Quantum correction to classical gravitational interaction between two polarizable objects

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A B S T R A C T

When gravity is quantized, there inevitably exist quantum gravitational vacuum fluctuations which induce quadrupole moments in gravitationally polarizable objects and produce a quantum correction to the classical Newtonian interaction between them. Here, based upon linearized quantum gravity and the leading-order perturbation theory, we study, from a quantum field-theoretic prospect, this quantum correction between a pair of gravitationally polarizable objects treated as two-level harmonic oscillators. We find that the interaction potential behaves like r^{-11} in the retarded regime and r^{-10} in the near regime. Our result agrees with what were recently obtained in different approaches. Our study seems to indicate that linearized quantum gravity is robust in dealing with quantum gravitational effects at low energies.

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Two gravitational wave signals GW150914 [1] and GW151226 [2] generated by black hole merging systems were detected recently by the Laser Interferometer Gravitational-Wave Observatory. This confirms directly a prediction of Einstein based on his classical theory of general relativity [3] regarding the existence of gravitational waves which are spacetime ripples propagating through the universe. Quantum mechanically, gravitational interaction is presumably mediated by gravitons when gravity is quantized. However, a full theory of quantum gravity is elusive at the present. Even though, general relativity as an effective field theory provides a framework to probe the low energy quantum gravity effects. In this respect, it has been found that there exists a quantum correction to the Newtonian force between two mass monopoles which behaves as r^{-3} [4], and this is obtained by summing one-loop Feynman diagrams with off-shell gravitons.

Recently, the quantum gravity correction to classical forces was extended to include the quadrupole–quadrupole interaction between a pair of polarizable objects from their induced quadrupole moments due to two-graviton exchange [5]. This correction is computed by first finding out the normal field modes which keep the cycle in phase in which at the outset one object is polarized and radiates a gravitational field which polarizes the second and induces a quadrupole of which the gravitational field in return polarizes the first, and then summing up the zero-point energy of all these normal modes to get the interaction potential between two quadrupoles. This is in close analogy to that in the computation of the Casimir–Polder and van der Waals (vdW) forces between a pair of atoms from their induced dipole moments due to two photon exchange [6]. An advantage of this method is that the details of quantization of the gravitational metric are not needed.

Although a full theory of quantum gravity is absent, one can still use linearized quantum gravity to find quantum gravitational corrections to classical physics which an ultimate quantum gravity theory must produce at low energies. One such example is the quantum light-cone fluctuations produced by gravitons propagating on a background spacetime [7–9]. When gravity is quantized, there inevitably exist quantum gravitational fluctuations which induce quadrupoles in gravitationally polarizable objects, thus giving rise to a quantum correction to the classical interaction between polarizable objects. In this paper, we present in the framework of linearized quantum gravity a different field-theoretic approach to the computation of the quantum gravity correction to the classical gravitational interaction between a pair of polarizable objects. Our approach, which is parallel to that used by Casimir and Polder in

Table 1
Intermediate states contributing to the two-objects potential and corresponding denominators.

