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Exactness of the Double Dual and Morita Duality for Grothendieck Categories

R. R. COLBY

Department of Mathematics, University of Hawaii, Honolulu, Hawaii 96822

AND

K. R. FULLER

Department of Mathematics, The University of Iowa, Iowa City, Iowa 52242

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Over any left and right self-injective ring R the R-dual functors $(\ )^*$ between R-Mod and Mod-R are exact, and hence so are the double R-dual functors $(\ )^{**}$ on each of these categories. In [3] we showed that if R is a one-sided artinian ring then $(\ )^{**}$ preserves monomorphisms if and only if R is a QF-3 ring, and $(\ )^{**}$ is left exact if and only if R is, in addition, its own maximal quotient ring $R = Q(R)$. The connection between Morita duality and artinian QF-3 rings was established in [4] and [7], and in [12] Tachikawa showed that it also holds for arbitrary QF-3 rings, i.e., for those rings with faithful injective finitely cogenerated left and right ideals; and also he characterized QF-3 maximal quotient rings as endomorphism rings of reflexive generator–cogenerators over rings with Morita duality.

Here we consider the left exactness of the double R-dual functors for rings without finiteness conditions and its connection with a version of Morita duality defined for abelian categories. We show that if $(\ )^{**}$ preserves monomorphisms in R-Mod then R is left QF-$3'$ in the sense [11] that the injective envelope $E(_RR)$ is R-torsionless, and that $(\ )^{**}$ is left exact on R-Mod if and only if, in addition, $R = Q(_RR)$. Then we characterize rings whose double dual functors preserve monomorphisms in both R-Mod and Mod-R as those such that the classes of torsion modules $\mathscr{T}$ and $\mathscr{T}'$ form localizing subcategories of R-Mod and Mod-R and the corresponding quotient categories admit contravariant exact functors $D: \mathcal{C} \rightleftarrows \mathcal{C}': D'$

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satisfying $DT = T' \circ (\ )^*$ and $D'T' = T \circ (\ )^*$, where $T: R\text{-Mod} \to \mathcal{C}$ and $T': \text{Mod-}R \to \mathcal{C}'$ are the canonical functors with kernels $\mathcal{E}$ and $\mathcal{E}'$, respectively. Moreover we show that $D: \mathcal{C} \rightleftarrows \mathcal{C}': D'$ defines a Morita duality. In a concluding section we prove that both functors $(\ )^{**}$ are left exact if and only if R is the endomorphism ring of a reflexive generator–cogenerator in a Grothendieck dategory with Morita duality (cf. [8, Section 8]), and we also characterize these rings in terms of a Morita duality between their categories of left and right modules of torsionless dominant dimension $\geqslant 2$. As a consequence, if both functors $(\ )^{**}$ preserve monomorphisms then both double dual functors over the maximal quotient ring of R are left exact.

1. Left Exactness of $(\ )^{**}$ on R-Mod

Let $\mathcal{M}$ $(=R\text{-Mod})$ denote the category of left R-modules. Recall that there is a natural transformation $\sigma: 1_{R\text{-Mod}} \to (\ )^{**}$, defined via the usual evaluation maps $\sigma_M: M \to M^{**}$. An R-module M is called torsionless (reflexive) in case σ_M is a monomorphism (an isomorphism). Also recall that R is left QF-3$'$ if the injective envelope $E({}_RR)$ of ${}_RR$ is torsionless.

Let $\mathcal{E}$ denote the full subcategory of $\mathcal{M}$ whose objects are those modules M such that $M^* = 0$. Then $\mathcal{E}$ is a localizing subcategory of $\mathcal{M}$ if and only if $\mathcal{E}$ is closed under submodules ([10, Proposition 4.6.3]). If $\mathcal{E}$ is localizing and $M \in |\mathcal{M}|$, we denote the unique largest submodule of M that belongs to $\mathcal{E}$ by $M_{\mathcal{E}}$. Note that if M is torsionless, then $M_{\mathcal{E}} = 0$.

1.1. Proposition. *The ring R is left QF-3$'$ if and only if $\mathcal{E}$ is localizing and for $M \in |\mathcal{M}|$, $M_{\mathcal{E}} = 0$ implies M is torsionless.*

Proof. Let $E = E({}_RR)$. If R is left QF-3$'$, then, since there is a monomorphism $i: E \to R^X$ for some set X, $M^* = 0$ if and only if $\operatorname{Hom}_R(M, E) = 0$. Hence if $M_0 \subseteq M$ and $M^* = 0$, then $\operatorname{Hom}_R(M_0, E) = 0$ so $M_0^* = 0$. Thus $\mathcal{E}$ is localizing. If $M_{\mathcal{E}} = 0$, consider the exact sequence

$$0 \longrightarrow K \hookrightarrow M \xrightarrow{\sigma_M} M^{**}.$$

If $\alpha \in K^*$, there exists $\beta: M \to E$ and a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, hook] \arrow[d, "\alpha"'] & M \arrow[d, "\beta"] \\ & R \arrow[r, hook] & E \arrow[d, "i"] \\ & & R^X. \end{tikzcd}$$

