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Poincare's Theorem and Uncertainty in Classical Mechanics

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In a recent paper (Brillouin, 1960) the present author emphasized the great importance of a famous Poincaré theorem (1892). This theorem was first stated for celestial mechanics, but it extends immediately to all problems of classical mechanics, provided such problems be conservative.

It was explained that these problems may (and generally will) exhibit discontinuous changes whenever any frequency becomes zero. Under such conditions, classical mechanics is unable to yield a definite answer and determinism becomes impossible.

To make the whole situation clear, a great variety of examples have been selected. For all of them there can be no determinism but only some statistical answer, when *experimental errors in the measurement of initial conditions* are taken into account. Once again the warning should be repeated: mathematicians too often assume "given initial conditions" as if these conditions could be exactly and rigorously defined. Such a point of view is absolutely untenable for physicists who deal only with experimental results known within empirical errors. This viewpoint is in full agreement with Max Born.

The Poincaré discontinuities also play an important role in applied mechanics or wave mechanics. Only statistical answers and Information Theory are valid in these problems.

I. INTRODUCTION

Poincaré's famous theorem was published in Volume I, Chapter V of his "Méthodes Nouvelles de la Mécanique Céleste" (1892); its fundamental result is:

In the most general conservative problem, the canonical equations of classical mechanics do not admit any analytical and uniform integral besides the energy integral.

The great importance of this theorem is well-known; it was recently

* Work done under contract Nonr 266(56).

re-emphasized by the present author (Brillouin, 1960) who showed that this condition results in discontinuities in the solutions of the Hamilton-Jacobi method; it applies to any kind of solution, and it may be explained by the following statement:

For a given mechanical problem with energy conservation and no dissipation, one may find a few variables that can be separated away from the system. When this has been done, he is left with the hard core of nonseparable variables. This is where the Poincaré theorem applies, and specifies that the total energy is the only expression represented by a well-behaved mathematical function. Many other quantities may appear as "constants" of a certain given motion, but they cannot be expressed as analytical and uniform integrals. This means that any kind of modifications in the definition of the problem may provoke an abrupt and sudden change of the "constants." This discontinuity may be the result of a very small change of any parameter in the mechanical equations, or also of any small change in the initial conditions. Max Born quoted this discussion of the present author in a very interesting paper published in the Jubilee Book offered to W. Heisenberg for his 60th birthday (Born, 1961).

Poincaré's original theorem applies to a mechanical *conservative* system (no dissipation, no feedback, no internal source of energy). If any parameter entering the equations is varied, then one might think of power series expansions with respect to this parameter, that should represent the perturbed motion, but according to Poincaré these power series will not be analytic and continuous, and uniform convergence will not obtain. Hence the possibility of discontinuities or sudden changes.

This situation, of course, corresponds to practical unstability. It is physically impossible to measure accurately all the parameters defining the system. The mathematician speaks of a parameter with "a certain given value" and admits that this value is strictly defined, without any possible error. The physicist knows that nothing can be measured accurately, that our theories, even the best ones, are only acceptable within certain limits and that we have to take into account errors of measurements in all possible applications. Under such conditions, the above-mentioned unstability corresponds to a situation where the theory is unable to yield a well-defined solution.

This general point of view was emphasized in the author's books (Brillouin, 1959, 1961) and many recent papers (Brillouin, 1957, 1959, 1960).

II. THE HAMILTON-JACOBI METHOD

Let us recall a simple example and consider the manner in which Born uses it (Born, 1925) in atomic problems. We take a conservative mechanical system, without friction and without energy supply. The total energy is the sum of kinetic and potential energies:

$$E_{\text{tot}} = E_{\text{kin}} + E_{\text{pot}} \tag{1}$$

Let us call $x_1 \cdots , x_K \cdots , x_N$ the variables, and $p_1 \cdots , p_K \cdots , p_N$ the corresponding momenta of the system under consideration; these quantities, moreover, may depend on a certain number of parameters $\alpha_1 , \alpha_2 \cdots , \alpha_n$ capable of varying from one case to the other. The initial conditions $x_{10} \cdots , x_{N0}$, and $p_{10} \cdots , p_{N0}$ are not rigorously "given" but we have been able to *determine* them experimentally with errors $\Delta x_{10} \cdots , \Delta x_{N0}$, and $\Delta p_{10} \cdots , \Delta p_{N0}$.

The Hamilton-Jacobi method seeks a transformation enabling us to replace the variables x_K , p_K by some new variables W_K , J_K (Delaunay) which satisfy the following conditions:

$$E_{\text{tot}} = F(J_1 \cdots J_K \cdots J_N) \tag{2}$$

The total energy is a function only of the new momenta $J_1 \cdots J_N$ which remain constant while the angular variables W_K have linear dependence on time:

$$W_K = \nu_K t + W_{0K} \qquad \nu_K = \frac{\partial F}{\partial J_K} \tag{3}$$

On the whole, we obtain a multiperiodic motion, with partial periods ν_K. We refer the reader to the classical book by Born (1925) for all additional explanations. This multiperiodic motion seems to present a character of stability, as Poincaré pointed out. Each coordinate x_K is defined by a relation:

$$x_K = f_K(W_1 \cdots W_N) \tag{4}$$

with a periodic function f_K (period, 2π) with respect to all W_K's. No physical coordinate x_K can increase till infinity, which is a character of stability.

All right, but! the *quantities* J_K , W_{0K} *are not continuous functions* of the x_{0K} , p_{0K} nor of the α_m. The smallest variation of the parameter α_m or the smallest change in the initial values may bring on discon-

tinuous variations of the J_K, W_{0K} and lead to a solution completely different from the previous one. And this is without contest a special type of unstability, imposed by Poincaré's theorem, though incompletely studied and not fully understood in all its consequences.

