On the existence of some new positive interior spike solutions to a semilinear Neumann problem

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A R T I C L E   I N F O

Article history:
Received 28 April 2009
Available online 5 August 2009

MSC:
35B40
35J20
35J65

Keywords:
Nonlinear elliptic equation
Multiple interior peaks
Finite-dimensional reduction

A B S T R A C T

In this paper we are concerned with the following Neumann problem

\[
\begin{align*}
\varepsilon^2 \Delta u - u + f(u) &= 0, \quad u > 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( \varepsilon \) is a small positive parameter, \( f \) is a superlinear and subcritical nonlinearity, \( \Omega \) is a smooth and bounded domain in \( \mathbb{R}^N \). Solutions with multiple boundary peaks have been established for this problem. It has also been proved that for any integer \( k \) there exists an interior \( k \)-peak solution which concentrates, as \( \varepsilon \to 0^+ \), at \( k \) sphere packing points in \( \Omega \).

In this paper we prove the existence of a second interior \( k \)-peak solution provided that \( k \) is large enough, and we conjecture that its peaks are located along a straight line. Moreover, when \( \Omega \) is a two-dimensional strictly convex domain, we also construct a third interior \( k \)-peak solution provided that \( k \) is large enough, whose peaks are aligned on a closed curve near \( \partial \Omega \).

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1. Introduction

Let us consider the following singularly perturbed elliptic problem:

\[
\begin{align*}
\varepsilon^2 \Delta u - u + f(u) &= 0, \quad u > 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( \varepsilon \) is a small positive parameter, \( f \) is a superlinear and subcritical nonlinearity, \( \Omega \) is a smooth and bounded domain in \( \mathbb{R}^N \). Solutions with multiple boundary peaks have been established for this problem. It has also been proved that for any integer \( k \) there exists an interior \( k \)-peak solution which concentrates, as \( \varepsilon \to 0^+ \), at \( k \) sphere packing points in \( \Omega \).

In this paper we prove the existence of a second interior \( k \)-peak solution provided that \( k \) is large enough, and we conjecture that its peaks are located along a straight line. Moreover, when \( \Omega \) is a two-dimensional strictly convex domain, we also construct a third interior \( k \)-peak solution provided that \( k \) is large enough, whose peaks are aligned on a closed curve near \( \partial \Omega \).

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\begin{align}
\begin{cases}
\varepsilon^2 \Delta u - u + u^{p-1} = 0 & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}
\end{align}

where \( \Omega \) is a smooth and bounded domain of \( \mathbb{R}^N \), \( N \geq 2 \), \( 2 < p < \frac{2N}{N-2} \) if \( N \geq 3 \) and \( p > 2 \) if \( N = 2 \), and \( \varepsilon > 0 \) is a small parameter. Here \( \nu \) denotes the unit outward normal at a point on \( \partial \Omega \). This problem arises from different mathematical models: for example, it appears in the study of stationary solutions for the Keller–Segal system in chemotaxis and the Gierer–Meinhardt system in biological pattern formation.

Denoting by \( \mathcal{H}(P) \), \( P \in \partial \Omega \), the mean curvature of the boundary, it is known that this problem has positive multiple boundary peak solutions with each peak concentrating at a different critical point of \( \mathcal{H} \) or with all the peaks approaching a local minimum point of \( \mathcal{H} \) (see [3,6,9,12,13,18,19,21] and the references therein). Furthermore, solutions with multiple interior peaks have been established, with each peak concentrating at a different point whose location depends on the geometry of the domain (see [1,5,8,10,22,23] and the references therein). It turns out that a general guideline is that while multiple boundary spikes tend to concentrate at the critical points of the boundary mean curvature \( \mathcal{H}(P) \), the location of the interior spikes is governed by the distance between the peaks as well as from the boundary \( \partial \Omega \). More specifically, the function \( \varphi_k : \Omega^k \to \mathbb{R} \) defined by

\begin{equation}
\varphi_k(P) := \min_{i,h=1,\ldots,k} \left\{ d_{\partial \Omega}(P_i), \frac{|P_i - P_h|}{2} \right\}, \quad P := (P_1, \ldots, P_k), \tag{1.2}
\end{equation}

appears naturally in the location of the interior spikes. Indeed in [10] Gui and Wei proved that for any \( k \geq 2 \) there exists a solution of (1.1) with \( k \) interior peaks at the global maximum points of the function \( \varphi_k \). One of the most general result, due to Gui and Wei [11], states that given two arbitrary integers \( l_1 \) and \( l_2 \) there exist solutions with \( l_1 \) peaks on the boundary and \( l_2 \) peaks in the interior.

The question of constructing higher dimensional concentration sets for (1.1) has been investigated only in recent years. In [16,17] Malchiodi and Montenegro proved that for \( N \geq 2 \) there exists a family of solutions concentrating at all the boundary \( \partial \Omega \) or at some of its components. Later, in [14] Malchiodi showed a concentration phenomenon along a closed nondegenerate geodesic of \( \partial \Omega \) in the three-dimensional case. Moreover in [24] the authors constructed a solution with a higher dimensional concentration set inside the domain: more precisely, assuming \( N = 2 \), given \( \Gamma \) a straight line intersecting orthogonally with \( \partial \Omega \) at exactly two points and satisfying a nondegeneracy condition, there exists a solution concentrating along \( \Gamma \). We mention here the paper [15], where new phenomena are presented for which concentration occurs at \( l \)-dimensional submanifolds of \( \partial \Omega \).

The aim of this paper is to construct a family of multiple interior peak solutions to the problem (1.1) which differ from that by Gui and Wei in [10] and can be considered as the analogous in the discrete case of the higher dimensional concentration results in [16] and [24].

We consider the more general problem

\begin{align}
\begin{cases}
\varepsilon^2 \Delta u - u + f(u) = 0, & u > 0 \text{ in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\end{align}

We will assume that \( f : \mathbb{R}_+ \to \mathbb{R} \) is of class \( C^{1+\sigma} \) and satisfies the following conditions:

\begin{enumerate}
\item[(f1)] \( f(0) = f'(0) = 0 \);
\item[(f2)] \( f(u) = O(|u|^p) \), \( f'(u) = O(|u|^{p-1}) \) as \( |u| \to \infty \) for some \( p_1, p_2 > 1 \) and there exists \( p_3 > 1 \) such that
\end{enumerate}

\footnote{Hereafter \( d_{\partial \Omega}(P) \) denotes the distance of \( P \) from \( \partial \Omega \).}
where \( \delta \) is sufficiently small and \( \bar{\delta} \) is sufficiently large, there exists a solution consisting of \( \bar{\delta} \) located at distance \( \bar{\delta} \) whose distance from the boundary is small, the problem

\[
\begin{aligned}
\Delta w - w + f(w) &= 0, \quad w > 0 \quad \text{in} \quad \mathbb{R}^N, \\
w(0) &= \max_{z \in \mathbb{R}^N} w(z), \quad \lim_{|z| \to +\infty} w(z) = 0
\end{aligned}
\]

has a unique solution \( w \), which is nondegenerate, i.e., denoting by \( L \) the linearized operator

\[
L : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N), \quad L[u] := \Delta u - u + f'(w)u,
\]

then

\[
\text{Kernel}(L) = \text{span} \left\{ \frac{\partial w}{\partial z_1}, \ldots, \frac{\partial w}{\partial z_N} \right\}.
\]

By the well-known result of Gidas, Ni and Nirenberg [7] \( w \) is radially symmetric and strictly decreasing in \( r = |z| \). Moreover, by classical regularity arguments, the following asymptotic result holds

\[
\lim_{|z| \to +\infty} |z|^{\frac{N-1}{2}} e^{|z|^2} w(|z|) = A > 0 \quad \text{and} \quad \lim_{|z| \to +\infty} \frac{w'(|z|)}{w(|z|)} = -1.
\]

The class of nonlinearities \( f \) satisfying (F1)–(F3) includes, and it is not restricted to, the model \( f(u) = u^{p-1} \) with \( p > 2 \) if \( N = 1, 2 \) and \( 2 < p < \frac{2N}{N-2} \) if \( N \geq 3 \), and also the model \( f(u) = u(u - a)(1 - u) \) where \( 0 < a < \frac{1}{2} \) and \( N \leq 8 \). Other nonlinearities can be found in [4].

