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Model order reduction for nonlinear dynamical systems based on trajectory piecewise-linear approximations

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Abstract

In this paper we analyze and expand a recently developed approach to Model Order Reduction (MOR) for nonlinear dynamical systems based on trajectory piecewise-linear (TPWL) approximations. Error estimates are given for solutions computed with TPWL reduced order models, and problems of preserving stability and passivity are examined. Since the TPWL method has limited a priori guarantees on global accuracy, its effectiveness is demonstrated on a range of examples including a micromachined switch, two nonlinear electronic circuits, and shock propagation modeled by Burgers' equation.
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1. Introduction

Today's highly engineered systems, such as e.g. electronic circuits or jet engines, integrate a large number of functionally and physically varied elements, often leading

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to complicated mixed-technology designs. The involved, multi-physical descriptions of the subsystems force the designers to move to higher levels of abstraction and use simplified descriptions, in order to efficiently perform system-level design. At the same time, however, a constant push toward higher performance of the systems (e.g. higher operating frequencies in electronic circuits, increased operating speeds in turbomachinery) generates a need to consider a more complete set of physical effects, typically described by large-scale (and often nonlinear) dynamical systems. In this context, Model Order Reduction (MOR) arises as an approach which links those contradicting trends, by providing methods for automatically extracting easily evaluated models of important dynamical characteristics from detailed physical descriptions.

So far, most of the research effort has focused on developing MOR techniques suitable for linear systems. The most popular approaches to linear MOR include using Krylov subspace projections [4], Hankel norm approximants and Truncated Balanced Realization (TBR) [5,12], and Proper Orthogonal Decomposition (or Karhunen–Loève expansion) [6,21].

Model Order Reduction techniques for nonlinear systems are much scarcer and include methods based on linearization or bilinearization of the initial system around the equilibrium point [1,11,13], algorithms using Proper Orthogonal Decomposition [7,20], and finally methods of balanced truncation [9,16]. Still, many problems arise while using existing nonlinear MOR algorithms. On one hand, simple methods based on polynomial expansions about a single state are effective only for weakly nonlinear systems and ‘small’ inputs [2,11]. On the other hand, algorithms based on balancing transformations, although accurate, either are characterized by high numerical cost of generating the models or inadequately address the problem of numerical cost of evaluating the final reduced order model. The last problem is associated with the fact that, unlike in the linear case, projection of a nonlinear system and reduction of its order does *not* automatically imply reduction of simulation costs for the obtained reduced order model [15].

What is needed is an approach for extracting nonlinear reduced order models which can be evaluated at a very low cost, yet are capable of capturing strongly nonlinear behavior of the original system. One approach is based on quasi-piecewise-linear approximation of the nonlinearity, with linearization points taken from a trajectory in the state space of the initial system [15], and was found to significantly outperform MOR methods based on polynomial expansions or bilinearization (cf. [1,2,15]). In this paper we aim at further analyzing this MOR approach, discussing in more detail its limitations, applicability, and issues related to stability and passivity preservation.

In Section 2, we describe the idea of quasi-piecewise-linear approximation of nonlinearity, and the corresponding order reduction strategies. We also give an approach for selecting linearization points from a state-space trajectory, leading to Trajectory Piecewise-Linear (TPWL) reduced order systems, and present examples of applying the TPWL MOR method. Then, in Sections 3 and 4, we consider the problems of estimating the errors in solutions computed with TPWL reduced models, and preserving stability with the discussed models, respectively. Section 5 shows how error estimates and stability analysis may be used to select subsequent linearization points. In Section 6 we briefly describe the issue of passivity of TPWL models. Section 7 provides additional computational results for more sophisticated examples from the engineering domain. Finally, in Section 8 we present our conclusions.

2. TPWL reduced order models

In this paper we focus on discussing model order reduction strategies for nonlinear dynamical systems in the following state-space form:

$$\begin{cases} \dot{x} = f(x) + Bu, \\ y = C^{\mathrm{T}}x, \end{cases} \tag{1}$$

where $x = x(t) \in R^N$ is a vector of states (evolving with time t) for a given system, $f : R^N \to R^N$ is a nonlinear vector-valued function, B is an $N \times M$ input matrix, $u = u(t) \in R^M$ is an input to the system, C is an $N \times K$ output matrix and $y = y(t) \in R^K$ is the output.

In order to perform order reduction, many MOR schemes construct an orthonormal basis $V = [v_1, \dots, v_q]$ (where $q \ll N$) which spans an 'important' part of the state space. Performing a projection $x = Vz$, and applying the same 'testing' basis V to the initial nonlinear system (1) yields

$$\begin{cases} \dot{z} = V^{\mathrm{T}}f(Vz) + V^{\mathrm{T}}Bu(t), \\ y = C^{\mathrm{T}}Vz, \end{cases} \tag{2}$$

where z is a reduced, q-th order vector of states.

2.1. Quasi-piecewise-linear approximation of nonlinearity

Although the order of system (2) is reduced to q, its numerical solution will typically remain costly, due to a high cost of evaluating the nonlinear term $V^{\mathrm{T}}f(Vz)$. Evaluation of this term will normally require $\mathrm{O}(N^\alpha)$ ($\alpha \geqslant 1$) operations, and will be as costly as evaluating $f(x)$ in the initial nonlinear system. In order to reduce the computational cost, the following quasi-piecewise-linear approximate representation of nonlinear function f has been proposed in [15]:

$$f(x) \approx \sum_{i=0}^{s-1} \tilde{w}_i(x)\left(f(x_i) + A_i(x - x_i)\right), \tag{3}$$

where x_i's ($i = 1, \dots, (s-1)$) are some linearization points (states), A_i's are the Jacobians of f evaluated at states x_i, and $\tilde{w}_i(x)$'s are state-dependent weights ($\sum_{i=0}^{s-1} \tilde{w}_i(x) = 1$, for all x). Applying the above approximation, and performing a projection of system (1) yields:

$$\begin{cases} \dot{z} = (\sum_{i=0}^{s-1} w_i(z)[V^{\mathrm{T}}f(x_i) + V^{\mathrm{T}}A_i(Vz - x_i)]) + V^{\mathrm{T}}Bu, \\ y = C^{\mathrm{T}}Vz, \end{cases} \tag{4}$$

where $w_i(z)$'s are weights dependent on the reduced order state z ($\sum_{i=0}^{s-1} w_i(z) = 1$ for all z). Unlike the cost of evaluating (2), the cost of evaluating the right hand side of Eq. (4) is typically less than $O(sq^2)$ (assuming that scalar weighting functions $w_i(\cdot)$ are inexpensive to compute), where s is the number of linearization points used. (Note also that in a typical situation only one or two of w_i's are non-zero at a time, and consequently the model evaluation time is rather insensitive to s.) Since, unlike in the reduced order system (2), the considered cost does *not* depend on the initial size of the problem N, large speedup of computations may be achieved if using the above reduced system.