Thus if $\alpha \neq 0$, there exists $\hat{\beta} \in M^*$ with $\hat{\beta}(K) \neq 0$. Since $K = \bigcap \{\mathrm{Ker} f: f \in M^*\}$, we conclude that $\alpha = 0$ so $K^* = 0$. Hence, since $M_{\mathscr{E}} = 0$, $K = 0$ and M is torsionless.

Conversely, $R_{\mathscr{E}} = 0$ and E is an essential extension of R. Thus, since $\mathscr{E}$ is closed under submodules, $E_{\mathscr{E}} = 0$ so E is torsionless.

1.2. Proposition. *If ( )** preserves monomorphisms in $\mathscr{M}$ then R is left QF-3$'$.*

Proof. In the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, "i"] \arrow[d, "\cong"'] & E(R) \arrow[d, "\sigma_{E(R)}"] \\ & R^{**} \arrow[r, "i^{**}"] & E(R)^{**}, \end{tikzcd}$$

i is an essential monomorphism and i^{**} is a monomorphism. Hence $\sigma_{E(R)}$ is a monomorphism.

One can show that if R is right coherent and left QF-3$'$, then ( )** preserves monomorphisms of finitely presented left R-modules. However, we do not know if the converse of Proposition 1.2 is true.

If $\mathscr{E}$ is a localizing subcategory of $\mathscr{M}$ we denote the quotient category $\mathscr{M}/\mathscr{E}$ by $\mathscr{Q}$ and the canonical functor $\mathscr{M} \to \mathscr{Q}$ by T ([10, Section 4.3]). We denote a right adjoint of T by S ([10, Section 4.4]) and let $u: 1_{\mathscr{M}} \to ST$, $v: TS \to 1_{\mathscr{Q}}$ be the associated arrows of adjunction ([10, Section 1.5]). Then v is an isomorphism and for any $M \in |\mathscr{M}|$, $\ker u_M$ and $\operatorname{coker} u_M$ are in $\mathscr{E}$ ([10, Proposition 4.4.3]). A module M is *closed* if u_M is an isomorphism. Equivalently, $M \cong S(A)$ for some $A \in |\mathscr{Q}|$.

We say $M \in |\mathscr{M}|$ has *torsionless dominant dimension* $\geqslant n$ and write t.dom.dim. $M \geqslant n$ if there is an injective resolution $0 \to M \to E_1 \to E_2 \to \cdots$ with E_i torsionless for $i \leqslant n$. Thus R is left QF-3$'$ if and only if t.dom.dim. $R \geqslant 1$.

1.3. Lemma. *If R is left QF-3$'$ then the following are equivalent for a left R-module M:*

(1) *M is closed;*

(2) *there is an exact sequence*

$$0 \longrightarrow M \longrightarrow E({}_RR)^{X_1} \longrightarrow E({}_RR)^{X_2};$$

(3) *t.dom.dim. $M \geqslant 2$.*

Proof. In ([8, Section 5]), since R is left QF-3$'$, we may take $V = E({}_RR)$. Then (1) is equivalent to (2) by ([8, Lemma 5.3]). That (1) is equivalent to

(3) follows from ([10, Proposition 4.5.6 and Lemma 4.5.1]) and Proposition 1.1.

We recall that the maximal left quotient ring $Q(_RR)$ of R is the ring BiEnd $E(_RR)$ and identify R with its canonical image in $Q(_RR)$. Then $R = Q(_RR)$ if and only if there is an exact sequence $0 \to E(R)/R \to E(R)^X$ ([5, Proposition 4.3.1]).

1.4. THEOREM. *The following are equivalent*:

(1) $(\ )^{**}$ *is left exact on* $\mathscr{M}$;

(2) $(\ )^{**}$ *preserves monomorphisms in* $\mathscr{M}$ *and t.dom.dim.* $R \geqslant 2$;

(3) $(\ )^{**}$ *preserves monomorphisms in* $\mathscr{M}$ *and* $R = Q(_RR)$.