The first result deals with the two-dimensional case and states that, if \( \tilde{ \delta } \) is sufficiently small and \( \bar{k} \) is sufficiently large, there exists a solution consisting of \( \bar{k} \) interior peaks which belong to a curve whose distance from the boundary is \( \bar{\delta} \). Roughly speaking, the limit profile of such solution resembles a crown of peaks surrounding the boundary, which recall the boundary layer of Malchiodi and Montenegro in [16].

**Theorem 1.1.** Assume that hypotheses (F1)–(F3) hold and that \( \Omega \subset \mathbb{R}^2 \) is a strictly convex, smooth and bounded domain. Then for any \( \delta_0 > 0 \) there exist \( \bar{\delta} \in (0, \delta_0) \) and an integer \( \bar{k} \) such that, for \( \varepsilon \) sufficiently small, the problem (1.3) has a solution \( u_\varepsilon \in H^2(\Omega) \) with \( \bar{k} \) interior peaks at \( P_{\varepsilon}^1, \ldots, P_{\varepsilon}^{\bar{k}} \in \Omega \). Moreover

\[
\varphi_\varepsilon(P_{\varepsilon}^1, \ldots, P_{\varepsilon}^{\bar{k}}) \to \bar{\delta} \quad \text{as} \quad \varepsilon \to 0^+.
\]

Furthermore, if \( (P_{\varepsilon}^1, \ldots, P_{\varepsilon}^{\bar{k}}) \) is the limit of a subsequence of \( (P_{\varepsilon}^1, \ldots, P_{\varepsilon}^{\bar{k}}) \) as \( \varepsilon \to 0^+ \), then the points \( P_{\varepsilon}^i \) are located at distance \( \bar{\delta} \) from the boundary and the distance between two successive \( P_{\varepsilon}^i \)'s is \( 2\bar{\delta} \). Finally \( k \to +\infty \) as \( \delta_0 \to 0^+ \) and

\[
\tilde{\delta} \approx \frac{\ell (\partial \Omega)}{2k} \frac{1}{\max \varphi_\varepsilon^\ast} \quad \text{as} \quad \delta_0 \to 0^+,
\]

where \( \ell (\partial \Omega) \) denotes the length of \( \partial \Omega \).

A couple of open questions naturally arise when \( \Omega \subset \mathbb{R}^N \) is a strictly convex, smooth and bounded domain with \( N \geq 3 \).

\[
|f'(u + \phi) - f'(u)| \leq \begin{cases} c|\phi|^{p_3-1} & \text{if } p_3 > 2, \\ c(|\phi| + |\phi|^{p_3-1}) & \text{if } p_3 \leq 2; \end{cases}
\]
(1) If $\delta$ is sufficiently small and $k$ is sufficiently large, is there a solution consisting of $k$ interior peaks which belong to a hypersurface whose distance from the boundary is $\delta$? (The limit profile would recall the boundary layer found in [17].)

(2) Let $N = 3$ and let $\Gamma$ be a closed geodesic of $\partial \Omega$. If $\delta$ is sufficiently small and $k$ is sufficiently large, is there a solution consisting of $k$ interior peaks located on a curve whose distance from $\Gamma$ is $\delta$? (The limit profile would recall the boundary layer found in [14].)

Let us point out that the arguments we use in the proof of Theorem 1.1 strongly rely on the assumption on the dimension.

In the next two results we prove the existence of another solution to problem (1.3) with multiple interior spikes in the general case $N \geq 2$.

**Theorem 1.2.** Let $f$ satisfy conditions (f1)–(f3) and let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a smooth and bounded domain. Then there exists an integer $k_0$ such that for any $k \geq k_0$ and for $\varepsilon > 0$ sufficiently small the problem (1.3) has a solution $u_\varepsilon \in H^2(\Omega)$ with $k$ interior peaks at $P_{1\varepsilon}^*, \ldots, P_{k\varepsilon}^* \in \Omega$. Moreover, if $(P_{1\varepsilon}^*, \ldots, P_{k\varepsilon}^*)$ is the limit of a subsequence of $(P_1^*, \ldots, P_k^*)$ as $\varepsilon \to 0^+$, then

$$\varphi_k(P_1^*, \ldots, P_k^*) \leq \frac{\text{diam}(\Omega)}{2k} \leq \max_{\Omega^k} \varphi_k. \quad (1.5)$$

**Theorem 1.3.** Let $f$ satisfy conditions (f1)–(f3) and let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a smooth and bounded domain. Assume that $\Omega$ is convex and there exist $P_0 \in \Omega$ with $d_{\partial \Omega}(P_0) := \max_{P \in \partial \Omega} d_{\partial \Omega}(P)$ and $\varepsilon \in \mathbb{R}^N$ with $|\varepsilon| = 1$ such that

2 \{ε, −ε\} \subset \partial d_{\partial \Omega}(P_0). Then, for any $k \geq 4$ and for $\varepsilon > 0$ sufficiently small, the problem (1.3) has a solution $u_\varepsilon \in H^2(\Omega)$, with $k$ interior peaks at $P_{1\varepsilon}^*, \ldots, P_{k\varepsilon}^* \in \Omega$. Moreover, if $(P_1^*, \ldots, P_k^*)$ is the limit of a subsequence of $(P_{1\varepsilon}^*, \ldots, P_{k\varepsilon}^*)$ as $\varepsilon \to 0^+$, then

$$\varphi_k(P_1^*, \ldots, P_k^*) \leq \frac{d_{\partial \Omega}(P_0)}{k} \leq \max_{\Omega^k} \varphi_k. \quad (1.6)$$

More detailed asymptotic behavior of the solutions $u_\varepsilon$ of the above theorems can be found in Sections 2 and 3. Roughly speaking each peak has a profile similar to a rescaled $w$, i.e.

$$u_\varepsilon(x) \approx \sum_i w\left(\frac{x - P_i^\varepsilon}{\varepsilon}\right) \quad \text{as } \varepsilon \to 0^+. \quad (\text{1.6})$$

Observe that, contrary to Theorem 1.1, Theorems 1.2 and 1.3 do not locate the interior peaks of their solutions. The reason is that the solution of Theorem 1.1 corresponds to a local maximum of the function $\varphi_k$ on a suitable neighborhood of the boundary $\partial \Omega^k$, while the solution of Theorems 1.2 and 1.3 is obtained as a saddle point of $\varphi_k$ by using a min–max theorem. More precisely, in order to apply a min–max argument, we provide $\varphi_k$ with a suitable local linking structure whose linking sets are formed by the configurations $(P_1, \ldots, P_k)$ which are aligned in $\Omega$ along a fixed direction. The construction suggests that the peaks should be arranged on a suitable straight line, as we conjecture, and in this sense they would represent the analogous in the discrete case of the layered solution of Wei and Yang in [24].

We note that in the case of a ball $\Omega = B_R$ one may use the symmetry to construct interior peaks located at the vertices of a regular $k$-polygon or along a diameter, obtaining the results of Theorems 1.1 and 1.3.

Let us point out that the multiple interior peak solutions provided by Theorem 1.1, Theorem 1.2 (Theorem 1.3) and by the result of Gui and Wei in [10], which corresponds to the maximum of $\varphi_k$,
are different thanks to (1.4), (1.5) and (1.6) (observe that \( \ell(\partial\Omega) > 2\text{diam}(\Omega) \geq 4\max_{\Omega} d_{\partial\Omega}(P) \)). So, in particular the following corollary holds.