The main three questions concerning reduced order system (4) refer to:

(1) computing weights $w_i(z)$,
(2) selecting projection basis V,
(3) selecting suitable linearization points x_i.

Generally speaking, weights w_i are computed using information on distances between the current state and the linearization points (or their projections). Also, it has been found advantageous to apply weights which transition rather rapidly as the state proceeds from the neighborhood of one linearization point to another, making a single linearized model dominant for most of the state space. This provides a rationale to refer to model (4) as to a piecewise-linear model. A detailed discussion on the method for computing weights may be found in [15].

Referring to the issue of selecting a suitable projection basis—various strategies for generating projection bases have been considered, including algorithms based on Krylov subspace methods (e.g. in which a collection of Krylov subspaces are computed for different linearized models and then merged together), Truncated Balanced Realization (TBR), or methods performing hybrid, two-step reduction using both previously mentioned approaches. Detailed descriptions of various effective procedures for obtaining the projection bases for piecewise-linear models may be found in [14,15,17].

2.2. *Selecting linearizations along a state-space trajectory*

One of the most crucial issues while trying to extract a reduced order model in form (4) is selecting the collection of linearization points x_i. It is a trivial observation that a linearization of f from state x_i accurately approximates the initial nonlinear function at some given state x, provided x is 'close enough' to the linearization point x_i, i.e. $\|x - x_i\| < \epsilon$, or x lies within an N-dimensional ball of radius ϵ, centered at x_i.

Consequently, in order to obtain a good global approximation of f, it is obviously desirable to 'cover' the entire N-dimensional state space with such balls, thereby assuring that any state is within ϵ of a certain linearization point (and a corresponding linearized model). The problem is that the number of balls will grow exponentially with order of the state space N. For example, the number of radius 0.1 balls required to fill a 1000-dimensional unit hypercube will equal roughly 10^{1000}. Generating and storing in computer's memory such a large number of linearized models would be inefficient, if not impossible.

Since it is computationally infeasible to cover the entire N-dimensional state space with suitable linear models, we have proposed to generate a collection of models selected from a single, fixed 'training' trajectory of the system, corresponding to some relevant 'training input.' In order to obtain the training trajectory (or its approximation) one performs simulation of the full order nonlinear system (1) or applies a fast approximate simulation proposed in [15].

After selecting linearization points from a 'training' trajectory, and generating the corresponding models, one obtains a 'trajectory piecewise-linear' (TPWL) reduced order model, which consists s linearizations, where $s \ll N$. Usually $s \approx q$, and consequently the cost of simulation with model (4) is substantially lower than an analogous cost for system (2). Nevertheless, in this approach linearized models will 'cover' only a region in the state-space around the training trajectory. Therefore, the scope of applicability of model (4) will be limited to those input signals which do not drive the operating point of the considered system 'too far' from the linearization points (and the 'training' trajectory). Before we attempt to more precisely discuss this issue by performing error analysis for TPWL approximations, and investigate some of the qualities of TPWL reduced order models, let us first illustrate their performance by presenting two simple application examples.

2.3. Application examples

As a first example consider modeling shock movement, as described by 1D Burgers' equation:

$$\frac{\partial U(x,t)}{\partial t} + \frac{\partial f(U(x,t))}{\partial x} = g(x), \tag{5}$$

where U is the unknown conserved quantity (e.g. mass, density, heat), and $f(U) = 0.5U^2$, $g(x) = 0.02\exp(0.02x)$, in this example. The initial and boundary conditions used with the above PDE are

$$U(x,0) \equiv 1, \qquad U(0,t) = u(t),$$

for all $x \in [0,l]$, $t > 0$, where u is the incoming flow, and l is the length of the modeled region. Discretizing U with respect to x yields $\underline{U} = [U_1, \ldots, U_N]^{\mathrm{T}}$, where U_i approximates U at point $x_i = i\Delta x$ ($\Delta x = l/N$, where N is the number of grid points). Then, using (5), and incorporating the boundary conditions results in the following dynamical system:

$$\frac{\mathrm{d}\underline{U}}{\mathrm{d}t} = F(\underline{U}) + G + Bu,$$

where $G = 0.02[\exp(0.02x_1)\cdots\exp(0.02x_N)]^{\mathrm{T}}$, $B = [1/(2\Delta x)0\cdots 0]^{\mathrm{T}}$, and

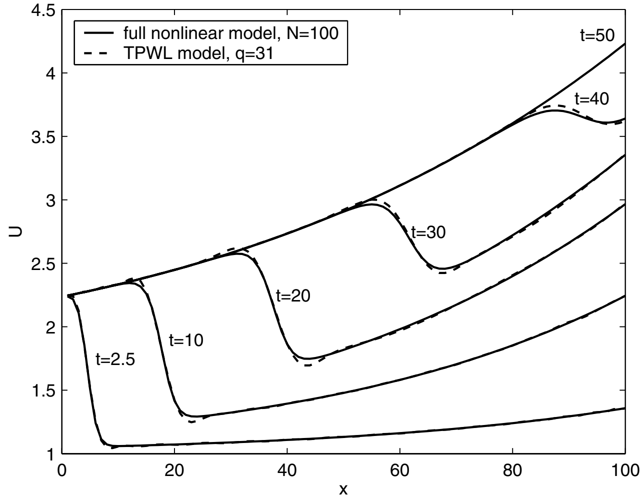

Fig. 1. Shock movement modeled by a TPWL reduced order model of order $q = 31$ (with $s = 21$ linearization points).

$$F(\underline{U}) = \frac{1}{\Delta x}\begin{bmatrix} -0.5U_1^2 & & & \\ & 0.5(U_1^2 - U_2^2) & & \\ & & \ddots & \\ & & & 0.5(U_{N-1}^2 - U_N^2) \end{bmatrix}.$$

Fig. 1 shows the shock movement, as modeled by Burgers' equation (5), for $l = 100$, $N = 100$, $\Delta x = 1$, and the incoming flow $u(t) \equiv \sqrt{5}$, computed for a full-order nonlinear model and a reduced TPWL model of order $q = 31$. One may note excellent agreement between the two discussed models. Nevertheless, one should also be aware that an approximately three-fold reduction of the problem size achieved in this case needs to be treated as very modest when compared to other application examples (cf. Section 7).