Proof. $(1) \Rightarrow (2)$. By Proposition 1.2 and Lemma 1.3 it suffices to prove that in a minimal injective resolution $0 \to R \to E_1 \to E_2$ of R. E_2 is torsionless. In the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r] \arrow[d, "\sigma"'] & E_1 \arrow[r, "d"] \arrow[d, "\sigma_1"'] & E_2 \arrow[d, "\sigma_2"'] \\ 0 \arrow[r] & R^{**} \arrow[r] & E_1^{**} \arrow[r] & E_2^{**} \end{tikzcd}$$

both rows are exact, σ is an isomorphism, and σ_1 is a monomorphism by Proposition 1.2. Hence $\operatorname{Ker}\sigma_2 \cap \operatorname{Im} d = 0$ so since E_2 is an essential extension of $\operatorname{Im} d$, $\operatorname{Ker}\sigma_2 = 0$. Hence E_2 is torsionless.

$(2) \Rightarrow (1)$. Suppose $0 \to K \to M \to N \to 0$ is exact in $\mathscr{M}$ and form the induced sequence $0 \to N^* \to M^* \to K^* \to Y \to 0$. Since $(\ )^{**}$ preserves monomorphisms, $Y^* = 0$. Let $0 \to R \to E_1 \to E_2$ be the given exact sequence with E_i torsionless and injective for $i = 1, 2$. Since E_i is torsionless and $Y^* = 0$ we see that $\operatorname{Hom}_R(Y, E_i) = 0$ for $i = 1, 2$. Thus the argument of ([3, Theorem 2]) can be applied to show that $0 \to K^{**} \to M^{**} \to N^{**}$ is exact.

$(2) \Leftrightarrow (3)$. Since $R = Q(_RR)$ if and only if there is an exact sequence $0 \to E(R)/R \to E(R)^X$ for some set X ([5, Proposition 4.3.1]), this equivalence follows from Proposition 1.2 and Lemma 1.3.

If R is (left and right) QF-3′, then $Q(_RR) = Q(R_R)$ ([12, Proposition 4.6]).

1.5. COROLLARY. *If the functors* $(\ )^{**}$ *preserve monomorphisms of left and right* R*-modules then* $(\ )^{**}$ *is left exact on* R-Mod *if and only if* $(\ )^{**}$ *is left exact on* Mod-R.

2. Morita Duality and the Functors ( )**

We continue the notation of Section 1 and denote by $\mathscr{M}'$, $\mathscr{E}'$, $\mathscr{C}'$, T', S', u', and v' the analogues of $\mathscr{M}$, $\mathscr{E}$, $\mathscr{C}$, T, S, u, and v, respectively, for right modules.

2.1. Lemma. *Suppose $\mathscr{E}'$ is localizing. Then $(\ )^{**}$ preserves monomorphisms in $\mathscr{M}$ if and only if $T' \circ (\ )^*: \mathscr{M} \to \mathscr{C}'$ is exact.*

Proof. Suppose $(\ )^{**}$ preserves monomorphisms in $\mathscr{M}$ and $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact in $\mathscr{M}$. Form the exact sequence $0 \to M_3^* \to M_2^* \to M_1^* \to N \to 0$ which, since T' is exact ([10, Theorem 4.3.8]), induces the exact sequence

$$0 \to T'(M_3^*) \to T'(M_2^*) \to T'(M_1^*) \to T'(N) \to 0.$$

Now since $(\ )^*$ is left exact on $\mathscr{M}'$ and since $(\ )^{**}$ preserves monomorphisms, it follows that $N^* = 0$. Hence $N \in \mathscr{E}'$ so $T'(N) = 0$ ([10, Lemma 4.3.4]). Thus $0 \to T'(M_3^*) \to T'(M_2^*) \to T'(M_1^*) \to 0$ is exact.

Conversely, suppose $T' \circ (\ )^*$ is exact and let $0 \to M_1 \to M$ be a monomorphism in $\mathscr{M}$. Then $T'(M^*) \to T'(M_1^*) \to 0$ is exact. If $M^* \to M_1^* \to N \to 0$ is exact, then $T'(M^*) \to T'(M_1^*) \to T'(N) \to 0$ is exact. From these it follows that $T'(N) = 0$, $N \in \mathscr{E}'$ and hence $N^* = 0$. Therefore $0 \to M_1^{**} \to M^{**}$ is exact as asserted.

Recall that the ring R is QF-3 if it contains idempotents e, f such that Re (fR) is faithful and is a direct summand of every faithful left (right) R-module. Suppose R is QF-3. Define

$$\hat{T} = fR \otimes \_ : \mathscr{M} \to fRf\text{-Mod},$$

$$\hat{S} = \operatorname{Hom}_{fRf}(fR, \_): fRf\text{-Mod} \to \mathscr{M},$$

$$\hat{\hat{T}} = \_ \otimes Re: \mathscr{M}' \to \text{Mod-}eRe,$$

$$\hat{\hat{S}} = \operatorname{Hom}_{eRe}(Re, \_): \text{Mod-}eRe \to \mathscr{M}'.$$