**Corollary 1.4.** Assume that hypotheses (f1)–(f3) hold and that \( \Omega \subset \mathbb{R}^2 \) is a strictly convex, smooth and bounded domain. Then for any \( k_0 > 0 \) there exists an integer \( k > k_0 \) such that for \( \epsilon \) sufficiently small the problem (1.3) has three different interior \( k \)-peak solutions.

The paper is organized as follows. Notation, preliminaries and some useful estimates are recalled in Section 2. In Section 3 we sketch the Lyapunov–Schmidt reduction method which reduces the problem to finding a critical point for a functional on a finite-dimensional space. In Section 4 we set up a maximizing problem and we show that the solution to this maximizing problem actually provides a solution to problem (1.3) (Theorem 1.1). Finally Section 5 is devoted to apply a min–max argument in order to catch a saddle point of \( \psi_k \) (Theorems 1.2 and 1.3).

### 2. Preliminaries

We need to fix some notation. For any smooth bounded domain \( U \), let \( P_U w \) be the unique solution of

\[
\begin{cases}
-\Delta u + u = f(w) & \text{in } U, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U.
\end{cases}
\]

For \( P \in \Omega \) we set

\[
\Omega_\epsilon := \{ y : \epsilon y \in \Omega \} \quad \text{and} \quad \Omega_{\epsilon, P} := \{ y : \epsilon y + P \in \Omega \}.
\]

Let us scale problem (1.3), so that we get an equivalent problem

\[
\begin{cases}
-\Delta u + u = f(u) & \text{in } \Omega_\epsilon, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega_\epsilon.
\end{cases}
\]

(2.1)

Associated with problem (2.1) is the rescaled energy functional

\[
J_\epsilon(u) := \frac{1}{2} \int_{\Omega_\epsilon} (|\nabla u|^2 + u^2) - \int_{\Omega_\epsilon} F(u), \quad u \in H^1(\Omega_\epsilon),
\]

where \( F(u) = \int_0^u f(s) \, ds \). Fix \( P = (P_1, \ldots, P_k) \in \Omega^k \) and set, for any \( i = 1, \ldots, k, \)

\[
w_i(y) = w_{P_i}(y) := w\left(y - \frac{P_i}{\epsilon}\right), \quad \mathcal{P}w_i(y) := \mathcal{P}_{\Omega_{\epsilon}} w_i(y), \quad y \in \Omega_\epsilon,
\]

and

\[
\varphi_{\epsilon, P}(x) := w\left(\frac{x - P}{\epsilon}\right) - \mathcal{P}_{\Omega_{\epsilon}, P} w\left(\frac{x - P}{\epsilon}\right), \quad \psi_{\epsilon, P}(x) := -\epsilon \log \varphi_{\epsilon, P}(x), \quad x \in \Omega.
\]

We have the following result (see [10] or [20]).
Lemma 2.1.

\[ \psi_{\varepsilon, P}(x) \to 2\psi_0(x) := \inf_{z \in \partial \Omega} \left\{ |z - P| + |z - x| \right\} \quad \text{as } \varepsilon \to 0 \]

uniformly for \( x \in \overline{\Omega} \) and \( P \) on a compact subset of \( \Omega \). In particular \( \psi_0(P) = 2d_{\partial \Omega}(P) \).

We look for a solution to (2.1) as

\[ u = \sum_{i=1}^{k} \mathcal{P}w_i + \Phi_{\varepsilon, P}, \]

where the rest term \( \Phi_{\varepsilon, P} = \Phi_{\varepsilon, P_1, \ldots, P_k} \) belongs to a suitable space.

3. The Lyapunov–Schmidt reduction

Let \( H^2_N(\Omega_\varepsilon) \) be the Hilbert space defined by

\[ H^2_N(\Omega_\varepsilon) := \left\{ u \in H^2(\Omega_\varepsilon): \frac{\partial u}{\partial v} = 0 \text{ on } \partial \Omega_\varepsilon \right\}. \]

We point out that solving problem (2.1) is equivalent to solve

\[ S_{\varepsilon}(u) := \Delta u + u - f(u) = 0, \quad u \in H^2_N(\Omega_\varepsilon). \]

Let \( k \geq 1 \) be a fixed integer and for small \( \delta > 0 \) consider the set

\[ A_\delta := \left\{ P \in \Omega^k: \varphi_k(P) \geq \delta \right\}, \]

where the function \( \varphi_k \) is defined in (1.2).

For \( P \in A_\delta \) let us set

\[ \mathcal{K}_{\varepsilon, P} = \text{span} \left\{ \frac{\partial \mathcal{P}w_i}{\partial P^j_i}: i = 1, \ldots, k, \; j = 1, \ldots, N \right\} \subset H^2_N(\Omega_\varepsilon), \]

\[ \mathcal{C}_{\varepsilon, P} = \text{span} \left\{ \frac{\partial \mathcal{P}w_i}{\partial P^j_i}: i = 1, \ldots, k, \; j = 1, \ldots, N \right\} \subset L^2(\Omega_\varepsilon), \]

denoting by \( P^j_i \) the \( j \)-th component of \( P_i \) for \( j = 1, \ldots, N \). We also need the orthogonal spaces

\[ \mathcal{K}^\perp_{\varepsilon, P} = \left\{ u \in H^2_N(\Omega_\varepsilon): \int_{\Omega_\varepsilon} u \frac{\partial \mathcal{P}w_i}{\partial P^j_i} = 0, \; i = 1, \ldots, k, \; j = 1, \ldots, N \right\}, \]

\[ \mathcal{C}^\perp_{\varepsilon, P} = \left\{ u \in L^2(\Omega_\varepsilon): \int_{\Omega_\varepsilon} u \frac{\partial \mathcal{P}w_i}{\partial P^j_i} = 0, \; i = 1, \ldots, k, \; j = 1, \ldots, N \right\}. \]

We have the following crucial result.
Proposition 3.1. For any $\delta > 0$ there exist $\varepsilon_0 > 0$ and $c > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and for any $P \in A_\delta$ there exists a unique $\Phi_{\varepsilon, P} \in K_{\varepsilon, P}^+$ such that

$$S_{\varepsilon} \left( \sum_{i=1}^{k} P_i w_i + \Phi_{\varepsilon, P} \right) \in C_{\varepsilon, P}^+.$$ 

Moreover

$$\|\Phi_{\varepsilon, P}\|_{H^2(\Omega_\varepsilon)} \leq c \varepsilon^{-\left(1 + \frac{2}{\gamma} \right)} e^{\|\Phi_{\varepsilon, P}\|}.$$ 

Finally, the function $P \to \Phi_{\varepsilon, P} \in H^1(\Omega_\varepsilon)$ is of class $C^1$.

Proof. We argue exactly as in [10, Section 3] and [1, Section 3].

Let us introduce the reduced energy $\tilde{J}_\varepsilon : A_\delta \to \mathbb{R}$ as

$$\tilde{J}_\varepsilon(P) := J_{\varepsilon} \left( \sum_{i=1}^{k} P_i w_i + \Phi_{\varepsilon, P} \right).$$

The following result holds.

Proposition 3.2. If $P \in A_\delta$ is a critical point of $\tilde{J}_\varepsilon$, then the function $u_\varepsilon = \sum_{i=1}^{k} P_i w_i + \Phi_{\varepsilon, P}$ is a critical point of $J_{\varepsilon}$, i.e. a solution to problem (2.1).

Proof. We argue exactly as in [1, Proposition 3.6].

Finally we need the expansion of the reduced energy.

