therefore, we can say that our results in this section confirm Kac’s conjecture for unital algebras (see also [BKL1]).
Note added in proof

After this article was submitted, P. Kolesnikov announced the complete proof of Kac’s conjecture.

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