Then $\hat{T}$ is exact and $\hat{S}$ is a full and faithful right adjoint of $\hat{T}$. Furthermore, from the proof of ([3, Theorem 1]) we have $\operatorname{Ker} \hat{T} = \mathscr{E}$ so by ([10, Theorem 4.4.9]), $T\hat{S}: fRf\text{-Mod} \to \mathscr{C}$ is an equivalence. Similarly Mod-eRe and $\mathscr{C}'$ are equivalent. Now let

$$D = \operatorname{Hom}_{fRf}(\_, fRe): fRf\text{-Mod} \to \text{Mod-}eRe$$

and

$$D' = \operatorname{Hom}_{eRe}(\_, fRe): \text{Mod-}eRe \to fRf\text{-Mod}.$$

Then $D\colon fRf\text{-Mod} \rightleftarrows \text{Mod-}eRe\colon D'$ establishes a Morita duality ([12, Theorem 5.1]) and using ([12, Proposition 4.8]) one checks that

$$D\hat{T} \cong \hat{\hat{T}} \circ (\ )^* \qquad \text{and} \qquad D'\hat{\hat{T}} \cong \hat{T} \circ (\ )^*.$$

2.2. THEOREM. *The functors $(\ )^{**}$ preserve monomorphisms of left and right modules if and only if $\mathscr{C}$ and $\mathscr{C}'$ are localizing and there are contravariant exact functors $D\colon \mathcal{Q} \rightleftarrows \mathcal{Q}'\colon D'$ such that $DT = T' \circ (\ )^*$ and $D'T' = T \circ (\ )^*$.*

Proof. Suppose the $(\ )^{**}$ functors preserve monomorphisms. Then $\mathscr{C}$ and $\mathscr{C}'$ are localizing by Propositions 1.1 and 1.2. Since $T' \circ (\ )^*\colon \mathscr{M} \to \mathcal{Q}'$ is exact by Lemma 2.1 and since $T'(M^*) = 0$ for any module M for which $T(M) = 0$, there exists an exact contravariant functor $D\colon \mathcal{Q} \to \mathcal{Q}'$ such that $DT = T' \circ (\ )^*$ ([10, Corollaries 4.3.11 and 4.3.12]). Symmetrically, D' exists.

The converse follows from Lemma 2.1.

Let $D\colon \mathscr{B} \rightleftarrows \mathscr{B}'\colon D'$ be a pair of contravariant functors between abelian categories $\mathscr{B}$ and $\mathscr{B}'$. Then D and D' *are adjoint on the right* in case there are isomorphisms

$$\eta_{AA'}\colon \operatorname{Hom}_{\mathscr{B}}(A, D'(A')) \to \operatorname{Hom}_{\mathscr{B}'}(A', D(A))$$

natural in $A \in |\mathscr{B}|$ and $A' \in |\mathscr{B}'|$. (Using dual categories, this notion can, of course, be formulated in terms of adjoint pairs of covariant functors (see [10, Chap. I)].) Associated with the $\eta_{AA'}$ *are the arrows of right adjunction* τ: $1_{\mathscr{B}} \to D'D$ and $\tau'\colon 1_{\mathscr{B}'} \to DD'$ defined by $\tau_A = \eta^{-1}_{AD(A)}(1_{D(A)})$ and $\tau'_{A'} = \eta_{D'(A')A'}(1_{D'(A')})$, respectively. These satisfy, for each $A \in |\mathscr{B}|$, $A' \in |\mathscr{B}'|$,

$$D(\tau_A) \circ \tau'_{D(A)} = 1_{D(A)} \qquad \text{and} \qquad D'(\tau'_{A'}) \circ \tau_{D'(A')} = 1_{D'(A')}.$$

Moreover, any pair τ, τ' of natural transformations satisfying these conditions determine natural isomorphisms $\eta_{AA'}\colon \operatorname{Hom}_{\mathscr{B}}(A, D'A') \to \operatorname{Hom}_{\mathscr{B}'}(A', DA)$, via $\eta_{AA'}\colon \lambda \mapsto D(\lambda) \circ \tau'_{A'}$, with arrows τ and τ'.

We call an object A of $\mathscr{B}$ (A' of $\mathscr{B}'$) *reflexive* in case τ_A (respectively, $\tau'_{A'}$) is an isomorphism; and we note that (as in [1, Section 23]) D and D' define a duality between the *full subcategories of reflexive objects* $\mathscr{B}_0 \subseteq \mathscr{B}$ and $\mathscr{B}_0 \subseteq \mathscr{B}'$. Then we say that the pair $D\colon \mathscr{B} \rightleftarrows \mathscr{B}'\colon D'$ *defines a Morita duality* in case D and D' are exact, and the subcategories $\mathscr{B}_0 \subseteq \mathscr{B}$ and $\mathscr{B}'_0 \subseteq \mathscr{B}'$ are closed under subjects and quotient objects and contain generating sets for $\mathscr{B}$ and $\mathscr{B}'$, respectively.