Lemma 3.3. It holds

$$\tilde{J}_\varepsilon(P) = k I(w) - \frac{1}{2} \left( \gamma + o(1) \right) e^{-\Phi_{\varepsilon, P}}$$

uniformly with respect to $P \in A_\delta$. Here

$$\Phi_{\varepsilon}(P) := -\varepsilon \ln \left[ -\sum_{i=1}^{k} \varphi_{\varepsilon, P_i}(P_i) + \sum_{i,h=1}^{k} \sum_{i \neq h} w \left( \frac{|P_i - P_h|}{\varepsilon} \right) \right]$$

satisfies

$$\Phi_{\varepsilon}(P) \to 2 \varphi_{\varepsilon}(P) \quad \text{uniformly in } A_\delta \text{ as } \varepsilon \to 0.$$ 

Moreover

$$I(w) := \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla w|^2 + w^2) \, dy - \int_{\mathbb{R}^N} F(w) \, dy$$
and
\[
\gamma := \int_{\mathbb{R}^N} f(w)e^{-y_1} \, dy.
\]

**Proof.** We argue exactly as in [8, Theorem 3.3]. □

4. The local maximization argument: Proof of Theorem 1.1

In order to set up a maximization problem, we need some auxiliary lemmas.

**Lemma 4.1.** Let \( \Omega \subset \mathbb{R}^2 \) be a smooth and bounded domain. Then, for any \( \delta_0 > 0 \) there exist \( \bar{\delta} := \bar{\delta}(\delta_0) \in (0, \delta_0) \) and an integer \( \bar{k} := \bar{k}(\delta_0) \) such that
\[
\sup_{d_{\partial \Omega}(P) = \bar{\delta}} \varphi_{\bar{k}}(P) = \bar{\delta}.
\]
Moreover, if \( P^* = (P^*_1, \ldots, P^*_\bar{k}) \in \Omega_{\bar{k}} \) is such that \( d_{\partial \Omega}(P^*_i) = \bar{\delta} \) for all \( i \) and \( \varphi_{\bar{k}}(P^*) = \bar{\delta} \), then the points \( P^*_1, \ldots, P^*_\bar{k} \) are located at distance \( \bar{\delta} \) from the boundary and the distance between two successive \( P^*_i \)'s is equal to \( 2\bar{\delta} \). Finally \( \bar{k} = \bar{k}(\delta_0) \to +\infty \) as \( \delta_0 \to 0 \).

**Proof.** Let \( \delta_0 > 0 \) be sufficiently small such that for any \( \delta \in (0, \delta_0] \) the set
\[
\gamma_\delta := \{ P \in \Omega \mid d_{\partial \Omega}(P) = \delta \}
\]
is a regular closed curve and
\[
\text{every point of } \gamma_\delta \text{ has exactly two points on } \gamma_\delta \text{ at distance } 2\delta. \tag{4.1}
\]

Then set \( \bar{k} = \ell(\partial \Omega) \frac{2\delta_0}{\delta_0} \).

For \( Q \in \partial \Omega \) let \( v_Q \) denote the unit inward normal to \( \partial \Omega \) at \( Q \). For any \( Q \in \partial \Omega \) and \( \delta \in (0, \delta_0] \) let us consider the polygonal \( \Sigma_{Q,\delta} = \{P^0_1, \ldots, P^0_{\bar{k}}\} \) where \( P^0_1 = Q + \delta v_Q \in \gamma_\delta \) and the point \( P^0_{i+1} \) is the next point (proceeding anticlockwise) on \( \gamma_\delta \) after \( P^0_i \) such that \( |P^0_{i+1} - P^0_i| = 2\delta \) for \( i = 1, \ldots, \bar{k} - 1 \).

Fixed \( Q \in \partial \Omega \), let us consider the continuous map
\[
\delta \to \Sigma_{Q,\delta} = (P^\delta_1, \ldots, P^\delta_{\bar{k}})
\]
(the continuity follows from (4.1)). The definition of \( \bar{k} \) implies that
\[
\min_{|i-j| > 1} |P^{\delta_0}_i - P^{\delta_0}_j| < 2\delta_0, \quad P^{\delta_0}_i \in \Sigma_{Q,\delta_0}.
\]

On the other hand, \( |P^\delta_i - P^\delta_j| \approx 2|i-j|\delta \) as \( \delta \to 0 \), therefore, if \( \delta \) is small enough, \( |P^\delta_i - P^\delta_j| > 2\delta \) for \( |i-j| > 1 \). By continuity, there exists \( \delta_Q \) such that
\[
\delta_Q = \max \{ \delta \leq \delta_0 \mid \min_{|i-j| > 1} |P^\delta_i - P^\delta_j| = 2\delta, \quad P^\delta_i \in \Sigma_{Q,\delta} \}.
\]

Next we set
\[ \delta = \sup_{Q \in \partial \Omega} \delta_Q. \]

For any \( \varepsilon > 0 \), there exists a \( Q_\varepsilon \) such that \( \delta \geq \delta_{Q_\varepsilon} > \delta - \varepsilon \); therefore the polygonal \( \Sigma_{Q_\varepsilon, \delta_{Q_\varepsilon}} = \{ P_1, \ldots, P_k \} \) has the property \( \min_{i \neq j} |P_i - P_j| = 2 \delta_{Q_\varepsilon} \geq 2 \delta - 2 \varepsilon \). We have thus proved

\[ \sup_{d_{\partial \Omega}(P_1) = \delta} \varphi_k(P) = \delta. \]

Finally let \( P^* = (P^*_1, \ldots, P^*_k) \in \Omega^\# \) be such that \( d_{\partial \Omega}(P^*_i) = \delta \) for all \( i \) and \( \varphi_k(P^*) = \delta \), which implies that \( P^*_1, \ldots, P^*_k \in \gamma_\delta \) and

\[ \min_{i \neq j} |P^*_i - P^*_j| \geq 2 \delta. \]

Let us consider a parametrization of \( \gamma_\delta : s \in [0, 1] \mapsto \gamma_\delta(s) \) and assume that

\[ P^*_1 = \gamma_\delta(s^*_1), \ldots, P^*_k = \gamma_\delta(s^*_k) \]

with

\[ 0 = s^*_1 < s^*_2 < \cdots < s^*_k < 1. \]

Set \( P^*_i = Q^* + \nu_Q \cdot \delta \), with \( Q^* \in \partial \Omega \).

Assume by contradiction that \(|P^*_1 - P^*_2| > 2 \delta\). Let us construct the polygonal \( \Sigma_{Q^*, \delta} = \{ \gamma_\delta(0), \gamma_\delta(s_2), \ldots, \gamma_\delta(s_k) \} \). Since \(|P^*_1 - P^*_2| = |\gamma_\delta(0) - \gamma_\delta(s_2)| > 2 \delta\), we have \( s_2 < s^*_2 \).

We claim that \(|\gamma_\delta(s_2) - P^*_3| = |\gamma_\delta(s_2) - \gamma_\delta(s^*_3)| > 2 \delta\) (otherwise, there would exist an \( s \) in \((0, s_2]\), another \( s \) in \([s^*_2, s^*_3)\) and another in \((s^*_3, s^*_k]\) such that \(|P^*_3 - \gamma_\delta(s)| = 2 \delta\), in contradiction with (4.1)). Consequently \( s_3 < s^*_3 \). By repeating the same argument we arrive at \( s_k < s^*_k \) and \( |\gamma_\delta(s_k) - \gamma_\delta(1)| = |\gamma_\delta(s_k) - \gamma_\delta(0)| > 2 \delta\). This implies that \( \min_{i \neq j > 1} |\gamma_\delta(s_i) - \gamma_\delta(s_j)| > 2 \delta\), and by the definition of \( \delta_{Q^*} \) we would derive \( \delta_{Q^*} > \delta \), and the contradiction follows. This concludes the proof. \( \square \)

**Lemma 4.2.** Let \( D \subset \mathbb{R}^2 \) be a strictly convex, smooth and bounded domain. Then, for any \( \delta > 0 \) there exists \( \eta > 0 \) such that, if \( P, Q \in \partial D \), \(|P - Q| \geq \delta \) and \( \eta_1, \eta_2 \in [0, \eta] \), \((\eta_1, \eta_2) \neq (0, 0), \) then

\[ |P + \nu_P \eta_1 - Q - \nu_Q \eta_2| < |P - Q|. \]