Case	$|I\rangle$	$|II\rangle$	$|III\rangle$	Denominator
(1)	$|1_A, 0_B\rangle|1^{(1)}\rangle$	$|0_A, 0_B\rangle|1^{(2)}, 1^{(3)}\rangle$	$|0_A, 1_B\rangle|1^{(4)}\rangle$	$D_i = (\omega_A + \omega')(\omega' + \omega)(\omega_B + \omega')$,
				$D_{ii} = (\omega_A + \omega')(\omega' + \omega)(\omega_B + \omega)$
(2)	$|1_A, 0_B\rangle|1^{(1)}\rangle$	$|1_A, 1_B\rangle|\{0\}\rangle$	$|0_A, 1_B\rangle|1^{(2)}\rangle$	$D_{iii} = (\omega_A + \omega')(\omega_A + \omega_B)(\omega_B + \omega)$
(3)	$|1_A, 0_B\rangle|1^{(1)}\rangle$	$|1_A, 1_B\rangle|\{0\}\rangle$	$|1_A, 0_B\rangle|1^{(2)}\rangle$	$D_{iv} = (\omega_A + \omega')(\omega_A + \omega_B)(\omega_A + \omega)$
(4)	$|1_A, 0_B\rangle|1^{(1)}\rangle$	$|1_A, 1_B\rangle|1^{(2)}, 1^{(3)}\rangle$	$|0_A, 1_B\rangle|1^{(4)}\rangle$	$D_v = (\omega_A + \omega')(\omega_A + \omega_B + \omega' + \omega)(\omega_B + \omega')$
(5)	$|1_A, 0_B\rangle|1^{(1)}\rangle$	$|1_A, 1_B\rangle|1^{(2)}, 1^{(3)}\rangle$	$|1_A, 0_B\rangle|1^{(4)}\rangle$	$D_{vi} = (\omega_A + \omega')(\omega_A + \omega_B + \omega' + \omega)(\omega_A + \omega)$
(6)	$|0_A, 1_B\rangle|1^{(1)}\rangle$	$|0_A, 0_B\rangle|1^{(2)}, 1^{(3)}\rangle$	$|1_A, 0_B\rangle|1^{(4)}\rangle$	$D_{vii} = (\omega_B + \omega')(\omega' + \omega)(\omega_A + \omega')$,
				$D_{viii} = (\omega_B + \omega')(\omega' + \omega)(\omega_A + \omega)$
(7)	$|0_A, 1_B\rangle|1^{(1)}\rangle$	$|1_A, 1_B\rangle|\{0\}\rangle$	$|1_A, 0_B\rangle|1^{(2)}\rangle$	$D_{ix} = (\omega_B + \omega')(\omega_A + \omega_B)(\omega_A + \omega)$
(8)	$|0_A, 1_B\rangle|1^{(1)}\rangle$	$|1_A, 1_B\rangle|\{0\}\rangle$	$|0_A, 1_B\rangle|1^{(2)}\rangle$	$D_x = (\omega_B + \omega')(\omega_A + \omega_B)(\omega_B + \omega)$
(9)	$|0_A, 1_B\rangle|1^{(1)}\rangle$	$|1_A, 1_B\rangle|1^{(2)}, 1^{(3)}\rangle$	$|1_A, 0_B\rangle|1^{(4)}\rangle$	$D_{xi} = (\omega_B + \omega')(\omega_A + \omega_B + \omega' + \omega)(\omega_A + \omega')$
(10)	$|0_A, 1_B\rangle|1^{(1)}\rangle$	$|1_A, 1_B\rangle|1^{(2)}, 1^{(3)}\rangle$	$|0_A, 1_B\rangle|1^{(4)}\rangle$	$D_{xii} = (\omega_B + \omega')(\omega_A + \omega_B + \omega' + \omega)(\omega_B + \omega)$

studying the quantum electromagnetic vacuum fluctuation induced electric dipole–dipole interaction between two neutral atoms in a quantum theory of electromagnetism [10], is based upon the leading-order perturbation theory. Let us note that a rather simple perturbative calculation of both the retarded and instantaneous electromagnetic interaction can also be found in [11].

The system we consider consists of two gravitationally polarizable objects in a bath of fluctuating quantum gravitational fields in vacuum. We assume that the two objects labeled as A and B can be treated as two-level harmonic oscillators. Then, the total Hamiltonian can be given by

$$H = H_F + H_A + H_B + H_{AF} + H_{BF} , \tag{1}$$

where H_F is the Hamiltonian of gravitational fields, the Hamiltonian of the object $H_{A(B)}$ takes the form

$$H_{A(B)} = E^0_{A(B)}|0_{A(B)}\rangle\langle0_{A(B)}| + E^1_{A(B)}|1_{A(B)}\rangle\langle1_{A(B)}| , \tag{2}$$

and $H_{A(B)F}$ represents the interactions between the objects and gravitational fields