We remark that any Morita duality as classically defined for module categories satisfies the definition above (see [1, Section 24]); so do the Morita dualities between functor categories studied by Yamagata [13]. In these two settings generating sets of small projectives are needed to show that the exactness hypothesis is implied by the other hypotheses, but we do not know whether this is so in general.

2.3. Proposition. *Let $D\colon \mathscr{B} \rightleftarrows \mathscr{B}'\colon D'$ be a Morita duality. Then*

(i) *D and D' are faithful*;

(ii) *if G is a generator (cogenerator) in $\mathscr{B}$, then $D(G)$ is a cogenerator (generator) in $\mathscr{B}'$.*

Proof. (i) Suppose $0 \neq f\colon A \to B$ in $\mathscr{B}$. Then, since $\mathscr{B}_0$ contains a generating set for $\mathscr{B}$, there exists $A_0 \in |\mathscr{B}_0|$ and $g\colon A_0 \to A$ such that $fg \neq 0$. Factor $fg = ih$ with i monic, h epic, and $\operatorname{Im} h = \operatorname{Im} fg \in |\mathscr{B}_0|$. The commutative diagram

$$\begin{tikzcd} A_0 \arrow[r, "g"] \arrow[dr, "h"'] & A \arrow[r, "f"] & B \\ & \operatorname{Im} fg \arrow[ur, "i"'] & \end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd} D(B) \arrow[r, "D(f)"] \arrow[dr, "D(i)"'] & D(A) \arrow[r, "D(g)"] & D(A_0) \\ & D(\operatorname{Im} fg) \arrow[ur, "D(h)"'] & \end{tikzcd}$$

with $D(h) \neq 0$ since A_0, $\operatorname{Im} fg \in |\mathscr{A}_0|$, and $D(i)$ epic since D is exact. Thus $D(f) \neq 0$.

(ii) Suppose G is a generator in $\mathscr{B}$. Let $0 \neq f\colon A \to B$ in $\mathscr{B}'$. There exists $A_0 \in |\mathscr{B}_0'|$ and $k\colon A_0 \to A$ such that $fk \neq 0$. Let $\varphi = fk$. Then since D' is faithful, $D'(\varphi) \neq 0$. Hence there exists $g\colon G \to D'(B)$ such that $D'(\varphi)\, g \neq 0$. Since D is faithful, $0 \neq D(D'(\varphi)\, g) = D(g)\, DD'(\varphi)$. In the commutative diagram

$$\begin{tikzcd} DD'(A_0) \arrow[r, "DD'(\varphi)"] & DD'(B) \arrow[r, "D(g)"] & D(G) \\ A_0 \arrow[u, "\tau'_{A_0}"] \arrow[r, "\varphi"] & B \arrow[u, "\tau'_B"'] & \end{tikzcd}$$

let $h = D(g)\, \tau'_B\colon B \to D(G)$. Then, since τ'_{A_0} is an isomorphism, it follows easily that $h\varphi \neq 0$. Hence $hf \neq 0$ also so $D(G)$ is a cogenerator. Similarly we see that $\mathscr{B}_0'$ contains a cogenerating set for $\mathscr{B}'$ so the dual of this argument shows that if G is a cogenerator, then $D(G)$ is a generator.

After proving the following lemmas we shall show that the functors $D: \mathcal{C} \rightleftarrows \mathcal{C}': D$ of Theorem 2.2 define a Morita duality, thus obtaining an analogue of ([12, Theorem 5.1]).

2.4. LEMMA. *Let $\mathscr{T}$ be localizing. If M is a closed left R-module and $T(\sigma_M)$: $T(M) \to T(M^{**})$ is an isomorphism, then M is reflexive.*

Proof (cf. [7, Theorem 2.3]). In the commutative square

$$\begin{tikzcd} ST(M) \arrow[r, "ST(\sigma_M)"] & ST(M^{**}) \\ M \arrow[u, "u_M"] \arrow[r, "\sigma_M"'] & M^{**}, \arrow[u, "u_{M^{**}}"'] \end{tikzcd}$$

u_M and $ST(\sigma_M)$ are isomorphisms so $u_{M^{**}}$ is epic. Since M^{**} is torsionless, $u_{M^{**}}$ is monic. Thus $u_{M^{**}}$ is an isomorphism so σ_M is also.