Let us go back to formula (2) and emphasize the paradox: the total energy is always a continuous function of the parameters of the movement. But the Hamilton-Jacobi method defines this energy as a function F of the quantities J_K which are not continuous. This looks like a strange challenge: it is necessary that the function F present discontinuities which exactly compensate those of the J's. What is then the meaning of the derivatives $\partial F/\partial J_K$ which yield frequencies ν_K? It seems very mysterious. It is not exaggerated to say that the Hamilton-Jacobi method has some very definite shortcomings. Its very elegant formalism is spoiled by Poincaré's discontinuities.

III. CONDITIONS OF DISCONTINUITY AND CASES OF RESONANCE

These general remarks leave an unfortunate impression and some examples are necessary to understand the exact meaning of the preceding statements.

First of all, let us specify *the conditions when* discontinuities may occur. Born's discussion (1925), Poincaré's theorem, and their comparison (Brillouin, 1960) clearly show that discontinuities may appear whenever a compound frequency is zero:

$$\nu_n = \sum_K n_K \nu_K = 0 \tag{5}$$

where the n_K's are positive or negative integers. Some authors believed it possible to overcome the difficulty by saying that everything is all right as long as the frequencies are incommensurable. But this is not acceptable in a real physical problem where all quantities (ν_K's included) are defined only within certain possible errors $\Delta\nu_K$. Moreover, it is sufficient that a certain compound frequencies ν_K be very small, without being identically naught, for divergent series to appear. These series may be semiconvergent and seem perfectly sufficient to yield approximate solutions, but they cannot supply an exact mathematical solution.

Let us note, for more precision, that condition (5) contains cases of *internal resonance*. Let us consider a system depending upon two angular variables W_1, W_2 and therefore two frequencies ν_1, ν_2. Condition (5)

is thus written:

$$n_1\nu_1 = -n_2\nu_2 \tag{6}$$

Let us take $n_2 = -n_2'$ with n_2' positive, and we obtain a condition of internal resonance between harmonic n_1 of ν_1 and harmonic n_2' of ν_2. The internal resonance gives a typical case of unstability through discontinuity.

IV. ONE DEGREE OF FREEDOM AND ONE SINGLE FREQUENCY EQUAL TO ZERO

The first example to consider corresponds to a problem with one single variable and one single frequency ν. This frequency may depend upon initial conditions and also upon a certain number of parameters in the definition of the problem. We are thus led to consider the meaning of the condition:

$$\nu = 0 \tag{7}$$

Let us immediately take an example: the *circular pendulum.*

A rigid pendulum is sustained by an axis around which it can turn. When slightly touched, the pendulum oscillates on both sides of the vertical. Violently thrown, it will turn indefinitely. The discontinuity occurs when the pendulum has been thrown with just enough energy to reach the top and stay in the unstable vertical position. Will it then fall to the left or will it fall to the right? In one case, it will start a series of long oscillations; in the other case, it will turn indefinitely (while slowing almost to zero each time it reaches the top). The top position yields an infinite period and zero frequency.

$$\nu = 0 \qquad \tau = \infty$$

This is a typical example of condition (7).

We discussed a variety of one-dimensional problems on previous occasions (Brillouin, 1959) where we found it useful to draw some figures, representing the momentum p as a function of the variable θ. In the present problem, the potential energy is represented by the curve of Fig. 1, where V_1 represents the potential energy for the vertical pendulum.

One type of motion yields small oscillations (A) about the origin 0 (Fig. 2). The second type of motion (B) corresponds to a continuous

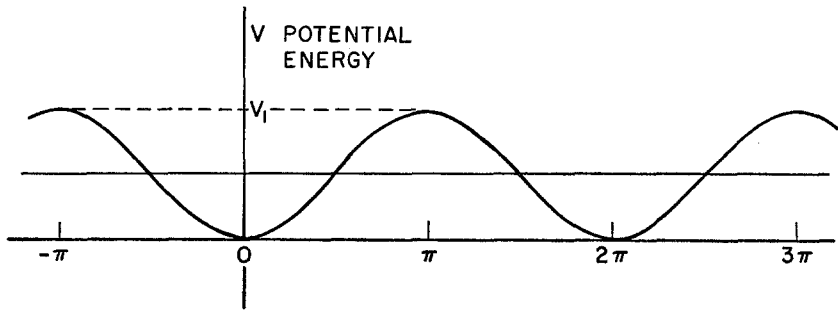

FIG. 1

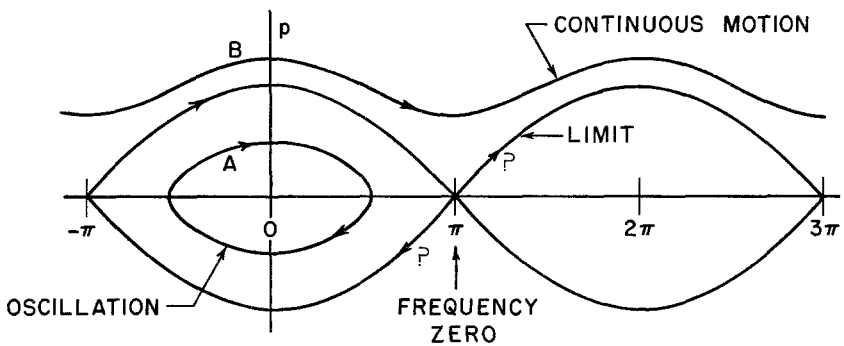

FIG. 2

increase of θ. The limiting curve is represented by motions where p is going to zero at $\theta = \pi, 3\pi, 5\pi$ and this motion has zero frequency.

Instead of the pendulum we may consider a ball running through a system of periodic hills and valleys represented by Fig. 1 (change θ to x and replace π by the distance from hill to valley). The motions of Fig. 2 correspond to oscillations within one valley (case A) or long distance motion through valleys and up above the hills (case B). The limiting case is that of a ball rolling up the hill and standing still, in this unstable position.