Another example we considered was a nonlinear transmission line circuit model with quadratic resistors, shown in Fig. 2. The current flowing through the resistors to the ground at each node is given by

$$i_n(v) = g \cdot \mathrm{sgn}(v)v^2,$$

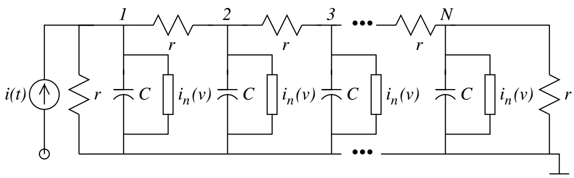

Fig. 2. Example of a transmission line model with quadratic nonlinearity.

where v is the voltage at the resistor terminals, g is the conductivity coefficient, $\mathrm{sgn}(v) = 1$ if $v \geqslant 0$, and $\mathrm{sgn}(v) = -1$ if $v < 0$. If we take $C = r = g = 1$, then the nonlinear operator f (cf. (1)) takes the following form:

$$f(\underline{v}) = A\underline{v} - n(\underline{v}), \tag{6}$$

where

$$A = \begin{bmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \ddots \\ & & 1 & -2 \end{bmatrix}, \qquad n(\underline{v}) = \begin{bmatrix} \mathrm{sgn}(v_1)v_1^2 \\ \mathrm{sgn}(v_2)v_2^2 \\ \cdots \\ \mathrm{sgn}(v_N)v_N^2 \end{bmatrix}, \tag{7}$$

and $\underline{v} = [v_1 \cdots v_N]^{\mathrm{T}}$ is the vector of states.

In a numerical test we generated a reduced order TPWL model of order $q = 25$ (with $s = 16$ linearization points), for the initial circuit with $N = 100$ nodes. Then, we simulated both original nonlinear system and TPWL reduced order model, with the input current $i(t)$ equal to unit step

$$i(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geqslant 0. \end{cases}$$

A sample comparison of voltage at node 5, computed with both models is shown in Fig. 3. Again, one may note a very good agreement of the transients computed with full and reduced order models. This example is also used in the following sections to test an error estimation procedure.

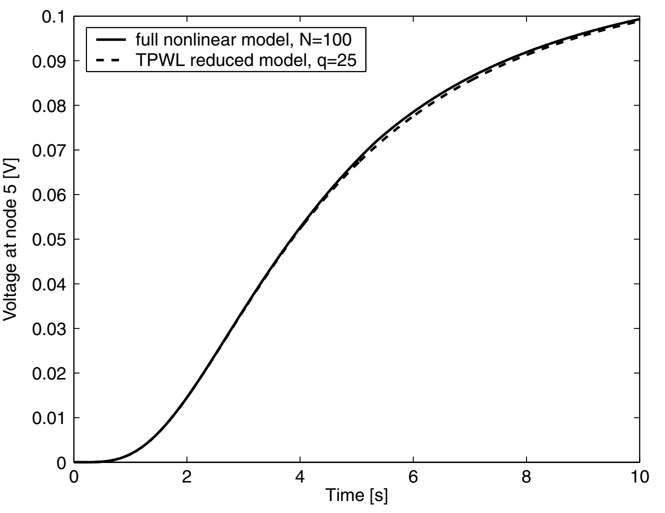

Fig. 3. Comparison of voltage at node 5 for the nonlinear transmission line model, computed with full nonlinear simulator, and the reduced order TPWL model.

3. Error analysis in TPWL models

In this section we analyze the errors which arise if a full-order nonlinear model is replaced by a TPWL reduced order model. First, we present a procedure for estimating the error in solution computed with a reduced order model. Then, we analyze the more general issue of error boundedness.

3.1. A posteriori error estimation

Recall that due to a local nature of approximations provided by linearized models, a TPWL model will adequately approximate a response of the initial nonlinear system to a given (testing) input signal, only if this input signal does not drive the operating point of the considered system too far from the linearization points x_i. In other words, if initial system's testing trajectory corresponding to the testing input stays close to the states visited by the training trajectory, the TPWL reduced model is expected to provide adequate approximation of the initial system. Yet, such *a priori* information on the testing trajectory is normally unavailable (since it would require simulation of the initial nonlinear system for the considered testing input). In this case one may use a posteriori error analysis, which exploits information from the testing trajectory computed by a TPWL reduced model to assess the approximation error. The a posteriori error analysis for TPWL models, first presented in [15] for the case of negative monotone function f, aims at estimating $\|\delta x(t)\|_2$, where

$$\delta x(t) \equiv x(t) - \hat{x}(t) = x(t) - Vz(t),$$

$\|\cdot\|_2$ is the Euclidean norm, and $x(t)$ and $z(t)$ are solutions at time t of (1) and (4), respectively (for the same initial condition x_o[1] and the same input signal u). As proven in [15], if f from Eq. (1) is negative monotone[2], i.e.

$$\exists_{\lambda>0}\forall_{x,y}\;\; (x-y)^{\mathrm{T}}(f(x)-f(y)) \leqslant -\lambda(x-y)^{\mathrm{T}}(x-y), \tag{8}$$

then $\|\delta x(t)\|_2$ satisfies the following inequality:

$$\begin{aligned}\|\delta x(t)\|_2 \leqslant \frac{1}{\lambda} & \sup_{\tau\in[t_i,t_{i+1}]} \|h(z(\tau)) + (I - VV^{\mathrm{T}})Bu(\tau)\|_2 \\ & \times(1-\exp(-\lambda(t-t_i))) + \|\delta x(t_i)\|_2 \exp(-\lambda(t-t_i)),\end{aligned} \tag{9}$$

for all $t \in [t_i, t_{i+1}]$, where

$$h(z) = f(Vz) - \sum_{i=0}^{s-1} w_i(z)[VV^{\mathrm{T}} f(x_i) + VV^{\mathrm{T}} A_i V(z - z_i)]. \tag{10}$$

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[1] If x_o cannot be represented exactly in basis V, the initial condition for the reduced system is taken as $z_o = V^{\mathrm{T}} x_o$.