Assuming that both of the ( )** functors preserve monomorphisms, we define for each $A \in |\mathcal{C}|$, $\tau_A: A \to D'D(A)$ to be the composition

$$A \xrightarrow{v_A^{-1}} TS(A) \xrightarrow{T(\sigma_{S(A)})} T(S(A)^{**}) = D'DTS(A) \xrightarrow{D'D(v_A)} D'D(A).$$

One checks that $\tau: 1_{\mathcal{C}} \to D'D$ is a natural transformation. Similarly, we define $\tau': 1_{\mathcal{C}'} \to DD'$.

2.5. LEMMA. *Suppose both functors ( )** preserve monomorphisms, and let $\tau: 1 \to D'D$ be the natural transformation defined above. Then:*

(i) *If $M \in |\mathscr{M}|$, $\tau_{T(M)} = T(\sigma_M)$. In particular, if M is reflexive then $\tau_{T(M)}$ is an isomorphism.*

(ii) *If $A \in |\mathcal{C}|$, then τ_A is an isomorphism if and only if $S(A) \in |\mathscr{M}|$ is reflexive.*

(iii) *τ_A is monic for all $A \in |\mathcal{C}|$.*

Proof. (i) First we note that, since v is an isomorphism, the arrows of adjunction u, v satisfy

$$T(u_M) = v_{T(M)}^{-1} \qquad \text{and} \qquad u_{S(A)} = S(v_A)^{-1}$$

for all $M \in |\mathscr{M}|$ and $A \in |\mathcal{C}|$ (see the equations on p. 12 of [10]). Let $M \in |\mathscr{M}|$ and $A = T(M)$. Then from the commutative diagram

$$\begin{tikzcd} T(M) \arrow[r, "T(\sigma_M)"] \arrow[d, "T(u_M)"'] & T(M^{**}) \arrow[d, "T(u_M^{**})"] \\ TST(M) \arrow[r, "T(\sigma_{ST(M)})"'] & T(ST(M)^{**}) \end{tikzcd}$$

and the fact that $T(u_M^{**}) = D'DT(u_M)$, we see that

$$T(\sigma_M) = D'DT(u_M)^{-1} \circ T(\sigma_{ST(M)}) \circ T(u_M)$$
$$= D'D(v_A) \circ T(\sigma_{S(A)}) \circ v_A^{-1} = \tau_A.$$

(ii) If $S(A)$ is reflexive, then since $A \cong TS(A)$, τ_A is an isomorphism by (i). If τ_A is an isomorphism, then since $S(A)$ is closed and $T(\sigma_{S(A)})$ is an isomorphism, $S(A)$ is reflexive by Lemma 2.4.

(iii) Since $S(A)$ is closed and hence torsionless, $\sigma_{S(A)}$ is monic so $T(\sigma_{S(A)})$ is also.

We are now ready to complete our analogue of ([12, Theorem 5.1]).

2.6. THEOREM. *If the functors* $(\ )^{**}$ *preserve monomorphisms of left and right R-modules, then the functors* $D\colon \mathcal{C} \rightleftarrows \mathcal{C}'\colon D'$ *of Theorem* 2.2 *define a Morita duality.*

Proof. Let $A \in |\mathcal{C}|$ and choose $M \in M$ with $T(M) = A$ (see the proof of [10, Theorem 4.3.3]). Then $D(A) = DT(M) = T'(M^*)$, and applying Lemma 2.5(i) and ([1, Proposition 20.14]) we have

$$\begin{aligned} D(\tau_A) \circ \tau'_{D(A)} &= D(\tau_{T(M)}) \circ \tau'_{T'(M^*)} \\ &= D(T(\sigma_M)) \circ T'(\sigma_{M^*}) \\ &= T'(\sigma_M^*) \circ T'(\sigma_{M^*}) \\ &= T'(\sigma_M^* \circ \sigma_{M^*}) \\ &= T'(1_{M^*}) = 1_{D(A)}. \end{aligned}$$

Similarly, for all $A' \in |\mathcal{C}'|$,

$$D'(\tau'_{A'}) \circ \tau_{D'(A')} = 1_{D'(A')}.$$

Thus D and D' are adjoint on the right with arrows τ and τ'.

By Lemma 2.5(i), $T({}_RR)$ and $T'(R_R)$ are reflexive, and by ([10, Lemma 4.4.8]) they are generators in $\mathcal{C}$ and $\mathcal{C}'$, respectively. To see that sub-objects and quotient objects of reflexive objects in $\mathcal{C}$ (and similarly in $\mathcal{C}'$) are reflexive, consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\tau_A"] & B \arrow[r] \arrow[d, "\tau_B"] & C \arrow[r] \arrow[d, "\tau_C"] & 0 \\ 0 \arrow[r] & D'D(A) \arrow[r] & D'D(B) \arrow[r] & D'D(C) \arrow[r] & 0. \end{tikzcd}$$

By Lemma 2.5(iii), τ_C in monic, so if τ_B is an isomorphism, then τ_C and τ_A are also by the 5-Lemma.