Figure 1 corresponds to an infinite succession of equidistant valleys and hills. We may equally well have only a finite number of valleys, with potential walls on both sides. Figure 3 shows two valleys with high potentials left and right. The motion A yields oscillations about the

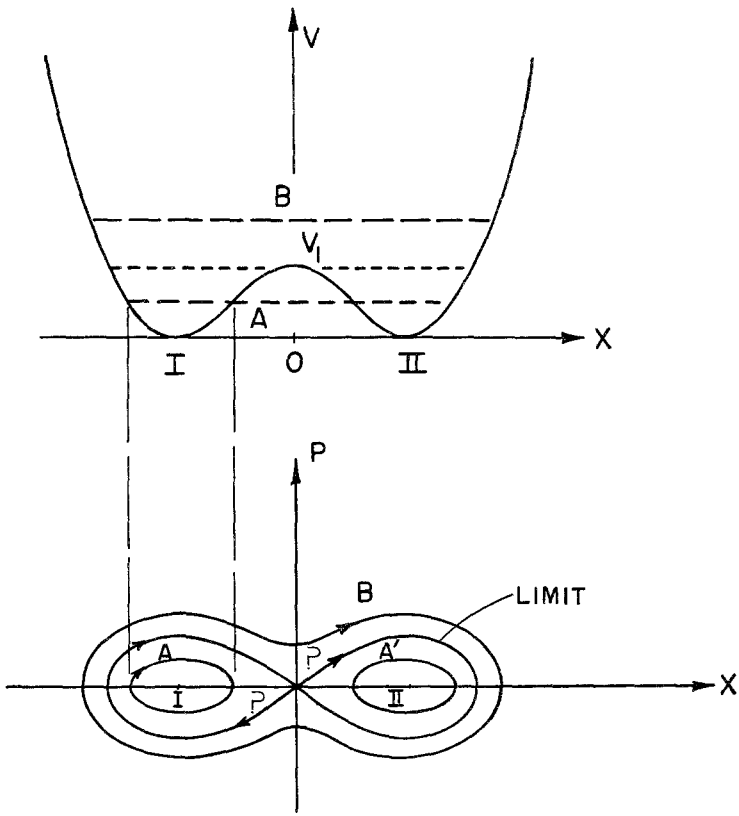

FIG. 3
FIG. 4

bottom of the valley on the left, while A′, with the same energy, oscillates in the valley on the right side. For energy just equal to V_1 (saddlepoint) the p, x trajectory has the ∞ shape; the velocity (or p) is zero at point 0, which means $\tau = \infty$ and $\nu = 0$. For higher energies (B) the diagram runs around both points I and II; it oscillates within the two valleys, from the left wall to the right wall. There is discontinuity at the ∞ shaped curve, of energy V_1.

In most physical problems, the potential energy drops to zero at infinite distance, when interaction between particles is naught. This yields a potential curve of the type drawn on Fig. 5. The point at in-

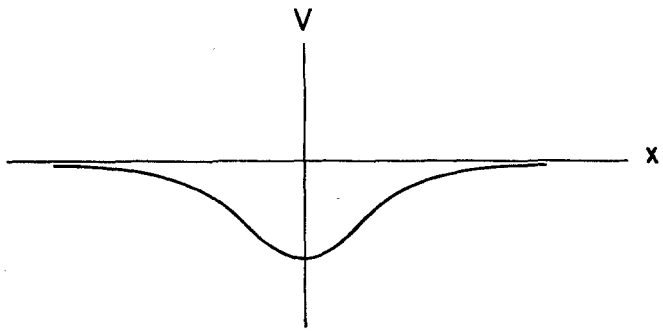

FIG. 5

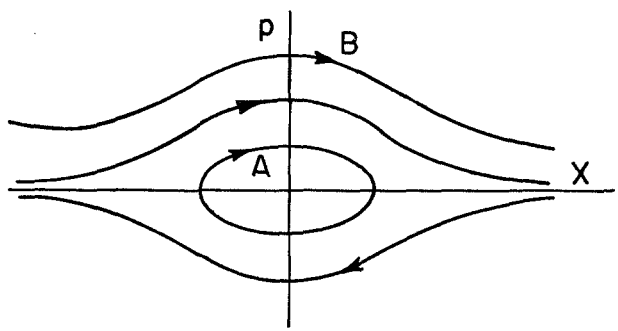

FIG. 6

finity is a maximum of potential, and zero energy corresponds to a Poincaré discontinuity (Fig. 6). The saddlepoint 0 of Fig. 4 is here removed to infinite distance.

Each point, in the p, x diagrams, where the trajectories intersect (point π on Fig. 2, or 0 on Fig. 4 or the point at infinity on Fig. 6) is a branching point where the trajectory may go one way or another (Fig. 7). Let us assume, as we did previously (Brillouin 1957, 1959) that initial conditions are not exactly given but have been experimentally measured with some degree of uncertainty. Initial data are contained in a small area α, which moves to the branching point and splits there in two parts, one part β following one trajectory and the other part β' going the opposite way. This is the physical meaning of Poincaré's discontinuities.

We may also understand from these figures, how it happens that the total energy E is a continuous function of the initial data, while the

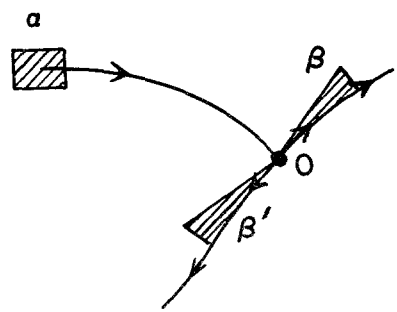

FIG. 7

integral J is not continuous. We emphasized this curious result at the end of Section II.