$$H_{A(B)F} = -\frac{1}{2}Q^{ij}_{A(B)}E_{ij} . \tag{3}$$

Here the Einstein convention is assumed for repeated indices, Latin indices run from 1 to 3, Q_{ij} is the gravitational vacuum fluctuation induced quadrupole moment of the object and $E_{ij} = R_{0i0j}$ is the gravito-electric tensor with $R_{\mu\nu\alpha\beta}$ being the Riemann tensor. E_{ij}, which is defined by an analogy between the linearized Einstein field equations and the Maxwell equations [12], determines the tidal gravitational acceleration between two nearby test particles in the classical Newtonian gravity. For a flat background spacetime with a linearized perturbation $h_{\mu\nu}$ propagating upon it, the metric can be expanded as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. From the definition of E_{ij} one can obtain

$$E_{ij} = \frac{1}{2}\ddot{h}_{ij} , \tag{4}$$

where a dot denotes a derivative with respect to time t.

In the transverse tracefree (TT) gauge, the gravitational field can be quantized as [7]

$$h_{ij}(\mathbf{r}, t) = \sum_{\mathbf{k}, \lambda}[a_\lambda(\omega, \mathbf{r})e_{ij}(\mathbf{k}, \lambda)f_\mathbf{k} + H.c.] , \tag{5}$$

where $H.c.$ denotes the Hermitian conjugate, $a_\lambda(\omega, \mathbf{r})$ is the gravitational field operator, λ labels the polarization states, $e_{ij}(\mathbf{k}, \lambda)$ are polarization tensors, and

$$f_\mathbf{k} = \frac{1}{\sqrt{2\omega(2\pi)^3}}e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \tag{6}$$

is the field mode with

$$\omega = |\mathbf{k}| = (k_x^2 + k_y^2 + k_z^2)^{1/2} . \tag{7}$$

Here we choose the units in which $32\pi G = 1$ and $\hbar = c = 1$, where G is the Newton's gravitational constant. The vacuum state of gravitational fields is defined as

$$a_\lambda(\omega, \mathbf{r})|\{0\}\rangle = 0 , \tag{8}$$

and the single- and two-graviton excited states are

$$a^\dagger_{\lambda_\alpha}(\omega_\alpha, \mathbf{r}_\alpha)|\{0\}\rangle = |1^{(\alpha)}\rangle , \tag{9}$$

$$\frac{1}{\sqrt{2}}a^\dagger_{\lambda_\alpha}(\omega_\alpha, \mathbf{r}_\alpha)a^\dagger_{\lambda_\beta}(\omega_\beta, \mathbf{r}_\beta)|\{0\}\rangle = |1^{(\alpha)}, 1^{(\beta)}\rangle . \tag{10}$$

Since the object-gravitational field coupling is linear in the object and the field variables, each object must interact with the field at least two times and return to its ground state. Thus, the position-dependent shift of the ground-state energy between two objects arises from fourth-order perturbations [10,13,14] in the leading-order perturbation theory

$$\Delta E_{AB} = -\sum_{I,II,III}' \frac{\langle0|\hat{H}_{AF} + \hat{H}_{BF}|I\rangle\langle I|\hat{H}_{AF} + \hat{H}_{BF}|II\rangle}{(E_I - E_0)(E_{II} - E_0)}$$

$$\times \frac{\langle II|\hat{H}_{AF} + \hat{H}_{BF}|III\rangle\langle III|\hat{H}_{AF} + \hat{H}_{BF}|0\rangle}{(E_{III} - E_0)} , \tag{11}$$

where the primed sum means that the ground state of the whole system $|0\rangle = |0_A\rangle|0_B\rangle|\{0\}\rangle$ is omitted and the summation includes position and frequency integrals. During each interaction, a graviton may either be emitted or absorbed by an object. The intermediate states $|I\rangle$ and $|III\rangle$ must consist of an excited object and a single graviton. While for the intermediate state $|II\rangle$ there are three possibilities, i.e., both objects in the ground state with two gravitons, both objects excited with no graviton or both objects excited with two gravitons. Therefore, there are ten possible combinations of intermediate states, which are listed in Table 1.