**Proof.** Fixed \( \delta > 0 \), the strict convexity implies

\[ \inf_{P \in \partial D, |Q - P| \geq \delta} \nu_P \cdot (Q - P) = \eta > 0, \]

where \( \nu_P \) is the unit inward normal to \( \partial D \) at \( P \). For \(|P - Q| \geq \delta \), \( \eta_1, \eta_2 \in [0, \eta] \) with \((\eta_1, \eta_2) \neq (0, 0)\), we compute

\[ \begin{align*}
|P + \nu_P \eta_1 - Q - \nu_Q \eta_2|^2 &- |P - Q|^2 \\
&= 2 \eta_1 (P - Q) \cdot \nu_P + 2 \eta_2 (Q - P) \cdot \nu_Q + \eta_1^2 + \eta_2^2 - 2 \eta_1 \eta_2 \nu_P \nu_Q \\
&\leq -2(\eta_1 + \eta_2) \eta + (\eta_1 + \eta_2)^2 < 0. \quad \square
\end{align*} \]
Lemma 4.3. Let $\Omega \subset \mathbb{R}^2$ be a strictly convex, smooth and bounded domain. Then, for any $\delta_0 > 0$ there exist $\bar{\delta} := \delta(\delta_0) \in (0, \delta_0)$ and an integer $k := k(\delta_0)$ such that, if $\eta$ is sufficiently small, then

\[
\sup_{P \in \partial U^k_\eta} \varphi_k(P) < \sup_{P \in U^k_\eta} \varphi_k(P) = \bar{\delta}
\]  

(4.2)

where

\[
U_\eta = \{ P \in \Omega \mid \bar{\delta} - \eta < d_{\partial \Omega}(P) < \bar{\delta} + \eta \}.
\]

Moreover, if $P_* = (P_1^*, \ldots, P_k^*) \in U^k_\eta$ is such that $\varphi_k(P_*) = \bar{\delta}$, then the points $P_1^*, \ldots, P_k^*$ are located at distance $\bar{\delta}$ from the boundary and the distance between two successive $P_i^*$'s is equal to $\bar{\delta}$. Finally, $\bar{k} \to +\infty$ and $2\bar{\delta} \cdot \bar{k} \to t(\partial \Omega)$ as $\delta_0 \to 0$. In particular

\[
\bar{\delta} \approx \frac{\ell(\partial \Omega)}{2 \bar{k}} < \max_{\Omega^k} \varphi_k \text{ as } \delta_0 \to 0.
\]  

(4.3)

Proof. Let $\bar{\delta} \in (0, \delta_0)$ and $\bar{k} = k(\delta_0)$ such that Lemma 4.1 holds. Set $D = \{ P \in \Omega \mid d_{\partial \Omega}(P) > \bar{\delta} \}$ and $\partial D = \gamma_{\bar{\delta}} = \{ P \in \Omega \mid d_{\partial \Omega}(P) = \bar{\delta} \}$. Assume $\delta_0$ sufficiently small so that $D$ is strictly convex and $\gamma_{\bar{\delta}}$ is a regular curve. Let us choose $\eta$ such that $\eta \in (0, \frac{\bar{\delta}}{2})$ and, according to Lemma 4.2, if $Q \in \gamma_{\bar{\delta}}$, $|Q - Q'| \geq \frac{\bar{\delta}}{2}$ and $\eta_1, \eta_2 \in [0, \eta]$, $(\eta_1, \eta_2) \neq (0, 0)$, then

\[
|Q + v_Q \eta_1 - Q' - v_Q \eta_2| < |Q - Q'|.
\]  

(4.4)

According to Lemma 4.1 we have $\sup_{P \in U^k_\eta} \varphi_k(P) \geq \bar{\delta}$. Estimate (4.2) will follow if we prove that

\[
P \in \partial U^k_\eta \Rightarrow \varphi_k(P) < \bar{\delta}.
\]  

(4.5)

It is immediate that, if $d_{\partial \Omega}(P_i) < \bar{\delta}$ for some $i$ or $|P_i - P_j| < 2\bar{\delta}$ for some $i \neq j$, then $\varphi_k(P) < \bar{\delta}$. Therefore, without loss of generality we may assume

\[
d_{\partial \Omega}(P_i) \geq \bar{\delta} \quad \forall i, \quad d_{\partial \Omega}(P_1) = \bar{\delta} + \eta, \quad |P_i - P_j| \geq 2\bar{\delta} \quad \forall i \neq j.
\]

Consider $Q_i$ the projections of $P_i$ on $\gamma_{\bar{\delta}}$, i.e.

\[
P_i = Q_i + \eta_i v_Q, \quad Q_i \in \gamma_{\bar{\delta}}, \quad \eta_i \in [0, \eta],
\]

denoting by $v_Q$ the unit inward normal to $D$ at $Q \in \gamma_{\bar{\delta}}$, and, considering a parametrization $s \in [0, 1] \mapsto \gamma_{\bar{\delta}}(s)$, assume that

\[
Q_1 = \gamma_{\bar{\delta}}(s_1), \quad \ldots, \quad Q_k = \gamma_{\bar{\delta}}(s_k) \quad \text{with} \quad 0 = s_1 < s_2 < \cdots < s_k < 1.
\]

If there exist $i \neq j$ such that $|Q_i - Q_j| \leq \frac{\bar{\delta}}{2}$, then
\[|P_i - P_j|^2 = |Q_i + v Q_i \eta_i - Q_j + v Q_j \eta_j|^2\]

\[= |Q_i - Q_j|^2 + |v Q_i \eta_i - v Q_j \eta_j|^2 + 2(Q_i - Q_j, v Q_i \eta_i - \eta_j v Q_j)\]

\[\leq |Q_i - Q_j|^2 + 4|Q_i - Q_j| + 4\eta^2 \leq \frac{\delta^2}{4} + 2\delta \eta + 4\eta^2 < 4\delta^2,\]

and so \(\varphi_k(P) < \delta\), by which (4.5) follows.

Finally assume \(|Q_i - Q_j| \geq \frac{\delta}{2}\) if \(i \neq j\). Then \(|Q_i - Q_j| \geq |P_i - P_j|\) by (4.4). If \(|Q_i - Q_j| < 2\delta\) for some \(i \neq j\), then \(\varphi_k(P) \leq \varphi_k(Q) < \delta\) and we have done. Now assume \(|Q_i - Q_j| > 2\delta\) for every \(i \neq j\), which means \(\varphi_k(Q) = \delta\). By Lemma 4.1 \(|Q_2 - Q_1| = 2\delta\). Then (4.4) implies \(|P_2 - P_1| < |Q_2 - Q_1| = 2\delta\), which implies \(\varphi_k(P) < \delta\), and (4.5) follows. We have thus proved (4.2).

From the arbitrariness of \(\eta\) in (4.2), if \(P^* = (P_1^*, \ldots, P_k^*) \in U_{\eta}^k\) is such that \(\varphi_k(P^*) = \delta\), then \(d_{\partial \Omega}(P_i^*) = \delta\) for every \(i\) and, consequently, according to Lemma 4.1 the points \(P_1^*, \ldots, P_k^*\) are located at a distance \(\delta\) from the boundary and the distance between two successive \(P_i^*\)’s is \(2\delta\), i.e. the points \(P_1^*, \ldots, P_k^*\) form a polygonal having vertices on \(\gamma_\delta\) with edge \(2\delta\), and the length of this polygon is \(2\delta k\). Since \(\gamma_\delta \to \partial \Omega\), then we get \(2\delta k \to \ell(\partial \Omega)\) as \(\delta_0 \to 0\).