Let us consider Fig. 4 and assume the motion to start from the left part of the x-axis:

$$x < x_I \qquad p = 0$$

The first trajectories are of type A and the integral

$$J = \int_0 p\, dx \tag{8}$$

measures the area enclosed in the A trajectory. There is obviously a sudden change of this area when we pass from trajectory A to trajectory B, and this does not yield any sudden change in energy E.

These new results complete and modify the description given in previous papers (Brillouin, 1959) and give a much greater importance to uncertainties in the definition of the motion. The *Poincaré discontinuities* correspond to conditions where *prediction is actually* impossible and determinism cannot exist.

V. MOTIONS IN SPACE

Problems in two or three dimensions lead to many generalizations of the previous results. To be realistic we must consider actual physical problems where interactions between particles tend to zero at infinite distance, and the potential energy at infinity can be taken as zero. Any trajectory reaching infinity with zero velocity is thus a Poincaré limit trajectory, and we have an infinite variety of such trajectories. Speaking of astronomy and trajectories about an attractive center (say, the sun), we note the existance of periodic motions (Kepler's ellipses);

the larger they become, the lower their frequency; in the limit, the ellipse reaching infinity is a *parabola*, of *frequency zero*, and marks the border between ellipses and hyperbolas. This is a typical example of instability to keep in mind.

In addition to these infinite trajectories, we may have all sorts of other complications when we consider many attractive centers instead of just one. Some saddlepoints of the potential energy will appear, each of them giving rise to trajectories of zero frequency, which represent the limit between different types of orbits: some orbits circle about one center, others move around two or more centers of attraction. We must emphasize again the sharp distinction between systems with (or without) *separation of variables.* This point had already been clearly indicated in a former paper (Brillouin, 1960) where the question was discussed in Sections 9 to 12, especially at the end of Section 12.

When variables can be *separated,* each one of them behaves like a system with one single variable, and the discussion of Section IV applies. When variables *cannot be separated,* the so-called general solution with Hamilton-Jacobi methods appears to be very artificial, and we must candidly admit that we cannot solve such problems exactly (I do not know of any such problem ever solved!) and that our series expansions for successive approximations are not too reliable. Poincaré's theorem definitely proves that these series expansions do not converge. At best they can be semiconvergent. Assuming this to be the case, we can compute approximate solutions for any limited time interval, but a strictly rigorous solution cannot be obtained. We may compute with a reasonable approximation what will happen to the solar system for one century, but we do not know the outcome for an infinite time interval.

There is *no strict determinism* in classical mechanics especially when we keep in mind the most important fact: the system is not "given" but "*measured with limited accuracy*" over past periods of time. There is always some experimental uncertainty on initial conditions and this uncertainty grows up progressively in future times.

We may return to a problem in two dimensions and the condition (6) for internal resonance:

$$n_1 \nu_1 = n_2' \nu_2 \tag{6}$$

Figure 8 visualizes the situation and corresponds to the well-known "Lissajoux curves." Coordinate x_1 oscillates from 0 to a_1, and back (period τ_1) while x_2 varies from 0 to a_2 with a period τ_2. The case

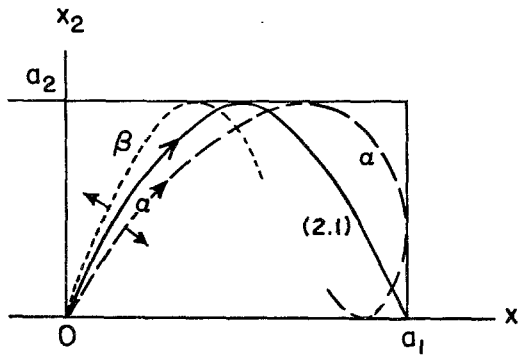

FIG. 8

$n_1 = 2$, $n_2 = 1$ of Eq. (6) means

$$\tau_1 = 2\tau_2 \qquad 2\nu_1 = \nu_2$$

When τ_1 is smaller, one gets the α curve; if on the contrary τ_1 is larger, curve β is obtained. In one case, the curve moves away from the original curve, to the right. In the other case, it moves to the left. Many similar problems may be found in atomic physics, with "rosette motions." One perturbation provokes a precession to the right; while an opposite perturbation yields precession to the left.

Furthermore, a condition (6) cannot be considered too strictly, as a mathematician would do. Frequencies ν_1 and ν_2 are experimentally determined with limited accuracy; their ratio also is known with a limited number of digits. Hence it can always be approximated by a decimal fraction. A mathematician would say that the ratio must be irrational, in order to prevent degeneracy (6), but an experimental measurement cannot distinguish irrational from rational.

Let us think of a TV spot scanning the screen. When the distance between lines is smaller than the diameter of the spot, it becomes impossible to say whether a relation (6) is satisfied or not!

VI. COUPLED OSCILLATORS

We explained in Section III (Eqs. (5) and (6)) that we had to beware of internal resonances. This can be best observed on systems of coupled oscillators. Let us first summarize briefly the well-known problem of two *coupled linear oscillators*. We assume a potential energy V and a kinetic

energy E_k :

$$\begin{aligned} V &= \tfrac{1}{2}a_{11}x_1^2 + a_{12}x_1x_2 + \tfrac{1}{2}a_{22}x_2^2 \\ E_k &= \tfrac{1}{2}m_{11}\dot{x}_1^2 + m_{12}\dot{x}_1\dot{x}_2 + \tfrac{1}{2}m_{22}\dot{x}_2^2 \end{aligned} \tag{9}$$

where x_1 and x_2 are the variables of both oscillators, while a_{12} and m_{12} represent *static and kinetic coupling* terms.