For the case (1) in Table 1, substituting $|I\rangle$, $|II\rangle$ and $|III\rangle$ into Eq. (11) yields

$$\Delta E_{AB(1)}(\mathbf{r}_A, \mathbf{r}_B) = -\frac{1}{16}\int_0^\infty d\omega \int_0^\infty d\omega'\left(\frac{1}{D_i} + \frac{1}{D_{ii}}\right)$$

$$\times \tilde{Q}_A^{ij}\tilde{Q}_A^{*kl}\tilde{Q}_B^{ab}\tilde{Q}_B^{*cd}G_{ijab}(\omega, \mathbf{r}_A, \mathbf{r}_B)$$

$$\times G_{klcd}(\omega', \mathbf{r}_A, \mathbf{r}_B) , \tag{12}$$

where $\tilde{Q}^{ij}_{A(B)} = \langle0_{A(B)}|Q^{ij}_{A(B)}|1_{A(B)}\rangle$, $\tilde{Q}^{*ij}_{A(B)} = \langle1_{A(B)}|Q^{ij}_{A(B)}|0_{A(B)}\rangle$, and

$$G_{klcd}(\omega, \mathbf{r}_A, \mathbf{r}_B) = \langle0|E_{kl}(\omega, \mathbf{r}_A)E_{cd}(\omega, \mathbf{r}_B)|0\rangle \tag{13}$$

is the two-point correlation function of gravito-electric fields. The expressions of D_i and D_{ii} are given in Table 1 in which $\omega_{A(B)} = (\omega^1_{A(B)} - \omega^0_{A(B)})$ with $\omega^1_{A(B)} = E^1_{A(B)}$ and $\omega^0_{A(B)} = E^0_{A(B)}$, which represents the transition frequency of the object. The contributions of other cases in Table 1 to ΔE_{AB} can be calculated similarly. Summing up all possible intermediate states, we obtain that

$$
\Delta E_{AB}(\mathbf{r}_A, \mathbf{r}_B) = -\frac{1}{16} \int_0^\infty d\omega \int_0^\infty d\omega' \sum_{n=i}^{xii} \frac{1}{D_n}
$$
$$
\times \tilde{Q}_A^{ij} \tilde{Q}_A^{*kl} \tilde{Q}_B^{ab} \tilde{Q}_B^{*cd} G_{ijab}(\omega, \mathbf{r}_A, \mathbf{r}_B)
$$
$$
\times G_{klcd}(\omega', \mathbf{r}_A, \mathbf{r}_B) . \tag{14}
$$

It is easy to show

$$
\sum_{n=i}^{xii} \frac{1}{D_n} = \frac{4(\omega_A + \omega_B + \omega)}{(\omega_A + \omega_B)(\omega_A + \omega)(\omega_B + \omega)} \left(\frac{1}{\omega + \omega'} - \frac{1}{\omega - \omega'} \right) . \tag{15}
$$

Since the shift of the ground-state energy is just the vdW-like potential $U_{AB}(\mathbf{r}_A, \mathbf{r}_B)$ between two polarizable objects, one has from the above two equations,

$$
U_{AB}(\mathbf{r}_A, \mathbf{r}_B) = -\frac{1}{4} \int_0^\infty d\omega \int_0^\infty d\omega' \frac{\omega_A + \omega_B + \omega}{(\omega_A + \omega_B)(\omega_A + \omega)(\omega_B + \omega)}
$$
$$
\times \left(\frac{1}{\omega + \omega'} - \frac{1}{\omega - \omega'} \right)
$$
$$
\times \tilde{Q}_A^{ij} \tilde{Q}_A^{*kl} \tilde{Q}_B^{ab} \tilde{Q}_B^{*cd} G_{ijab}(\omega, \mathbf{r}_A, \mathbf{r}_B)
$$
$$
\times G_{klcd}(\omega', \mathbf{r}_A, \mathbf{r}_B). \tag{16}
$$

Assuming that the objects A and B are isotropically polarizable, we have

$$
\tilde{Q}_{A(B)}^{ij} \tilde{Q}_{A(B)}^{*kl} = 2|\tilde{Q}_{A(B)}^{ij}|^2 \delta_{ik} \delta_{jl} \equiv \frac{1}{2} \tilde{\alpha}_{A(B)} \delta_{ik} \delta_{jl} . \tag{17}
$$