[2] One may easily prove that e.g. function f given by (6) is negative monotone.

The above inequality leads us to proposing the following iterative scheme of computing error bounds $e(t_i)$ for $\|\delta x(t_i)\|_2$ at timesteps $t_0, t_1, t_2, \ldots$:

(1) At initial time t_0 take
$e(t_0) = \|I - VV^{\mathrm{T}}\|_2 \|x_o\|_2,$
where x_o is a known initial condition.
(2) For $i = 1, 2, \ldots$ iteratively compute

$$e(t_i) = \frac{1}{\lambda} \sup_{\tau \in [t_{i-1}, t_i]} \|h(z(\tau)) + (I - VV^{\mathrm{T}})Bu(\tau)\|_2 \times (1 - \exp(-\lambda(t_i - t_{i-1}))) + e(t_{i-1}) \exp(-\lambda(t_i - t_{i-1})). \tag{11}$$

Clearly, $\|\delta x(t_0)\|_2 = \|(I - VV^{\mathrm{T}})x_o\|_2 \leqslant e(t_0)$, and also it follows from (9) that $e(t_i) \geqslant \|\delta x(t_i)\|_2$ for every $t_1, t_2, \ldots,$ i.e. $e(t_i)$ provides a desired error bound. In practice, we replace the supremum in the above formula by a maximum over a discrete set of timesteps between t_{i-1} and t_i, corresponding to a certain numerical time integration scheme. (If t_i are the same as subsequent integration steps, we take a maximum of the two values at the ends of the considered time interval.) Clearly, this method of evaluating the supremum implicitly assumes that neither $h(z(t))$ nor $u(t)$ behave pathologically between subsequent integration timesteps.

Note, that in general the cost of evaluating $h(z)$ given by (10) could be as high as the cost of evaluating the initial nonlinear model, which would make using the above error estimation procedure impractical. However, in many cases we may compute the discussed estimates at a significantly lower cost. Note that if we include x_i in the projection basis V, then we have $Vz_i = x_i$, for every i. Furthermore, if we include $(f(x_i) - A_i x_i)$ $(\forall_i)$ in V, then $VV^{\mathrm{T}}(f(x_i) - A_i x_i) = (f(x_i) - A_i x_i)$, and $\|h(z)\|_2$ may be estimated with

$$\|h(z)\|_2 \leqslant \sum_{i=0}^{s-1} w_i(z) \left[\frac{1}{2} \sup_x \|W(x)\|_2 \, \|z - z_i\|_2^2 + \|(I - VV^{\mathrm{T}})A_i V\|_2 \, \|z\|_2 \right],$$

where $W(x)$ is the Hessian of f at x: $W(x) = [w_{ij}^k] \in R^{N \times N \times N}$, and $w_{ij}^k = \frac{\partial^2 f_i(x)}{\partial x_i \partial x_j}$. Then, we may replace (11) with

$$e(t_i) = \frac{1}{\lambda} \sup_{\tau \in [t_{i-1}, t_i]} \sum_{i=0}^{s-1} w_i(z(\tau)) \Bigg[\frac{1}{2} \sup_x \|W(x)\|_2 \, \|z(\tau) - z_i\|_2^2 + \|(I - VV^{\mathrm{T}})A_i V\|_2 \, \|z(\tau)\|_2 + \|(I - VV^{\mathrm{T}})B\|_2 \|u(\tau)\|_2 \Bigg] \times (1 - \exp(-\lambda(t_i - t_{i-1}))) + e(t_{i-1}) \exp(-\lambda(t_i - t_{i-1})). \tag{12}$$

One should note that since the values of norms $\|(I - VV^{\mathrm{T}})A_i V\|_2$ (for every i) and $\|(I - VV^{\mathrm{T}})B\|_2$ can be computed during construction of the reduced model, the cost of evaluating (12) is $\mathrm{O}(sq)$ only. This means that error estimation may be performed 'on the fly', along with the reduced order simulation, without increasing the complexity of the fast solver.

The main challenges associated with using the above scheme are related to: (1) Finding λ (cf. (8)) which should be as precise as possible. (Quality of the error estimates heavily depends on this parameter, therefore one could consider using different λ's in different regions of the state space, if at all possible and computationally feasible.); (2) Finding estimates of $\|h(z(t))\|_2$, given by (10), which typically requires estimating $\|W(x)\|_2$—the norm of the Hessian of f.

In some instances both λ and $\|W(x)\|_2$ are readily available, as in the nonlinear transmission line example considered in Section 2.3 (cf. Fig. 2). It is trivial to show that function f describing nonlinearity of this system (given by Eq. (6)) is negative monotone, provided all v_i are nonnegative at all times (which is satisfied if the input current $i(t) \geqslant 0$ for all t).

In the considered example the value of λ (cf. (8)) may then be taken as $\lambda = -1 \cdot \max_i\{\lambda_i \in \sigma(A)\}$, where $\sigma(A)$ is the spectrum of matrix A, given by (7). For the number of nodes $N = 100$, $\lambda = 9.67 \times 10^{-4}$. We also have that $\|W(x)\|_2 = 2g = 2$ for all x. Knowing λ and $\|W(x)\|_2$ we are ready to use formula (11) to compute error estimates.