It remains to show that

\[
\frac{\ell(\partial \Omega)}{2k} < \max_{P \in \Omega} \varphi_k(P) \quad \text{as} \quad k \to +\infty. \tag{4.6}
\]

(4.3) will follow by (4.6) taking into account that \(k \to +\infty\) and \(2\delta k \to \ell(\partial \Omega)\) as \(\delta_0 \to 0\). Let \(Q\) be a cube contained in \(\Omega\) and set \(l\) the length of its edge. Then \(Q\) contains \((\frac{\ell(\partial \Omega)}{\ell(\partial \Omega)} )^N\) disjoint balls of radius \(\frac{\ell(\partial \Omega)}{2k}\). Then (4.6) follows since \((\frac{\ell(\partial \Omega)}{\ell(\partial \Omega)} )^N > k\) for large \(k\).

**Proof of Theorem 1.1.** By Lemma 4.3, we deduce that, for any \(\delta_0 > 0\) there exist \(\delta \in (0, \delta_0)\) and an integer \(k\) such that, for, if \(\eta > 0\) is sufficiently small,

\[
a := \sup_{P \in U_{\eta}^k} \varphi_k(P) < \sup_{P \in U_{\eta}^k} \varphi_k(P) = \delta,
\]

where \(U_{\eta} = \{P \in \Omega \mid \delta - \eta < d_{\partial \Omega}(P) < \delta + \eta\}\). By (3.2) we obtain that for any \(\mu > 0\) there exists \(\epsilon_0 > 0\) such that for \(\epsilon \in (0, \epsilon_0)\)

\[
\max_{P \in U_{\eta}^k} |\Phi(\theta) - 2\varphi_k(P)| \leq \mu.
\]

Assume \(\epsilon_0\) sufficiently small such that \(|o(1)| \leq \frac{\epsilon}{2}\) for any \(P \in U_{\eta}^k\), where \(o(1)\) is the function in (3.1). Then by Lemma 3.3, using (3.2), we deduce that for \(\epsilon \in (0, \epsilon_0)\)

\[
\sup_{P \in U_{\eta}^k} \tilde{J}_\epsilon(P) \leq \tilde{k} l(w) - \frac{\gamma'}{4} \inf_{P \in U_{\eta}^k} e^{-\frac{\varphi_k(P)}{\epsilon}} \leq \tilde{k} l(w) - \frac{\gamma'}{4} e^{-2\varphi_k(P) + \mu} = \tilde{k} l(w) - \frac{\gamma'}{4} e^{-2\varphi_k(P) + \mu}
\]

while
\[ \sup_{P \in U^k_\eta} \tilde{J}_\varepsilon(P) > \tilde{k}I(w) - \gamma \inf_{P \in U^k_\eta} e^{-\phi_\varepsilon(P) / \varepsilon} \geq \tilde{k}I(w) - \gamma e^{-2\varepsilon} = \tilde{k}I(w) - \gamma e^{-\frac{2\varepsilon}{\varepsilon}}. \]

Consequently, choosing \( \mu < \delta - \alpha \), we have

\[
\sup_{P \in \mathcal{D}_\eta} \tilde{J}_\varepsilon(P) < \sup_{P \in U^k_\eta} \tilde{J}_\varepsilon(P)
\]

for \( \varepsilon \) small enough. Therefore, \( \tilde{J}_\varepsilon \) has a local maximum point \( P^* \in U^k_\eta \). Let \( P^* \) be such that, up to a subsequence, \( P^* \to P^* \) as \( \varepsilon \to 0 \). We claim that \( \varphi_k(P^*) = \delta \). Indeed, since \( P^* \in U^k_\eta \) we immediately have \( \varphi_k(P^*) \leq \delta \). On the other hand, if by contradiction it was \( \varphi_k(P^*) < \alpha < \alpha' < \delta \) for some suitable \( \alpha, \alpha' \), then by (3.2) for \( \varepsilon \) sufficiently small it would result \( \Phi_\varepsilon(P^*) < 2\alpha \), and, consequently, using again Lemma 3.3,

\[
\tilde{J}_\varepsilon(P^*) = \tilde{k}I(w) - \frac{1}{2} (\gamma + o(1))e^{-\phi_\varepsilon(P^*) / \varepsilon} \leq \tilde{k}I(w) - \frac{\gamma}{4} e^{-\frac{2\varepsilon}{\varepsilon}}.
\]

On the other hand, if \( P_0 \in U^k_\eta \) is such that \( \varphi_k(P_0) = \delta \), then for small \( \varepsilon \) we have \( \Phi_\varepsilon(P_0) > 2\alpha' \), and consequently

\[
\tilde{J}_\varepsilon(P_0) = \tilde{k}I(w) - \frac{1}{2} (\gamma + o(1))e^{-\phi_\varepsilon(P_0) / \varepsilon} \geq \tilde{k}I(w) - \gamma e^{-\frac{2\varepsilon}{\varepsilon}}.
\]

Since \( P^* \) is the maximum point of \( \tilde{J}_\varepsilon \) on \( U^k_\eta \), then \( \tilde{J}_\varepsilon(P^*) > \tilde{J}_\varepsilon(P_0) \), by which \( \alpha > \alpha' \), and the contradiction follows. Hence we have proved that \( \varphi_k(P^*) = \delta \).

Then Theorem 1.1 follows from Lemma 4.3 and Proposition 3.2. \( \square \)

5. The min–max argument: Proof of Theorems 1.2 and 1.3

We want to apply a min–max argument to characterize a topologically nontrivial critical value of \( \varphi_k \) which will produces a critical point of \( \tilde{J}_\varepsilon \) provided \( \varepsilon \) is small enough.

Let \( k \) be a fixed integer. We fix \( \varepsilon \in \mathbb{R}^N \) with \( |\varepsilon| = 1 \) and we define the continuous function \( S : \mathbb{R}^N \times (0, \infty)^{k-1} \to \mathbb{R}^{kN} \) by

\[
S_1(P, r) := P, \quad S_2(P, r) := P + r_2\varepsilon, \quad \ldots, \quad S_k(P, r) := P + r_2\varepsilon + \cdots + r_k\varepsilon,
\]

where we set \( r = (r_2, \ldots, r_k) \). It is useful to point out that

\[
r_i = \min_{j < i} |S_i(P, r) - S_j(P, r)|, \quad i = 2, \ldots, k.
\]

Let \( \delta > 0 \) be fixed and let us define

\[
\mathcal{D} := \left\{ P \in \Omega^k : \varphi_k(P) \geq \frac{\delta}{2} \right\}.
\]

Consider \( W \) the following open set of \( \mathbb{R}^{N+k-1} \):
\[ W := \{(P, r) \mid S(P, r) \in \Omega^k, \varphi_k(S(P, r)) > \delta \}. \]

Now choose \( P_0 \in \Omega \) such that \( d_{\partial \Omega}(P_0) = \max_{P \in \Omega} d_{\partial \Omega}(P) \) and set \( r_0 = d_{\partial \Omega}(P_0)/(k + 1) \). We claim that \((P_0, r_0) \in W\), where \( r_0 := (r_0, \ldots, r_0) \in \mathbb{R}^{k-1} \), provided \( \delta \) is small enough. Indeed by (5.1) we immediately deduce that \( \min_{i \neq j} |S_i(P_0, r_0) - S_j(P_0, r_0)| = r_0 \). Moreover, (5.1) also implies that \( |S_i(P_0, r_0) - P_0| \leq kr_0 \) for any \( i = 2, \ldots, k \). Therefore

\[ d_{\partial \Omega}(S_i(P_0, r_0)) \geq d_{\partial \Omega}(P_0) - |S_i(P_0, r_0) - P_0| \geq (k + 1)r_0 - kr_0 = r_0. \]

Finally, we deduce that

\[ \varphi_k(S(P_0, r_0)) = r_0 \leq \frac{d_{\partial \Omega}(P_0)}{k + 1} > \delta, \]

provided \( \delta \) is small enough and so \((P_0, r_0) \in W\).