Here $\tilde{\alpha}(\omega) \equiv \frac{1}{4} |\tilde{Q}_{ij}(\omega)|^2$. Then, Eq. (16) can be simplified as

$$
U_{AB}(\mathbf{r}_A, \mathbf{r}_B) = -\frac{1}{16(\omega_A + \omega_B)} \int_0^\infty d\omega \int_0^\infty d\omega' \frac{\tilde{\alpha}_A \tilde{\alpha}_B (\omega_A + \omega_B + \omega)}{(\omega_A + \omega)(\omega_B + \omega)}
$$
$$
\times \left(\frac{1}{\omega + \omega'} - \frac{1}{\omega - \omega'} \right)
$$
$$
\times G_{ijab}(\omega, \mathbf{r}_A, \mathbf{r}_B) G_{ijab}(\omega', \mathbf{r}_A, \mathbf{r}_B) . \tag{18}
$$

The two-point correlation function $G_{ijab}(\omega, \mathbf{r}_A, \mathbf{r}_B)$ can be obtained from $G_{ijab}(\mathbf{r}_A, \mathbf{r}_B, t_A, t_B)$ by Fourier transform. From Eqs. (4), (5), and (13), one finds

$$
G_{ijkl}(\mathbf{r}, \mathbf{r}', t, t') = \frac{1}{4} \langle 0|\ddot{h}_{ij}(\mathbf{r}, t)\ddot{h}_{kl}(\mathbf{r}', t')|0 \rangle
$$
$$
= \frac{1}{4(2\pi)^3} \int d^3\mathbf{k} \sum_\lambda e_{ij}(\mathbf{k}, \lambda) e_{kl}(\mathbf{k}, \lambda)
$$
$$
\times \frac{\omega^3}{2} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}') - i\omega(t-t')} . \tag{19}
$$

The summation of polarization tensors in the TT gauge gives [7]

$$
\sum_\lambda e_{ij}(\mathbf{k}, \lambda) e_{kl}(\mathbf{k}, \lambda) = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl} + \hat{k}_i\hat{k}_j\hat{k}_k\hat{k}_l + \hat{k}_i\hat{k}_j\delta_{kl}
$$
$$
+ \hat{k}_k\hat{k}_l\delta_{ij} - \hat{k}_i\hat{k}_l\delta_{jk} - \hat{k}_i\hat{k}_k\delta_{jl} - \hat{k}_j\hat{k}_l\delta_{ik}
$$
$$
- \hat{k}_j\hat{k}_k\delta_{il} , \tag{20}
$$

where

$$
\hat{k}_i = \frac{k_i}{k} . \tag{21}
$$

Transforming to the spherical coordinate: $\hat{k}_1 = \sin\theta \cos\varphi$, $\hat{k}_2 = \sin\theta \sin\varphi$ and $\hat{k}_3 = \cos\theta$, and letting

$$
\sum_\lambda e_{ij}(\mathbf{k}, \lambda) e_{kl}(\mathbf{k}, \lambda) = g_{ijkl}(\theta, \varphi) , \tag{22}
$$

we arrive at

$$
G_{ijkl}(r, \Delta t) \tag{23}
$$
$$
= \frac{1}{8(2\pi)^3} \int_0^\infty \omega^5 d\omega \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi \, g_{ijkl}(\theta, \varphi) e^{i\omega(r\cos\theta - \Delta t)} ,
$$

where $r = \Delta\mathbf{r} = |\mathbf{r}_A - \mathbf{r}_B|$ is the distance between objects A and B. Fourier transforming the above expression yields

$$
G_{ijkl}(\omega, \mathbf{r}_A, \mathbf{r}_B) = \frac{1}{2\pi} \int_{-\infty}^\infty d\Delta t e^{i\omega\Delta t} G_{ijkl}(r, \Delta t) \tag{24}
$$
$$
= \frac{\omega^5}{8(2\pi)^3} \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi \, g_{ijkl}(\theta, \varphi) e^{iwr\cos\theta} ,
$$