In a numerical test we used the same reduced order TPWL model of order $q = 25$ as the one described in Section 2.3, and simulated both original nonlinear system and TPWL reduced order model, with the input current $i(t)$ equal to unit step. (It should be stressed that q is relatively large, as compared to N for this example, and therefore it may be inefficient to use the extracted TPWL model in practice. Still, this reduced model provides useful insight while considering the problem of error estimation.) The actual error $\|\delta x\|_2$ and its estimate were computed at every timestep. Fig. 4 shows a comparison of this error and its estimate for the considered case. One may note that formula (12) gives reasonable estimates of the error of approximating the original nonlinear system with a TPWL reduced order model.

3.2. *Error boundedness*

Although a posteriori error analysis is helpful in assessing quality of solutions computed with a reduced order TPWL system, in response to given, finite input signals $u(t)$, it does not provide us with information about global properties of the discussed reduced order system.

Suppose the initial nonlinear system is L_p-stable (for any $p \in [1, \infty]$). In particular this means that trajectories of this system will be bounded, for bounded input signals. We may ask whether this property also holds for the corresponding TPWL reduced order model. (Equivalently, one may ask if the respective trajectories of the initial system and the TPWL model do not diverge, i.e. the error $\delta x(t) = x(t) - \hat{x}(t) = x(t) - Vz(t)$ (cf. (1) and (4)) is bounded for all times.) Furthermore, we may ask if, or under what conditions, a TPWL reduced order model is L_p-stable (for any p) or passive. Those issues are discussed below.

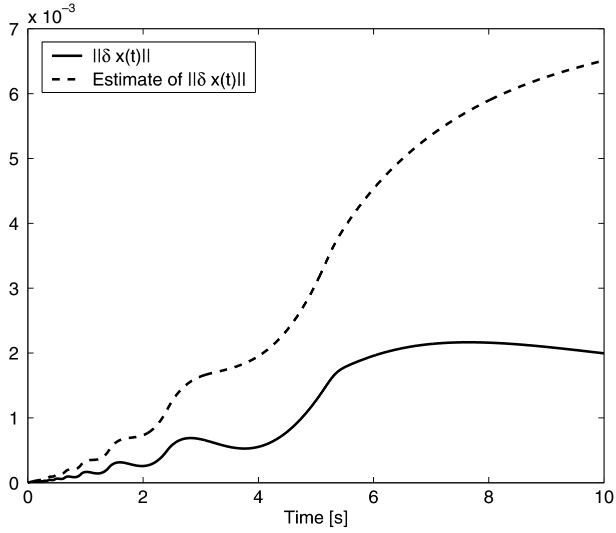

Fig. 4. A posteriori error estimates for a TPWL reduced order model of a nonlinear transmission line circuit model.

First consider the problem of error boundedness. For simplicity, we compare the full nonlinear model (1) with a corresponding full-order quasi-piecewise-linear model:[3]

$$\dot{\hat{x}} = \sum_{i=0}^{s-1} \tilde{w}_i(\hat{x})[f(x_i) + A_i(\hat{x} - x_i)] + Bu. \tag{13}$$

The first of Eq. (1) may be transformed as follows:

$$\begin{aligned}\dot{x} = f(x) + Bu = \sum_{i=0}^{s-1}(\tilde{w}_i(\hat{x})f(x)) + Bu = \sum_{i=0}^{s-1} \tilde{w}_i(\hat{x})\Bigg\{[f(x_i) + A_i(x - x_i)]\\ + \left[\int_0^1 (1-s)W(x+s(x-x_i))\,\mathrm{d}s \cdot (x-x_i) \otimes (x-x_i)\right]\Bigg\} + Bu.\end{aligned} \tag{14}$$

Subtracting Eq. (13) from (14) yields an error equation:

$$\begin{aligned}\dot{\delta x} = \left(\sum_{i=0}^{s-1} \tilde{w}_i(\hat{x})A_i\right)\delta x + \sum_{i=0}^{s-1} \tilde{w}_i(\hat{x})\\ \times \int_0^1 (1-s)W(x+s(x-x_i))\,\mathrm{d}s \cdot (x-x_i) \otimes (x-x_i),\end{aligned} \tag{15}$$

where $\delta x = x - \hat{x}$. We now make use of the following Theorem [18]:

[3] The following derivation can be easily extended to the case when we compare model given by (1) with a reduced order TPWL model (4).

Theorem 1. *Given a linear time-varying differential equation*:

$$\dot{x} = A(t)x(t) + v(t),$$

where $t \geqslant 0$, $x(t), v(t) \in R^N$, $A(t) \in R^{N\times N}$, *and* $v(\cdot)$ *and* $A(\cdot)$ *are piecewise-continuous*, $\|x(t)\|$ *is bounded as follows*:

$$\|x(t)\| \leqslant \exp\left\{\int_{t_0}^{t} \mu_{\text{ind}}[A(\tau)]\,\mathrm{d}\tau\right\}\|x_0\| + \int_{t_0}^{t} \exp\left\{\int_{\tau}^{t} \mu_{\text{ind}}[A(s)]\,\mathrm{d}s\right\}\|v(\tau)\|\,\mathrm{d}\tau,$$

for $t \geqslant t_0 \geqslant 0$, *where* t_0 *is the initial time,* $x_0 = x(t_0)$, *and* $\mu_{\text{ind}}(\cdot)$ *denotes a matrix measure* (*cf.* [18]) *induced by an* R^N *norm* $\|\cdot\|$ *defined by*

$$\mu_{\text{ind}}(A) = \lim_{\epsilon\to 0^+} \frac{\|I + \epsilon A\| - 1}{\epsilon}.$$