Equations of motion are easily obtained:

$$\begin{aligned} \left(m_{11}\frac{\partial^2}{\partial t^2} + a_{11}\right)x_1 + \left(m_{12}\frac{\partial^2}{\partial t^2} + a_{12}\right)x_2 &= 0 \\ \left(m_{12}\frac{\partial^2}{\partial t^2} + a_{12}\right)x_1 + \left(m_{22}\frac{\partial^2}{\partial t^2} + a_{22}\right)x_2 &= 0 \end{aligned} \tag{10}$$

We look for a "proper" vibration of frequency ω; hence we simply replace $\partial^2/\partial t^2$ by $-\omega^2$ and get

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 &= 0 \\ A_{12}x_1 + A_{22}x_2 &= 0 \end{aligned} \qquad A_{ik} = A_{ki} = -m_{ik}\omega^2 + a_{ik} \tag{11}$$

The linear equations (11) have a solution when the determinant is zero.

$$\begin{aligned} 0 = |A| &= A_{11}A_{22} - A_{12}^2 \\ &= m_{11}m_{22}[(\Omega_{11}^2 - \omega^2)(\Omega_{22}^2 - \omega^2) - K^2(\Omega_{12}^2 - \omega^2)^2] \end{aligned} \tag{12}$$

with

$$\Omega_{ik}^2 = \frac{a_{ik}}{m_{ik}} \quad \text{and} \quad K^2 = \frac{m_{12}}{m_{11}m_{22}}$$

Ω_{11} and Ω_{22} are the proper frequencies of uncoupled oscillators and K represents the coupling coefficient. The vibrations in the coupled system are not localized in one single oscillator; on the contrary both oscillators are vibrating together

$$\begin{aligned} x_1 &= X_1 \cos \omega t \qquad & x_2 &= X_2 \cos \omega t \\ -\frac{X_1}{X_2} &= +\frac{A_{12}}{A_{11}} = +\frac{A_{22}}{A_{12}} \end{aligned} \tag{13}$$

Internal resonance occurs when Ω_{11} and Ω_{22} are almost equal. The net result is best seen on the graph of Fig. 9. The frequency ω in the coupled system is plotted as a function of Ω_{22} (proper frequency 2 without

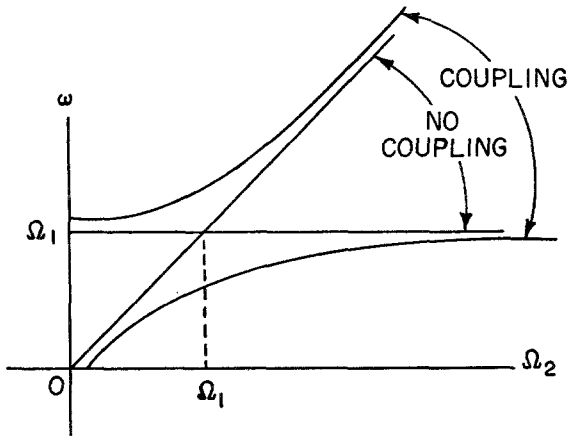

FIG. 9

coupling). When coupling K is zero, the plot consists of the horizontal line ($\omega = \Omega_{11}$) and the line at 45° ($\omega = \Omega_{22}$). Both lines intersect at

$$\omega = \Omega_{11} = \Omega_{22}$$

With coupling ($K \neq 0$) the frequency is given by Eq. (12) and is represented by two hyperbolic looking curves which do not intersect anymore. When Ω_{22} is equal to Ω_{11}, the frequency ω in Eq. (12) is given by

$$\Omega_{11} = \Omega_{22} \qquad \omega^2 = \Omega_{11}^2 \pm K(\Omega_{12}^2 - \omega^2) \tag{14}$$

hence

$$\omega^2 = \frac{\Omega_{11}^2 \pm K\Omega_{12}^2}{1 \pm K}$$

These formulas yield two frequencies ω, ω', one above and the other one below Ω_{11}. For one of these frequencies, oscillations are in phase (X_1 and X_2 positive) while opposition of phases obtains for the other frequency ($X_1 > 0$, $X_2 < 0$). If the system is excited without special care, both ω and ω' values will be excited together, and the motion will exhibit *beats* (or modulation) at frequency $\omega - \omega'$. This is schematically explained in Fig. 10. Experimental textbooks describe these beats under the nickname of "Sympathetic pendulums."

We may now sketch what happens when *one of the oscillators*, the

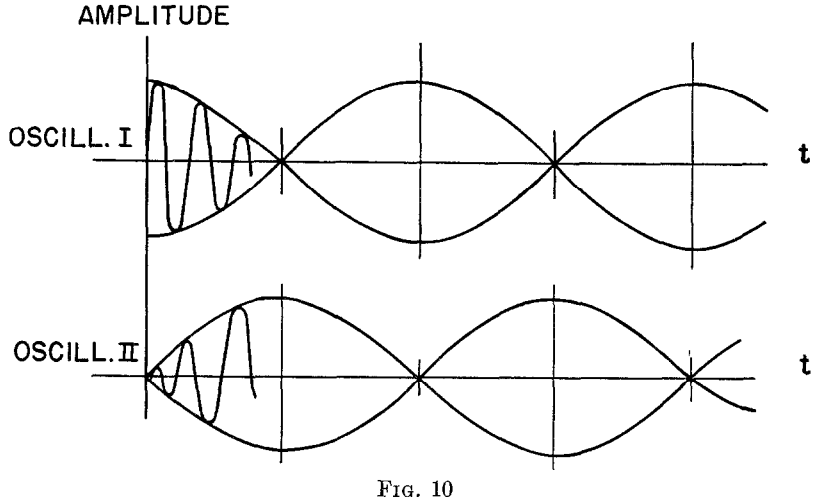

FIG. 10

first one, for instance, is *nonlinear*. This means replacing Eq. (10) by

$$\begin{aligned} \left(m_{11}\frac{\partial^2}{\partial t^2}+a_{11}\right)x_1+\left(m_{12}\frac{\partial^2}{\partial t^2}+a_{12}\right)x_2 &= \gamma\varphi(x_1) \\ \left(m_{12}\frac{\partial^2}{\partial t^2}+a_{12}\right)x_1+\left(m_{22}\frac{\partial^2}{\partial t^2}+a_{22}\right)x_2 &= 0 \end{aligned} \tag{15}$$

where φ can be any function of x_1, and γ measures the amount of nonlinearity. We shall not discuss the general problem, but we shall assume small coupling and very small nonlinearity.