2.7. COROLLARY. *If the functors ( )** preserve monomorphisms of left and right modules then the functors D and D' of Theorem* 2.2 *are faithful and $T(R)$ and $T'(R)$ are reflexive generator–cogenerators in $\mathcal{C}$ and $\mathcal{C}'$, respectively.*

Proof. Since $T(R)$ and $T'(R)$ are generators by ([10, Lemma 4.4.8]) and since $DT(R) \cong T'(R)$ and $D'T'(R) \cong T(R)$ the assertions are immediate from Theorem 2.6 and Proposition 2.3.

3. LEFT EXACTNESS OF BOTH FUNCTORS ( )**

As we proved in [3], a left artinian ring is a QF-3 maximal quotient ring if and only if both of its double dual functors are left exact. According to Tachikawa ([12, Theorem 5.3 and Proposition 5.2]), QF-3 maximal quotient rings are precisely those rings that are isomorphic to the endomorphism ring of a reflexive generator–cogenerator in a category Δ-Mod that has Morita duality with Mod-Γ for some rings Δ and Γ. Here we provide an analogous characterization of arbitrary rings whose double dual functors are both left exact, and consider some of its consequences. We continue the notations established in the first two sections in case both double duals preserve monomorphisms.

Recall that a Grothendieck category is an abelian category with exact direct limits which contains a generator. The Grothendieck categories are precisely those categories which are equivalent to W-Mod$/\mathcal{S}$ for some ring W and some localizing subcategory $\mathcal{S}$ of W-Mod ([10, Corollary 4.4.10]).

Following established custom for left modules, we let $\mathrm{End}_{\mathcal{C}}(A) = \mathrm{Hom}_{\mathcal{C}}(A, A)^{\mathrm{op}}$.

3.1. THEOREM. *Both R-double dual functors ( )** are left exact if and only if R is the endomorphism ring of a reflexive generator–cogenerator U in $\mathcal{C}$ for some Grothendieck categories $\mathcal{C}$ and $\mathcal{C}'$ with Morita duality $D:\mathcal{C} \rightleftarrows \mathcal{C}':D'$.*

Proof. ($\Rightarrow$) By Theorem 1.4 and Lemma 1.3, ${}_RR$ is closed. Thus ${}_RR = ST(R) = \mathrm{Hom}_{\mathcal{C}}(TR, TR)$. Moreover, $D:\mathcal{C} \rightleftarrows \mathcal{C}':D'$ is a Morita duality by Theorem 2.6 and $U = T(R)$ is a reflexive generator–cogenerator by Corollary 2.7.

($\Leftarrow$) As in the proof of ([8, Theorem 8.1]) there exist covariant functors

$$S = \mathrm{Hom}_{\mathcal{C}}(U, \_): \mathcal{C} \to \mathcal{M}$$

and

$$T: \mathscr{M} \to \mathcal{A},$$

such that T is a left adjoint of S, T is exact, and $TS \cong 1_{\mathcal{A}}$. Since U is a cogenerator in $\mathcal{A}$, $D(U)$ is a generator in $\mathcal{A}'$ by Proposition 2.3 so since $\mathrm{End}_{\mathcal{A}}(U) \cong \mathrm{End}_{\mathcal{A}'}(D(U))^{\mathrm{op}}$, we have covariant functors.

$$S' = \mathrm{Hom}_{\mathcal{A}'}(D(U), \_): \mathcal{A}' \to \mathscr{M}'$$

and

$$T': \mathscr{M}' \to \mathcal{A}',$$

such that T' is a left adjoint of S', T' is exact, and $T'S' \cong 1_{\mathcal{A}'}$. Now for $M \in |\mathscr{M}|$, we have R-isomorphisms

$$\begin{aligned} S'(D(T(M))) &= \mathrm{Hom}_{\mathcal{A}'}(D(U), DT(M)) \\ &\cong \mathrm{Hom}_{\mathcal{A}}(T(M), D'D(U)) \\ &\cong \mathrm{Hom}_{\mathcal{A}}(T(M), U) \\ &\cong \mathrm{Hom}_R(M, S(U)) \\ &\cong \mathrm{Hom}_R(M, R). \end{aligned}$$

The first two isomorphisms follow from the adjointness of D and D' and the reflexivity of U, and the third (since U is a cogenerator) shows that $\mathscr{T}$ is localizing. Also

$$DT(M) \cong T'S'DT(M) \cong T'(M^*)$$

so $T' \circ (\ )^*$ is exact. Similarly, $\mathscr{T}'$ is localizing and $T \circ (\ )^*$ is exact so both functors $(\ )^{**}$ preserve monomorphisms by Lemma 2.1. The proof is now complete by ([8, Corollary 8.4]) and Theorem 1.4.