Let \( U \) be the connected component of \( W \) containing \((P_0, r_0)\) and let us define

\[ K := \{S(P, r) \mid (P, r) \in \widetilde{U}\} \quad \text{and} \quad K_0 := \{S(P, r) \mid (P, r) \in \partial U\}. \]

We remark that \( D \) is an open set, \( K_0 \) and \( K \) are compact sets, \( K \) is connected (since \( U \) is connected) and

\[ K_0 \subset K \subset D \subset \overline{D} \subset \Omega^k. \]

\( K_0 \) can be rewritten as

\[ K_0 = \{S(P, r) \mid (P, r) \in \widetilde{U}, \varphi_k(S(P, r)) = \delta\}. \]

It is trivial that

\[ \max_{P \in \partial D} \varphi_k(P) = \frac{\delta}{2}, \quad \max_{P \in K_0} \varphi_k(P) = \delta. \quad (5.2) \]

Let \( \mathcal{F} \) be the complete metric space defined by

\[ \mathcal{F} = \{\eta : K \to D \mid \eta \text{ continuous, } \eta(P) = P \forall P \in K_0\}. \]

**Lemma 5.1.** There exists \( \delta_0 = \delta_0(k) \) such that if \( \delta \in (0, \delta_0) \)

\[ \inf_{\eta \in \mathcal{F}} \max_{P \in K} \varphi_k(\eta(P)) > \delta. \quad (5.3) \]

**Proof.** Let \( \eta \in \mathcal{F} \), namely \( \eta : K \to D \) is a continuous function such that \( \eta(P) = P \) for any \( P \in K_0 \). Setting \( \eta = (\eta_1, \ldots, \eta_k) \) where \( \eta_i : K \to \mathbb{R}^N \), let \( \tilde{\eta} : U \to \mathbb{R}^N \times \mathbb{R}^{k-1} \) be defined by

\[ \tilde{\eta}_1(P, r) = \eta_1(S(P, r)) \quad \text{and} \quad \tilde{\eta}_i(P, r) = \min_{j < i} |\eta_j(S(P, r)) - \eta_j(S(P, r))| \quad \text{if } i = 2, \ldots, k. \]

First of all \( \tilde{\eta} \) is a continuous function, because of the continuity of \( \eta \). Secondly, we claim that \( \tilde{\eta}(P, r) = (P, r) \) for any \((P, r) \in \partial U\). In fact, if \((P, r) \in \partial U\), then by definition \( S(P, r) \in K_0 \) and consequently \( \eta(S(P, r)) = S(P, r) \). Then
\[ \tilde{\eta}_1(P, r) = \eta_1(S(P, r)) = S_1(P, r) = P \]

and, by (5.1), if \( i \geq 2 \)
\[ \tilde{\eta}_i(P, r) = \min_{j < i} |\eta_i(S(P, r)) - \eta_j(S(P, r))| = \min_{j < i} |S_i(P, r) - S_j(P, r)| = r_i. \]

Hence the theory of the topological degree ensures that there exists \((P_0, r_0) \in U\) such that \( \tilde{\eta}(P_0, r_0) = (P_0, r_0) \), that is
\[ \eta_1(S(P_0, r_0)) = P_0 \quad \text{and} \quad \min_{j < i} |\eta_i(S(P_0, r_0)) - \eta_j(S(P_0, r_0))| = r_0, \quad i = 2, \ldots, k. \]

In particular
\[ \min_{j \neq i} |\eta_i(S(P_0, r_0)) - \eta_j(S(P_0, r_0))| \geq r_0. \]

Moreover, it is not difficult to check that
\[ |\eta_i(S(P, r_0)) - P_0| = |\eta_i(S(P, r_0)) - \eta_1(S(P, r_0))| \leq kr_0, \quad i = 2, \ldots, k, \]
which implies
\[ d_{S\Omega}(\eta_i(S(P, r_0))) \geq d_{S\Omega}(P_0) - |\eta_i(S(P, r_0)) - P_0| \geq (k + 1)r_0 - kr_0 = r_0. \]

Therefore, \( \varphi_k(\eta_i(S(P, r_0))) = r_0 \) and so \( \max_{P \in K} \varphi_k(\eta(P)) \geq r_0 \). Finally, by taking the infimum for all \( \eta \in F \),
\[ \inf_{\eta \in F} \max_{P \in K} \varphi_k(\eta(P)) \geq r_0 > \delta, \]
provided \( \delta \) is small enough. \( \square \)

**Lemma 5.2.** It holds
\[ c_0 := \inf_{\eta \in F} \max_{P \in K} \varphi_k(\eta(P)) > \max \left\{ \max_{P \in K_0} \varphi_k(P), \max_{P \in \partial D} \varphi_k(P) \right\}. \quad (5.4) \]

In particular, there exists a critical point \( P_0 \in D \) of \( \varphi_k \) with \( \varphi_k(P_0) = c_0 \).

**Proof.** Estimate (5.4) follows by (5.2) and (5.3). The existence of a critical point follows for example by [2]. \( \square \)

We are going to prove that the critical level \( c_0 \) found in Lemma 5.2 is different from the maximum level of the function \( \varphi_k \).

**Lemma 5.3.** There exists \( k_0 > 0 \) such that for any \( k \geq k_0 \) it holds \( c_0 \leq \frac{\text{diam}(Q)}{2k} < \max_{\Omega} \varphi_k \).
Proof. Let \( e \) be the direction of the diameter of \( \Omega \). Then for any \( P \in \Omega \) it holds
\[
\varphi_k(P, P + r_1 e, \ldots, P + (r_1 + \cdots + r_k) e) \leq \frac{\text{diam}(\Omega)}{2k}.
\]
Therefore, if we choose \( \eta \) as the identity map, we have
\[
c_0 = \inf_{\eta \in \mathcal{F}} \max_{P \in K} \varphi_k(\eta(P)) \leq \max_{P \in K} \varphi_k(P) \leq \frac{\text{diam}(\Omega)}{k}.
\]

Let \( Q \) be a cube contained in \( \Omega \) and set \( l \) the length of its edge. Then \( Q \) contains \( (\frac{k l}{2 \text{diam}(\Omega)})^N \) disjoint balls of radius \( \frac{\text{diam}(\Omega)}{k} \). Therefore, since \( (\frac{k l}{2 \text{diam}(\Omega)})^N > k \) for large \( k \), then we conclude
\[
\max_{P \in \Omega} \varphi_k(P) \geq \frac{\text{diam}(\Omega)}{k}. \tag{5.5}
\]

On the other hand, we can choose \( k \) points \( P_1^*, \ldots, P_k^* \in \Omega \) which are vertices of a regular polygon centered at \( P_0 \) \( d_{\partial \Omega}(P_1^*) = |P_1^* - P_2^*|/2 = d_{\partial \Omega}(P_0) \frac{\sin(\pi/k)}{1 + \sin(\pi/k)} \). We get
\[
\varphi_k(P_1^*, \ldots, P_k^*) = d_{\partial \Omega}(P_0) \frac{\sin(\pi/k)}{1 + \sin(\pi/k)},
\]
so that
\[
\max_{P \in \Omega^k} \varphi_k(P) \geq d_{\partial \Omega}(P_0) \frac{\sin(\pi/k)}{1 + \sin(\pi/k)}. \tag{5.6}
\]

Finally, it is easy to check that
\[
d_{\partial \Omega}(P_0) \frac{\sin(\pi/k)}{1 + \sin(\pi/k)} > \frac{d_{\partial \Omega}(P_0)}{k} \quad \text{if} \ k \geq 4. \tag{5.7}
\]
The claim follows by (5.5), (5.6) and (5.7). □