$$K \text{ small} \qquad \gamma \text{ very small} \qquad \gamma^2 \text{ negligible} \tag{15a}$$

The function φ can be expanded in power series

$$\varphi = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_1^2 + \alpha_3 x_1^3 \cdots$$

If x_1 is $X_1 \cos \omega t$ then x_1^2 gives a term in $X_1^2 \cos 2\omega t$, x_1^3 yields a term in $X_1^3 \cos 3\omega t \cdots$. As an example, let us assume

$$\varphi = \alpha_0 + \alpha_2 x_1^2 \tag{15b}$$

We may try the following system of successive approximations: For the zero order, we set $\gamma = 0$ and solve the system (10) as previously discussed. We select one of the two possible frequencies, say ω, and

obtain

$$x_{10} = X_1 \cos \omega t \qquad x_{20} = X_2 \cos \omega t \tag{16}$$

as in (13).

This is our zero order approximation with *dominant frequency* ω. For the *first order* approximation we use x_{10} in φ and assume a small γ coefficient

$$\varphi = \alpha_0 + \alpha_2 X_1^2 \cos^2 \omega t = \alpha_0 + \tfrac{1}{2}\alpha_2 X_1^2 + \tfrac{1}{2}\alpha_2 X_1^2 \cos 2\omega t \tag{17}$$

This expression appears on the right-hand side of Eq. (15). The constant term $(\alpha_0 - \tfrac{1}{2}\alpha_2 X_1^2)$ is of no special interest, and the term in $2\omega t$ will excite vibrations of frequency 2ω in x_1 and x_2. We thus look for solutions x_{11} and x_{21} in this first order approximation

$$\begin{aligned} x_1 &= x_{10} + x_{11} \qquad x_{11} = Y_1 \cos 2\omega t \\ x_2 &= x_{20} + x_{21} \qquad x_{21} = Y_2 \cos 2\omega t \end{aligned} \tag{18}$$

and we have linear equations with right hand term:

$$\begin{aligned} A_{11}(2\omega) Y_1 + A_{12}(2\omega) Y_2 &= \tfrac{1}{2}\gamma\alpha_2 X_1^2 \\ A_{21}(2\omega) Y_1 + A_{22}(2\omega) Y_2 &= 0 \end{aligned} \tag{19}$$

When the determinant is not zero, with

$$\begin{aligned} A_{ik}(2\omega) &= -4m_{ik}\omega^2 + a_{ik} \\ &= m_{ik}[\Omega_{ik}^2 - 4\omega^2] \\ |A_{ik}(2\omega)| &\neq 0 \end{aligned} \tag{20}$$

this set of equations is easily solved

$$Y_1 = \frac{\tfrac{1}{2}\gamma\alpha_2 A_{22} X_1^2}{|A|} \qquad Y_2 = -\frac{\tfrac{1}{2}\gamma\alpha_2 A_{12} X_1^2}{|A|} \tag{21}$$

We thus obtain small corrections $Y_1 Y_2$ of order γ to be used in Eqs. (18).

This situation however changes completely when the determinant is very small

$$|A(2\omega)| \approx 0 \tag{22}$$

We already discussed the determinant $|A|$ in Eq. (12) and called ω, ω' its two roots. We may use either ω or ω' in the zero order approxima-

tion in these two cases we have the following

$$\begin{aligned} &\text{using } \omega \qquad |A(2\omega)| = 0 \qquad \text{giving } 2\omega = \omega' \\ &\text{using } \omega' \qquad |A(2\omega')| = 0 \qquad \text{giving } 2\omega' = \omega \end{aligned} \tag{23}$$

These conditions represent internal resonances between one of the frequencies and the second harmonic of the other frequency. This anomaly corresponds exactly to the Poincaré condition (6). It requires a special treatment because Y_1 (or Y_2) may become of the same order of magnitude as X_1 (or X_2), and our assumption of one single dominant term is no more acceptable.

This problem has been discussed more completely by Magiros (1960, 1961) who has been able to obtain solutions for these abnormal conditions. Discontinuities again result from the Poincaré condition in this example.

In all problems of vibration, it is a general result that *coupling between oscillators results in an increase of the frequency differences.* In other words, frequencies seem to be repelling each other.

VII. SOME EXAMPLES IN ASTRONOMY

A. First Example: Case of Collisions

This example was discussed by Poincaré himself (Poincaré, 1892) and leads to obvious discontinuities: a particle A may pass a particle B without collision, and normally follow its way; or A may strike B and be deviated. The two motions are completely different, and the smallest change in the initial conditions is sufficient to pass from one trajectory to another.

B. Satellite Near the Earth

The earth is not a sphere but has the shape of a flat ellipsoid. The trajectory of an artificial satellite has been computed by a method of successive approximations by Brouwer (1959) who found an unstable motion for the inclination of 63°. A special case, permitting a separation of the variables has been discovered by Vinti (1959, 1961) who obtained no unstable trajectory except for the ones passing by the poles. There is a lack of convergence in the development computed by Brouwer when the inclination reaches 63° and this anomaly corresponds to an internal resonance (similar to Eq. (6)) between the frequencies of the zero order approximation. This internal resonance is shifted to a different inclination in the first order approximation, and is again shifted

in the following approximations; it would certainly reach the polar trajectory in the final stage.