Note, that the above inequality based on matrix measures will give tighter bounds than inequalities derived using Gronwall's inequality. In our case we take $A(t) = (\sum_{i=0}^{s-1} \tilde{w}_i(\hat{x}(t))A_i)$, since $\hat{x}(t)$ is a fixed, although possibly unknown trajectory in the state space. Applying the above theorem to the error equation (15), and keeping in mind that $\delta x(t_0) = 0$, yields:

$$\|\delta x(t)\| \leqslant \int_{t_0}^{t} \exp\left\{\int_{\tau}^{t} \mu_{\text{ind}}\left[\sum_{i=0}^{s-1} \tilde{w}_i(\hat{x}(s))A_i\right]\mathrm{d}s\right\} \times \left\|\sum_{i=0}^{s-1} \tilde{w}_i(\hat{x}(\tau))\int_0^1 (1-s)W(\theta(x(\tau),s))\,\mathrm{d}s(x(\tau)-x_i)\otimes(x(\tau)-x_i)\right\|\mathrm{d}\tau \tag{16}$$

where $\otimes$ denotes the Kronecker product, and $\theta(x,s) = x + s(x - x_i)$. Since

$$\mu_{\text{ind}}\left[\sum_{i=0}^{s-1} \tilde{w}_i(\hat{x}(s))A_i\right] \leqslant \sum_{i=0}^{s-1} \tilde{w}_i(\hat{x}(s))\mu_{\text{ind}}[A_i]$$

and

$$\left\|\int_0^1 (1-s)W(\theta(x,s))\,\mathrm{d}s\cdot(x-x_i)\otimes(x-x_i)\right\| \leqslant \frac{1}{2}\sup_x \|W(x)\|\,\|x-x_i\|^2,$$

where $\theta(x,s) = x + s(x - x_i)$, inequality (16) may be further transformed as follows:

$$\|\delta x(t)\|_{1,2,\infty} \leqslant \int_{t_0}^{t} \exp\left\{\int_{\tau}^{t} \sum_{i=0}^{s-1} \tilde{w}_i(\hat{x}(s))\mu_{\text{ind}1,2,\infty}[A_i]\,\mathrm{d}s\right\} \times \sum_{i=0}^{s-1}\left[\tilde{w}_i(\hat{x}(\tau))\frac{1}{2}\sup_x \|W(x)\|_{1,2,\infty}\|x(\tau)-x_i\|^2_{1,2,\infty}\right]\mathrm{d}\tau, \tag{17}$$

where $\mu_{1,2,\infty}$ denotes a matrix measure corresponding to the 1-, 2-, or ∞-norm used to estimate $\delta x(t)$. Moreover, since $0 \leqslant \tilde{w}_i(\hat{x}) \leqslant 1$ and $\sum_{i=0}^{s-1} \tilde{w}_i(\hat{x}) = 1$, for every $\hat{x}$ we have:

$$\begin{aligned}\|\delta x(t)\|_{1,2,\infty} \leqslant & \int_{t_0}^{t} \exp\{\mu_{\max}(t-\tau)\}\frac{1}{2}\sup_x \|W(x)\|_{1,2,\infty} \\ & \times \max_{i\in\{0,\ldots,(s-1)\}} \|x(\tau)-x_i\|^2_{1,2,\infty}\,\mathrm{d}\tau \\ \leqslant & \frac{1}{2}\sup_x \|W(x)\|_{1,2,\infty} \sup_{\tau\in[t_0,t]}\left[\max_{i\in\{0,\ldots,(s-1)\}} \|x(\tau)-x_i\|^2_{1,2,\infty}\right] \\ & \times \frac{1}{-\mu_{\max}}\left[1-\exp(\mu_{\max}(t-t_0))\right], \end{aligned} \tag{18}$$

where $\mu_{\max} = \max_{i\in\{0,\ldots,(s-1)\}} \mu_{\text{ind}1,2,\infty}[A_i]$. From the above it follows that the error δx will be bounded for all times $t \geqslant t_0$ provided: 1) the trajectory $x(t)$ is bounded for all times, 2) $\mu_{\max} < 0$. Note, that for a given matrix $A = [a_{ij}]$ we have: $\mu_{\text{ind}1} = \max_j\{a_{jj} + \sum_{i\neq j}|a_{ij}|\}$, $\mu_{\text{ind}2} = \lambda_{\max}[A^H + A]/2$, and $\mu_{\text{ind}\infty} = \max_i\{a_{ii} + \sum_{j\neq i}|a_{ij}|\}$.

Consequently, since the initial nonlinear system is assumed to be L_p-stable, one may immediately infer from the above analysis that, provided the input signal is bounded, δx will be bounded for all times in the following cases: (1) All A_i's are symmetric and strictly stable; (2) All A_i's are strictly diagonally dominant, with all diagonal elements being negative. Note, that if we were comparing solutions of the initial nonlinear system and the *reduced* TPWL model, then the above conclusions hold, provided we replace Jacobians A_i with $VV^{\mathrm{T}}A_i$.

Knowing that trajectories of the initial nonlinear, stable system, and the respective TPWL reduced order model, corresponding to the same bounded input signal, do not diverge may not be enough. Often, we need to know whether the response of a given TPWL model to a finite energy signal will also have finite energy. This leads us to a question of whether a TPWL model preserves the property of L_2 (or more generally L_p) stability of the initial nonlinear system. This issue is discussed in detail in Section 4.

4. Stability of TPWL models

4.1. Exponential stability of unforced systems

We start our discussion by considering unforced (autonomous) dynamical systems, for which the input term Bu is eliminated. First, we note that MOR algorithms like PRIMA [10] build stable reduced order models by exploiting the fact that congruence transformations preserve definiteness of system's "A" matrix. Let us consider an unforced, stable linear dynamical system $\dot{x} = Ax$, which implies that A is a stable

(Hurwitz) matrix. Then, a reduced order system $\dot{z} = V^{\mathrm{T}} A V z$ is not guaranteed to be stable, unless A is negative definite (which implies that matrix $V^{\mathrm{T}} A V$ is also negative definite). If A is an indefinite stable matrix, then only for some special choices of V (e.g. if every column of V is an eigenvector of A) matrix $V^{\mathrm{T}} A V$ will remain Hurwitz.

Below, we will build an analogy with the linear case, by considering systems in form (1) with a nonlinear function f satisfying the following condition:

$$\exists_{\lambda>0}\forall_x \;\; x^{\mathrm{T}} f(x) \leqslant -\lambda x^{\mathrm{T}} x. \tag{19}$$

Without loss of generality we also assume throughout this section that $f(0) = 0$, i.e. that system (1) has an equilibrium at the origin. If f is Lipschitz continuous, the above condition is equivalent to the fact that solution of the Cauchy problem: $\dot{x} = f(x)$ with initial condition $x(t_0) = x_0$ satisfies the inequality

$$\|x(t)\|_2 \leqslant \|x(t_0)\|_2 \, \mathrm{e}^{-\lambda(t-t_0)}, \tag{20}$$

for $t > 0$. This in turn implies that 0 is an exponentially stable equilibrium point for system $\dot{x} = f(x)$. Clearly, projected function $\hat{f}(\cdot) = V^{\mathrm{T}} f(V\cdot)$ also satisfies condition (19) (with the same λ if V is an orthonormal basis), and therefore the projected system: $\dot{z} = V^{\mathrm{T}} f(Vz)$ also has an exponentially stable equilibrium point at the origin.