Substituting Eq. (24) into Eq. (18) and integrating over $(\theta, \varphi, \theta', \varphi')$, we obtain

$$
U_{AB}(r) = -\frac{1}{2^9\pi^4(\omega_A + \omega_B)r^{10}}
$$
$$
\times \int_0^\infty d\omega \int_0^\infty d\omega' \frac{\tilde{\alpha}_A \tilde{\alpha}_B (\omega_A + \omega_B + \omega)}{(\omega_A + \omega)(\omega_B + \omega)}
$$
$$
\times \left(\frac{1}{\omega + \omega'} - \frac{1}{\omega - \omega'} \right)
$$
$$
\times [A_1(\omega r, \omega' r) \cos(\omega' r) + B_1(\omega r, \omega' r) \sin(\omega' r)] , \tag{25}
$$

where

$$
A_1(x, x') = xx'(315 + 8x^2x'^2 - 30x^2 - 30x'^2) \cos x \tag{26}
$$
$$
+ x'(-315 + 135x^2 + 30x'^2 - 18x^2x'^2
$$
$$
- 3x^4 + 2x^4x'^2) \sin x ,
$$

and

$$
B_1(x, x') = x(-315 + 135x'^2 + 30x^2 - 18x^2x'^2 - 3x'^4 \tag{27}
$$
$$
+ 2x^2x'^4) \cos x + (315 - 135x^2 + 3x^4 - 135x'^2
$$
$$
+ 63x^2x'^2 - 3x^4x'^2 + 3x'^4 - 3x^2x'^4 + x^4x'^4) \sin x .
$$

Since $A_1(x, -x') = -A_1(x, x')$ and $B_1(x, -x') = B_1(x, x')$, Eq. (25) can be re-expressed as

$$
U_{AB}(r) = -\frac{1}{2^9\pi^4(\omega_A + \omega_B)r^{10}} \int_0^\infty d\omega \frac{\tilde{\alpha}_A \tilde{\alpha}_B (\omega_A + \omega_B + \omega)}{(\omega_A + \omega)(\omega_B + \omega)}
$$
$$
\times \int_{-\infty}^\infty d\omega' \left(\frac{1}{2(\omega + \omega')} + \frac{1}{2(-\omega + \omega')} \right)
$$
$$
\times [A_1(\omega r, \omega' r) - iB_1(\omega r, \omega' r)] e^{i\omega' r}. \tag{28}
$$

Performing the principle value integral on ω', one has

$$U_{AB}(r) = -\frac{1}{32^2\pi^3(\omega_A+\omega_B)r^{10}} \int_0^\infty d\omega \, \frac{\tilde{\alpha}_A\tilde{\alpha}_B(\omega_A+\omega_B+\omega)}{(\omega_A+\omega)(\omega_B+\omega)}$$
$$\times [A_2(\omega r)\cos(2\omega r) + B_2(\omega r)\sin(2\omega r)] \,, \qquad (29)$$

where

$$A_2(x) = -630x + 330x^3 - 42x^5 + 4x^7 \,,$$
$$B_2(x) = 315 - 585x^2 + 129x^4 - 14x^6 + x^8 \,.$$

Eq. (29) can be re-written as

$$U_{AB}(r) \qquad (30)$$
$$= -\frac{1}{32^2\pi^3(\omega_A+\omega_B)r^{10}}$$
$$\times \left(\int_0^\infty d\omega \, \frac{\tilde{\alpha}_A\tilde{\alpha}_B(\omega_A+\omega_B+\omega)}{(\omega_A+\omega)(\omega_B+\omega)} \left[\frac{A_2(\omega r)}{2} + \frac{B_2(\omega r)}{2i} \right] e^{2i\omega r} \right.$$
$$\left. + \int_0^{-\infty} d\omega \, \frac{\tilde{\alpha}_A\tilde{\alpha}_B(\omega_A+\omega_B-\omega)}{(\omega_A-\omega)(\omega_B-\omega)} \left[\frac{A_2(\omega r)}{2} + \frac{B_2(\omega r)}{2i} \right] e^{2i\omega r} \right).$$