In other words, this is a case where a relation

$$n_1\nu_{10} = n_2'\nu_{20} \tag{24}$$

where ν_{10}, ν_{20} are zero approximation frequencies, is progressively changed into

$$n_1\nu_1 = n_2'\nu_2$$

where $\nu_1\nu_2$ are actual final frequencies.

This change was predicted in the author's paper (Brillouin, 1960). This type of problem was discussed in general terms in Section 9 (p. 87, especially footnote 6) and it was emphasized that the actual Poincaré condition should contain the frequencies ν_k of the final problem, while the discussion based on a set of successive approximations seemed (by mistake) to contain the original unperturbed frequencies ν_{k0}. It is obvious, at first sight, that any condition based on the arbitrary model selected for the zero order approximation should be modified after approximations of higher order have been introduced.

In all cases, what we had foreseen actually occurs: a very small modification in the initial conditions or in the structure of the system is enough to produce a finite change of the trajectories. Such is the kind of discontinuity which corresponds to the absence of uniform convergence demonstrated by Poincaré.

These anomalies make it impossible to predict exactly the behavior of the system for any distant time, and it leads directly to statistical mechanics. We may repeat here an example given by Borel a long time ago (1914).

A displacement of 1 cm for a mass of 1 gm located on the star Sirius results in a variation of the earth field of gravitation greater than 10^{-100}. This seems negligible at first sight, but such a perturbation gives us the possibility to compute the motion of the molecules of a gas for a duration of about 1 millionth of a second; prediction for any longer time becomes impossible. The example chosen by Borel supplies him with a wonderful justification for the statistical methods of Boltzmann; but at the same time, it raises the problem of cases of unstability, less obvious but no less real. Those are the examples which Poincaré's conditions clearly specify.

VIII. PROBLEMS OF APPLIED MECHANICS

All the problems discussed in the preceding sections corresponded to the usual situation in classical mechanics: no friction, no viscosity, no damping of any sort. This applies to celestial mechanics, and this assumption is presupposed in Poincaré's papers. The conditions of unstability discovered by Poincaré have, however, a far-reaching validity, and extend to problems of applied mechanics, where all sorts of damping terms (positive or negative) may be essential. We will not discuss such problems in details; they can be found in many books or papers on mechanical vibrations. Let us only sketch some of the mechanical examples and show how internal resonances yield unstability.

Coupled oscillators without resistances were discussed in Section VI; but actual oscillators are always damped. Continuous oscillations can be generated if the first oscillator contains a *negative resistance* (sustaining the motion) while the second oscillator is damped by a positive resistance. This problem can be discussed in general terms and leads to results sketched in Figs. 11 and 12. Figure 11 corresponds to small coupling; the hyperbolic curves obtain for zero resistance and the heavy curve represents the frequency of sustained oscillations. The ratio of intensities in both oscillators is plotted on the lower curve. If

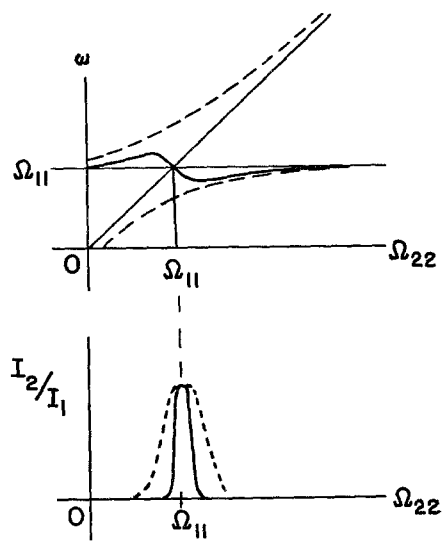

FIG. 11

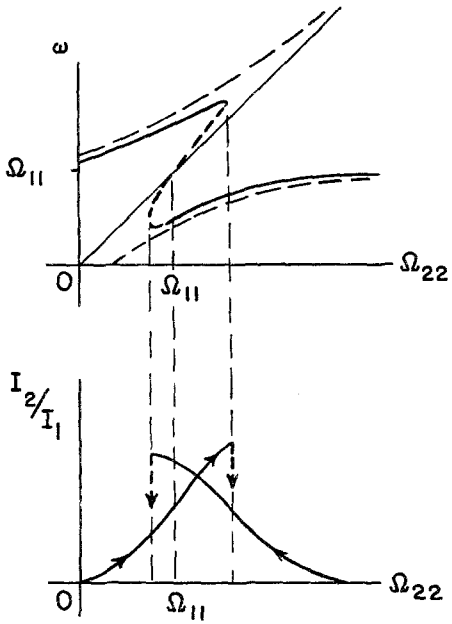

FIG. 12

the frequency of oscillations kept a constant value Ω_1 we would obtain a correct resonance curve (dotted line), but the reaction of the second oscillator on the first one provokes a displacement of the frequency ω which seems to be repelled away from Ω_2. As a result, the actual resonance curve looks sharper than normal.

Figure 12 corresponds to strong coupling; the frequency ω is now represented by a Z-shaped curve, with the medium part unstable. It extends quite a distance along both branches of the original (no resistance) curve. The resonance curve, in this case, splits into two different branches, with sudden transitions from the upper branches to the lower branches. There is no more continuity but sharp discontinuities between different types of motion (Brillouin, 1923; Cruft Electronic Staff, 1947; see also Appendix). Many examples of this kind are found in electronically sustained circuits. Let us quote a mechanical problem: a watch hanging on a nail. The first (sustained) oscillator is the watch's balance-wheel; the second (damped) oscillator is the pendulum consisting of the watch on the nail. When both are nearly in resonance, the watch goes fast or slow, depending on circumstances.