Once we replace the nonlinear function f with its trajectory quasi-piecewise approximation the situation becomes much more involved. Let us consider first an unforced TPWL system

$$\dot{z} = \sum_{i=0}^{s-1} w_i(z)[V^{\mathrm{T}} f(x_i) + V^{\mathrm{T}} A_i V z - V^{\mathrm{T}} A_i x_i)], \tag{21}$$

where A_i are Jacobians of f at linearization points x_i. Let us also assume that one of the linearized models is generated at the equilibrium of f. For instance we may assume that $x_0 = 0$.

One of substantial problems which arise is a possible existence of artificial, non-physical equilibria in TPWL system (21). Suppose that, for a given $j \in \{1, \ldots, (s-1)\}$, there exists a nonempty set D in the state space defined as follows: $D = \{z \in R^q : w_j(z) = 1\}$. If $z \in D$ then TPWL system (21) reduces to a single, linearized dynamical model

$$\dot{z} = V^{\mathrm{T}} f(x_j) + V^{\mathrm{T}} A_j (Vz - x_j).$$

Let us now consider $\tilde{z} = (V^{\mathrm{T}} A_j V)^{-1}(V^{\mathrm{T}} A_j x_j - V^{\mathrm{T}} f(x_j))$. It is trivial to check that if $\tilde{z} \in D$ then $\tilde{z}$ will be an equilibrium point of TPWL system (21). In order to illustrate this situation better let us consider the following example of a simple 1D dynamical system: $\dot{x} = -\tan(x)$, where $x \in (-\pi/2, \pi/2)$. Now, let us generate a corresponding TPWL system with two linearization points $x_0 = 0$ and $x_1 = \pi/4$. It is easy to check that the resulting system takes the following form:

$$\dot{x} = w_0(x) \cdot (-1 \cdot x) + w_1(x) \cdot (-1 - 2 \cdot (x - \pi/4)). \tag{22}$$

The initial system has a single equilibrium point at $x = 0$. In the TPWL model an 'artificial' equilibrium point may appear at $\tilde{x} = \pi/4 - 1/2$, depending on the applied weighting procedure.

If we apply e.g. the following weighting procedure:

$$w_1(x) = \begin{cases} 1 & \text{if } |x - \pi/4| < 1/4, \\ 0 & \text{otherwise} \end{cases}$$

and $w_0(x) = 1 - w_1(x)$ then for the initial condition $x(0) = \pi/4$ the solution $x(t)$ of (22) will monotonically tend to 0 as $t \to \infty$. On the other hand, if we apply the following weights:

$$w_1(x) = \begin{cases} 1 & \text{if } |x - \pi/4| < 3/5, \\ 0 & \text{otherwise} \end{cases}$$

and $w_0(z) = 1 - w_1(z)$ then the solution of (22) (with the same initial condition $x(0) = \pi/4$) will monotonically tend to the artificial equilibrium point at $(\pi/4 - 1/2)$ as $t \to \infty$ (cf. Fig. 5).

In order to examine the problem of artificial equilibria, consider the 'negative definiteness' condition also for TPWL reduced order system (21):

$$z^{\mathrm{T}} \cdot \left(\sum_{i=0}^{s-1} w_i(z)[V^{\mathrm{T}} f(x_i) + V^{\mathrm{T}} A_i (Vz - x_i)] \right) \leqslant -k_3 z^{\mathrm{T}} z, \tag{23}$$

where k_3 is some positive constant (for instance, we could possibly take $k_3 = \lambda$). The above simple condition clearly guarantees exponential stability of system (21). The important question is whether this condition can be satisfied e.g. for a certain distribution of linearization points or weights.

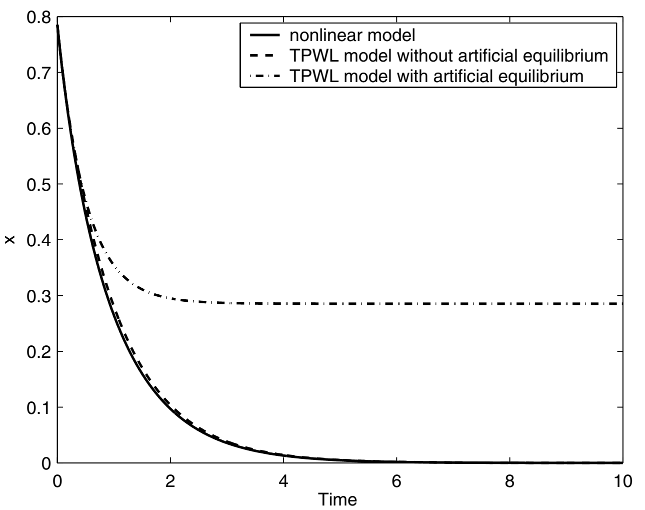

Fig. 5. The effect of artificial equilibrium in a 1D TPWL system.

Note, that if f satisfies condition (19) then $x^{\mathrm{T}}A_0x \leqslant -\lambda x^{\mathrm{T}}x$, i.e. the Jacobian at the origin is negative definite. Unfortunately, such condition does not hold for the rest of Jacobians $A_1, \ldots, A_{s-1}$. Consequently, condition (23) will not be satisfied automatically for any distribution of weights and/or linearization points. First, we may ask if there exists a weighing procedure which will stabilize the TPWL system, i.e. provide such distribution of weights that condition (19) will be satisfied for any x. The answer is clearly positive, since one may take: $w_0(z) \equiv 1$ and $w_1(z) \equiv \cdots \equiv w_{s-1}(z) \equiv 0$. Still, applying this trivial weighting procedure is equivalent to using only a single linearized model at the origin. In this case the TPWL model would be no more accurate that a simple linear reduced order model.