**Proof of Theorem 1.2.** Let us show how, by Lemmas 3.3 and 5.2, we may deduce that, if \( \varepsilon \) is small enough, the function \( \tilde{J}_\varepsilon \) satisfies

\[
c_\varepsilon := \inf_{\eta \in \mathcal{F}} \max_{P \in K} \tilde{J}_\varepsilon(\eta(P)) > \max\left\{ \max_{P \in K_0} \tilde{J}_\varepsilon(\eta(P)), \max_{P \in \partial \mathcal{D}} \tilde{J}_\varepsilon(\eta(P)) \right\}. \tag{5.8}
\]

By (5.4) we get that there exists \( a > b \) such that

\[
\inf_{\eta \in \mathcal{F}} \max_{P \in K} \varphi_k(\eta(P)) \geq a > b = \max\left\{ \max_{P \in K_0} \varphi_k(\eta(P)), \max_{P \in \partial \mathcal{D}} \varphi_k(\eta(P)) \right\}. \tag{5.9}
\]

Moreover, by (3.2) we obtain that for any \( \mu > 0 \) there exists \( \varepsilon_0 > 0 \) such that for \( \varepsilon \in (0, \varepsilon_0) \)

\[
\max_{P \in \mathcal{D}} |\Phi_\varepsilon(P) - 2\varphi_k(P)| \leq \mu. \tag{5.10}
\]

Since \( \eta(K) \subset \mathcal{D} \) for any \( \eta \in \mathcal{F} \), by (5.9) and (5.10), choosing \( \mu < a - b \), we easily get that for any \( \varepsilon \in (0, \varepsilon_0) \)

\[
\inf_{\eta \in \mathcal{F}} \max_{P \in K} \Phi_\varepsilon(\eta(P)) \geq 2a - \mu > 2b + \mu \geq \max\left\{ \max_{P \in K_0} \Phi_\varepsilon(\eta(P)), \max_{P \in \partial \mathcal{D}} \Phi_\varepsilon(\eta(P)) \right\}. \tag{5.11}
\]

Let us see how by (5.11) we get (5.8), using Lemma 3.3. First of all, we remark that we may assume \( \varepsilon_0 \) sufficiently small such that for any \( \varepsilon \in (0, \varepsilon_0) \) \( |o(1)| \leq \frac{\varepsilon}{2} \) for any \( P \in \mathcal{D} \), where \( o(1) \) is the function in formula (3.1). For any \( \eta \in \mathcal{F} \) let \( P_\varepsilon \in \eta(K) \) be such that \( \max_{P \in K} \Phi_\varepsilon(\eta(P)) = \Phi_\varepsilon(P_\varepsilon) \). Therefore by (5.10) we get

\[
\max_{P \in K} \tilde{J}_\varepsilon(\eta(P)) \geq \tilde{J}_\varepsilon(P_\varepsilon) = kl(w) - \frac{1}{2}(\gamma + o(1))e^{-\frac{\varphi_k(P_\varepsilon)}{\varepsilon}}
\geq kl(w) - e^{-\frac{\varphi_k(P_\varepsilon)}{\varepsilon}} \geq kl(w) - \gamma e^{-\frac{2a-\mu}{\varepsilon}},
\]

which implies

\[
\inf_{\eta \in \mathcal{F}} \max_{P \in K} \tilde{J}_\varepsilon(\eta(P)) \geq kl(w) - \gamma e^{-\frac{2a-\mu}{\varepsilon}}. \tag{5.12}
\]

On the other hand, let \( Q_\varepsilon \in K_0 \cup \partial \mathcal{D} \) be such that

\[
\max\left\{ \max_{P \in K_0} \tilde{J}_\varepsilon(P), \max_{P \in \partial \mathcal{D}} \tilde{J}_\varepsilon(P) \right\} = \tilde{J}_\varepsilon(Q_\varepsilon).
\]

Therefore by (5.10) we get

\[
\max\left\{ \max_{P \in K_0} \tilde{J}_\varepsilon(P), \max_{P \in \partial \mathcal{D}} \tilde{J}_\varepsilon(P) \right\} = \tilde{J}_\varepsilon(Q_\varepsilon) = kl(w) - \frac{1}{2}(\gamma + o(1))e^{-\frac{\varphi_k(Q_\varepsilon)}{\varepsilon}}
\leq kl(w) - \frac{\gamma}{4} e^{-\frac{\varphi_k(Q_\varepsilon)}{\varepsilon}} \leq kl(w) - \frac{\gamma}{4} e^{-\frac{2b+\mu}{\varepsilon}}, \tag{5.13}
\]

since \( \Phi_\varepsilon(Q_\varepsilon) \leq \max\{\max_{P \in K_0} \Phi_\varepsilon(P), \max_{P \in \partial \mathcal{D}} \Phi_\varepsilon(P)\} \). Finally, by (5.12) and (5.13), taking into account that \( 2a - \mu > 2b + \mu \), we deduce that for \( \varepsilon \) small enough estimate (5.8) holds.
Now, by (5.8) we obtain that $\bar{J}_\varepsilon$ has a critical point $P^* \in D$ such that $\bar{J}_\varepsilon(P^*) = c_\varepsilon$.

Let $P^* \in D$ be such that, up to a subsequence, $P^* \rightarrow P^*$. It only remains to prove that $\varphi_k(P^*) \leq c_0$, so that Theorem 1.2 will follow from Lemma 5.3 and Proposition 3.2. Assume by contradiction that $\varphi_k(P^*) = c^* > c_0$. By (5.4), we deduce that there exists $\eta_0 \in \mathcal{F}$ such that $\max_{P^* \in \mathcal{F}} \varphi_k(\eta_0(P)) \leq c_0 + \mu$.

By (5.10) we get that for $\varepsilon \in (0, \varepsilon_0)$

$$\max_{P^* \in \mathcal{F}} \varphi_k(\eta_0(P)) \leq 2c_0 + 3\mu. \tag{5.14}$$

Let $Q_\varepsilon \in \eta_0(K)$ be such that $\max_{P^* \in \mathcal{F}} \bar{J}_\varepsilon(\eta_0(P)) = \bar{J}_\varepsilon(Q_\varepsilon)$. Then by Lemma 3.3 and (5.14), for $\varepsilon \in (0, \varepsilon_0)$ we obtain

$$\max_{P^* \in \mathcal{F}} \bar{J}_\varepsilon(\eta_0(P)) = \bar{J}_\varepsilon(Q_\varepsilon) = kl(w) - \frac{1}{2}(\gamma + o(1))e^{-\varphi_k(Q_\varepsilon)} \leq kl(w) - \frac{\gamma}{4}e^{-\frac{2c_0 + 3\mu}{\varepsilon}},$$

which implies

$$\inf_{\eta \in \mathcal{F}} \max_{P^* \in \mathcal{F}} \bar{J}_\varepsilon(\eta(P)) \leq kl(w) - \frac{\gamma}{4}e^{-\frac{2c_0 + 3\mu}{\varepsilon}}. \tag{5.15}$$

On the other hand, by Lemma 3.3 we also get

$$\bar{J}_\varepsilon(P^*) = kl(w) - \frac{1}{2}(\gamma + o(1))e^{-\varphi_k(P^*)} \geq kl(w) - \gamma e^{-\frac{\varphi_k(P^*)}{\varepsilon}} \geq kl(w) - \gamma e^{-\frac{2c^* - 3\mu}{\varepsilon}}, \tag{5.16}$$

because $P^* \rightarrow P^*$ and by (5.10) $\varphi_k(P^*) \geq \varphi_k(P^*) - \mu \geq 2c^* - 3\mu$ for $\varepsilon$ small enough. If we choose $\mu > 0$ so that $2c_0 + 3\mu < 2c^* - 3\mu$, then by (5.15) and (5.16) it follows

$$c_\varepsilon = \bar{J}_\varepsilon(P^*) > \inf_{\eta \in \mathcal{F}} \max_{P^* \in \mathcal{F}} \bar{J}_\varepsilon(\eta(P))$$

and a contradiction arises. □

**Proof of Theorem 1.3.** We argue as above using Lemma 5.4 instead of Lemma 5.3. □

**References**