Another situation is found in many well-known mechanical problems, such as *shimmy* (for a car) or *wing-flutter* (for airplanes). Let us consider flutter: we have two coupled oscillators, both with damping, but *energy is fed* into the system *through the coupling device*. This is obvious in the coupling equations. When we discussed a problem of passive coupling (Eq. (11)) we had the same coefficient A_{12} in both equations. A typical *active coupling* would contain A_{12} in the first equation and another coefficient A_{21} in the second. This dissymmetry in the coupling terms is essential and results in a supply of energy from the outside into both oscillators (this energy coming from the wind on the wings). This energy supply is most intensive when oscillators are in resonance conditions. The classical theories need not be repeated here. The reader may just open the familiar book of Karman and Biot (1940).

IX. TRANSITION FROM CLASSICAL TO WAVE MECHANICS

All problems with internal resonances and Poincaré's unstabilities lead to special difficulties in wave mechanics. In most usual problems, the transition from classical to wave mechanics can be easily understood with the help of the well-known B.W.K. approximation method. When Poincaré conditions arise, there are some difficulties in the discussion.

Let us consider for instance the motion of a particle in the periodic potential of Fig. 1. Wave mechanics leads to Mathieu equations, which give very unexpected results for energies near V_1. In a similar way, for the problem of Fig. 3, wave mechanics introduces changes for energies near V_1, since oscillations in valleys I or II are now intercoupled by "tunnel effect" through the potential hill.

In both cases, conditions of zero frequency lead to very special conditions that require careful discussion.

As a conclusion, let us state that Poincaré's conditions specify situations where classical mechanics cannot give an exact answer: the slightest perturbation may bring about a sudden change of the type of motion. Determinism is not proven for such conditions.

These examples lead directly to statistical mechanics and are best discussed with the help of Information Theory.

APPENDIX. ON NEGATIVE RESISTANCES IN OSCILLATORS

It may be useful to sketch briefly the meaning of *negative resistance*. A great many electric, electronic, or semiconductor devices yield some rather complicated characteristics, where the current I, plotted as a

function of voltage V, may show a negative resistance R along some branches of the curve:

$$\frac{1}{R} = \frac{dI}{dV} \qquad R = -R' \quad R' > 0 \quad \text{(A.1)}$$

Such a device, if conveniently connected (or coupled) to a circuit, may yield a negative resistance in the circuit, and compensate the normal positive resistance, thus making it possible to obtain sustained oscillations. All negative resistance devices are strongly nonlinear, and the average negative resistance R depends upon the amplitude of oscillations. Altogether these devices are able to yield (depending upon the amplitude) a negative resistance

$$R_m' \leqq R \leqq R_M' \qquad R = -R' \quad \text{(A.2)}$$

comprised between two limits R_m' and R_M'. This general statement suffices for an elementary discussion.

Let us assume a negative resistance in the first oscillator of Eq. (10) and a normal positive resistance in the second oscillator. This means adding a term in $-R'\dot{x}_1$ in the first equation and a term $+r\dot{x}_2$ in the second. Let us also assume no potential coupling ($\alpha_{12} = 0$, hence $\Omega_{12} = 0$). We investigate the possibility of sustained oscillations in $e^{i\omega t}$, and we obtain both Eqs. (11) with:

$$\begin{aligned} A_{11} &= m_{11}(\Omega_{11}^2 - i\rho_1\omega - \omega^2) \qquad m_{11}\rho_1 = R' \\ A_{12} &= -m_{12}\omega^2 \\ A_{22} &= m_{22}(\Omega_{22}^2 + i\rho_2\omega - \omega^2) \qquad m_{22}\rho_2 = r \end{aligned} \quad \text{(A.3)}$$

where ρ_2 has a given value, while ρ_1 can take any value between the limits resulting from (A.2). Condition (12) now reads

$$(\Omega_{11}^2 - i\rho_1\omega - \omega^2)(\Omega_{22}^2 + i\rho_2\omega - \omega^2) - K^2\omega^4 = 0 \quad \text{(A.4)}$$

We must split real and imaginary terms

$$\begin{aligned} -\rho_1(\Omega_{22}^2 - \omega^2) + \rho_2(\Omega_{11}^2 - \omega^2) &= 0 \\ (\Omega_{11}^2 - \omega^2)(\Omega_{22}^2 - \omega^2) + \rho_1\rho_2\omega^2 - K^2\omega^4 &= 0 \end{aligned} \quad \text{(A.5)}$$

The first condition gives the value of ρ_1 for which sustained oscillations may occur

$$\rho_1 = \rho_2 \frac{\Omega_{11}^2 - \omega^2}{\Omega_{22}^2 - \omega^2} \quad \text{(A.6)}$$

This value, substituted in the second equation, yields an equation of third order in ω^2 (after multiplying by $\Omega_{22}^2 - \omega^2$)

$$(\Omega_{11}^2 - \omega^2)(\Omega_{22}^2 - \omega^2)^2 + \rho_2{}^2(\Omega_{11}^2 - \omega^2)\omega^2 - K^2\omega^4(\Omega_{22}^2 - \omega^2) = 0 \quad \text{(A.7)}$$

The ratio of amplitudes (13) X_1/X_2 is complex. The amplitude of oscillations depends upon nonlinear terms in the negative resistance characteristics, and this nonlinearity will provoke some small harmonics in the oscillation.

The discussion is based on the following remarks:
At *exact resonance* ($\Omega_{11} = \Omega_{22}$) there is always a solution

$$\Omega_{11}^2 = \Omega_{22}^2 = \omega^2$$

Far from resonance the solution is close to Ω_{11} (or more explicity to the solution of a system with zero resistance)

$$\Omega_{22} \gg \Omega_{11} \quad \text{or} \quad \Omega_{22} \ll \Omega_{11} \qquad \omega \approx \Omega_{11}$$

This is obvious from (A.6) which gives a small ρ_1 value and makes the $\rho_1\rho_2\omega^2$ term in the second Eq. (A.5) negligible if ρ_2 is small. These remarks lead directly to the two situations sketched in Figs. 11 and 12.

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