It follows from ([3, Theorems 1 and 2]) that if R is left or right artinian and the R-double dual functors preserve monomorphisms then the double dual functors over the maximal quotient ring $Q(R)$ are left exact. This is true without any finiteness conditions.

3.2. COROLLARY. *If both R-double dual functors $(\ )^{**}$ preserve monomorphisms then both double dual functors over $Q(R)$ are left exact.*

Proof. The corollary follows from Theoem 2.6, Corollary 2.7, and Theorem 3.1 since $Q(R) \cong ST(R) \cong \mathrm{Hom}_R(R, ST(R)) \cong \mathrm{Hom}_{\mathcal{A}}(T(R), T(R))$ (see [10, Theorem 4.13.4]).

A ring R is a left and right injective cogenerator if and only if its dual functors define a Morita duality $(\ )^*: \mathscr{M} \rightleftarrows \mathscr{M}' : (\ )^*$ ([2, 6, 9]). For any ring R, let $\mathscr{D}$ denote the full subcategory of $\mathscr{M}$ whose objects are the modules M with t.dom.dim.$(M) \geqslant 2$, and define $\mathscr{D}'$ similarly in $\mathscr{M}'$. We also have an application of the preceding results regarding Morita duality between $\mathscr{D}$ and $\mathscr{D}'$.

3.3. THEOREM. *Both R-double dual functors ( )** are left exact if and only if* $_RR \in |\mathscr{D}|$, $R_R \in |\mathscr{D}'|$, *and the R-dual functors define a Morita duality* $(\ )^* : \mathscr{D} \rightleftarrows \mathscr{D}' : (\ )^*$.

Proof. ($\Rightarrow$) By Proposition 1.2 and Lemma 1.3, $T: \mathscr{D} \rightleftarrows \mathscr{C} : S$ and $T' : \mathscr{D}' \rightleftarrows \mathscr{C}' : S'$ define equivalences and by Theorem 2.6, $D : \mathscr{C} \rightleftarrows \mathscr{C}' : D'$ define a Morita duality. But $(\ )^* \cong S'DT$ on $\mathscr{D}$ and $(\ )^* \cong SD'T'$ on $\mathscr{D}'$ as in the proof of Theorem 3.1 so $(\ )^* : \mathscr{D} \rightleftarrows \mathscr{D}' : (\ )^*$ defines a Morita duality: and by Lemma 2.5(ii) the reflexive R-modules of t.dom.dim. $\geqslant 2$ are the reflexive objects in $\mathscr{D}$ and $\mathscr{D}'$.

($\Leftarrow$) Since $_RR \in |\mathscr{D}|$, R must be left QF-3$'$ so $\mathscr{F}$ is localizing by Proposition 1.1. Similarly so is $\mathscr{F}'$. Thus by Lemma 1.3, $\mathscr{D} \approx \mathscr{M}/\mathscr{F}$ and $\mathscr{D}' \approx \mathscr{M}'/\mathscr{F}'$ are Grothendieck categories. Since $(\ )^* : \mathscr{D} \rightleftarrows \mathscr{D}' : (\ )^*$ define a Morita duality with $_RR \in |\mathscr{D}|$ and $R_R \in |\mathscr{D}'|$, $R \cong (R_R)^*$ is a generator–cogenerator in $\mathscr{D}$ by Proposition 2.3; and $R = \mathrm{End}(_RR)$.

In conclusion, viewing the proof of Theorem 3.1 from a different perspective, we have the following characterization of Morita dualities between Grothendieck categories which contain reflexive generators.

3.4. THEOREM. *Let* $D : \mathscr{B} \rightleftarrows \mathscr{B}' : D'$ *be a Morita duality where* $\mathscr{B}$ *and* $\mathscr{B}'$ *are Grothendieck categories. Assume that* $\mathscr{B}$ *contains a reflexive generator–cogenerator U and let* $R = \mathrm{End}_{\mathscr{B}}(U)$. *Then both R-double dual functors ( )** are left exact. Furthermore, there exist exact functors* T: R-Mod $\to \mathscr{B}$ *and* T': Mod-$R \to \mathscr{B}'$ *with full faithful right adjoints S and S' and kernels the subcategories of torsion left and right modules, respectively, such that*

$$D \circ T \cong T' \circ (\ )^* \qquad \textit{and} \qquad D' \circ T' \cong T \circ (\ )^*.$$

In particular,

$$D \cong T' \circ (\ )^* \circ S \qquad \textit{and} \qquad D' \cong T \circ (\ )^* \circ S'.$$

Note added in proof. There is an unpublished paper of K. Morita and H. Tachikawa entitled "QF-3 rings" which contains material on semiprimary QF-3 rings and Morita duality that should be included in the related literature.

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