Simplifying the above equation by contour-integral techniques, we obtain

$$U_{AB}(r) = -\frac{1}{32^2\pi^3 r^{10}} \int_0^\infty du \, \alpha_A(iu)\alpha_B(iu)[iA_2(iur)$$
$$+ B_2(iur)]e^{-2ur}$$
$$= -\frac{1}{32^2\pi^3 r^{10}} \int_0^\infty du \, \alpha_A(iu)\alpha_B(iu)S(ur)e^{-2ur} \,, \qquad (31)$$

where

$$\alpha_{A(B)}(\omega) = \lim_{\epsilon\to 0^+} \frac{\tilde{\alpha}_{A(B)}\omega_{A(B)}}{\omega_{A(B)}^2 - \omega^2 - i\epsilon\omega}$$
$$= \lim_{\epsilon\to 0^+} \frac{1}{4} \frac{|\tilde{Q}_{A(B)}^{ij}|^2\omega_{A(B)}}{\omega_{A(B)}^2 - \omega^2 - i\epsilon\omega} \,, \qquad (32)$$

is the object's ground-state polarizability, which is defined in analogy to the electric polarizability of atoms [15] and satisfies

$$Q_{ij}(\omega) = \alpha(\omega)E_{ij}(\omega, \mathbf{r}) \,, \qquad (33)$$

and

$$S(x) = 315 + 630x + 585x^2 + 330x^3 + 129x^4 + 42x^5$$
$$+ 14x^6 + 4x^7 + x^8 \,. \qquad (34)$$

In the far regime $r \gg \omega_{A(B)}^{-1}$, since there is an exponential in the integrand in Eq. (31), small values of u provide the dominating contribution. Thus, we can use approximately the static polarizability $\tilde{\alpha}_{A(B)}(0)$ and obtain

$$U_{AB}(r) = -\frac{3987}{4\pi(32\pi)^2 r^{11}} \alpha_A(0)\alpha_B(0)$$
$$= -\frac{3987\hbar c G^2}{4\pi r^{11}} \alpha_A(0)\alpha_B(0) \,. \qquad (35)$$

In the near regime $r \ll \omega_{A(B)}^{-1}$, the integral in Eq. (31) is effectively limited to a region where $e^{-2ur} \simeq 1$ and so all terms in $S(x)$ dependent on x can be neglected. Thus, the potential becomes

$$U_{AB}(r) = -\frac{315}{32^2\pi^3 r^{10}} \int_0^\infty du \, \alpha_A(iu)\alpha_B(iu)$$
$$= -\frac{315\hbar G^2}{\pi r^{10}} \int_0^\infty du \, \alpha_A(iu)\alpha_B(iu) \,. \qquad (36)$$

In the second line of Eqs. (35) and (36), we return to the SI units. These results agree with that obtained with a normal mode evaluation [5] and a two-graviton exchange calculation [16]. It is interesting to note here that the qualitative behavior of the effective interaction between two gravitationally polarizable objects can be inferred from a simple dimensional analysis. Dimensionally, the object's polarizability scales as $1/M^5$. Since the retarded potential energy must go as M and is quadratic in the polarizability, the r dependence must be r^{-11}. In the instantaneous case there is an additional factor of the excitation energy involved, so the scaling is as r^{-10}.

In conclusion, based upon the quantum theory of linearized gravity and the leading-order perturbation theory, we have studied the quantum gravitational vacuum fluctuation induced quadrupole–quadrupole quantum correction to the classical Newtonian force between a pair of polarizable objects. We find that the interaction potential decays as r^{-11} in the retarded region and r^{-10} in the near region. Our approach parallels that of Casimir and Polder in the investigation of the quantum electromagnetic vacuum fluctuation induced dipole–dipole interaction between two neutral atoms in a quantum theory of electromagnetism and suggests that it may be robust to use linearized quantum gravity to study quantum gravitational effects at low energies, such as those associated with quantized gravitational waves in the same way as one usually does with quantized electromagnetic waves.

Acknowledgements

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