Therefore we may ask the following question: Is it possible to generate a weighting procedure such that, for all z, condition (23) is satisfied *and*, for every i, there exists ϵ_i, such that, for all z, if $z \in B_{\epsilon_j}(z_j)$ (i.e. if z is in a ball with radius ϵ_j centered at z_j, where $z_j = V^{\mathrm{T}}x_j$ is the projection of linearization point x_j) for some particular $j = 0, \ldots, (s-1)$ then $w_j(z) = 1$? In other words, is there a stability-preserving weighting procedure which will also select the 'optimal' linearized model, in particular – the model which is the closest to the current state of the system z, provided z is close enough to the linearization points? As proved below, the answer to this question is positive if f satisfies condition (19).

Theorem 2. *Suppose $f : R^N \to R^N$ is Lipschitz continuous, $f(0) = 0$, and f satisfies condition (19) with some positive constant λ. Suppose also that all linearization points x_i can be represented exactly in basis V, i.e. $x_i = Vz_i$ for $i = 1, \ldots, (s-1)$. Then the origin is an exponentially stable equilibrium point for TPWL system (21) if, for all $i = 1, \ldots, (s-1)$, $w_i(z) = 0$ for $z \in (R^q \backslash B_{\epsilon_i}(z_i))$, where*

$$\epsilon_i \leqslant \left(\sqrt{\frac{\tilde{\lambda}^2}{W^2} + \frac{2\tilde{\lambda}}{W}\|x_i\|_2} - \frac{\tilde{\lambda}}{W}\right), \tag{24}$$

and $\tilde{\lambda}$ is any positive constant such that $\tilde{\lambda} < \lambda$, $W = \sup_x \|W(x)\|_2$, $W(x)$ is the Hessian of f evaluated at x, and $B_{\epsilon_i}(z_i) = \{z \in R^q : \|z - z_i\| \leqslant \epsilon_i\}$ (cf. Fig. 6).

Proof. We will prove that assumptions of the theorem on weights and function f imply that condition (23) is satisfied. Since for every z, $\sum_{i=0}^{s-1} w_i(z) = 1$ and $x_i = Vz_i$ this condition may be written in the following form:

$$\left(\sum_{i=0}^{s-1} w_i(z)z^{\mathrm{T}}[V^{\mathrm{T}}f(x_i) + V^{\mathrm{T}}A_iV(z - z_i)]\right) \leqslant -k_3\sum_{i=0}^{s-1}(w_i(z)z^{\mathrm{T}}z). \tag{25}$$

Therefore, the condition which clearly implies (23) is the following:

$$\exists_{\lambda > k_3 > 0}\forall_{i=0,\ldots,(s-1)}\forall_{z\in D_i}\ z^{\mathrm{T}}(V^{\mathrm{T}}f(x_i) + V^{\mathrm{T}}A_iV(z - z_i)) \leqslant -k_3z^{\mathrm{T}}z \tag{26}$$

where $D_i = \{z : w_i(z) \neq 0\}$. First, we note that if $z \in D_0$ then, since A_0 is negative definite, we have

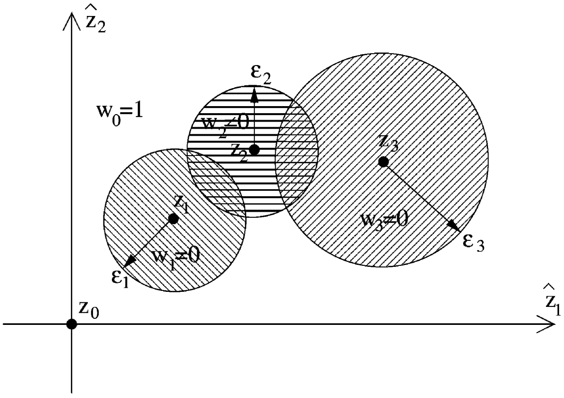

Fig. 6. Admissible weight distribution in a stability-preserving weighting procedure.

$$z^{\mathrm{T}}(V^{\mathrm{T}}f(x_0)+V^{\mathrm{T}}A_iV(z-z_0))=z^{\mathrm{T}}V^{\mathrm{T}}A_0Vz\leqslant-\lambda z^{\mathrm{T}}z<-k_3z^{\mathrm{T}}z. \tag{27}$$

Therefore condition (25) needs to be considered only for $i=1,\ldots,(s-1)$. If we subtract $z^{\mathrm{T}}V^{\mathrm{T}}f(Vz)$ from both sides of (26) and use (19) then we obtain the following condition equivalent to (26):

$$\exists_{\lambda>\tilde{\lambda}>0}\forall_{z\in D_i}\ z^{\mathrm{T}}(V^{\mathrm{T}}f(x_i)+V^{\mathrm{T}}A_iV(z-z_i)-V^{\mathrm{T}}f(Vz))\leqslant\tilde{\lambda}z^{\mathrm{T}}z, \tag{28}$$

for all $i=1,\ldots,(s-1)$. Since

$$\begin{aligned}V^{\mathrm{T}}f(Vz)&=V^{\mathrm{T}}f(x_i)+V^{\mathrm{T}}A_i(Vz-x_i)\\&\quad+\int_0^1(1-s)V^{\mathrm{T}}W(x_i+s(Vz-x_i))\,\mathrm{d}s\,(Vz-x_i)\otimes(Vz-x_i),\end{aligned} \tag{29}$$

for all i and z, (28) is equivalent to condition:

$$\begin{aligned}&\exists_{\lambda>\tilde{\lambda}>0}\ -z^{\mathrm{T}}\left(\int_0^1(1-s)V^{\mathrm{T}}W(\theta(z,s))\,\mathrm{d}s(V(z-z_i))\otimes(V(z-z_i))\right)\\&\quad\leqslant\tilde{\lambda}z^{\mathrm{T}}z\end{aligned}$$

for all $z\in D_i$, and $i=1,\ldots,(s-1)$, where $\theta(z,s)=x_i+s(Vz-x_i)$. Below we write two subsequently stronger conditions implying the above inequality:

$$\begin{aligned}&\exists_{\lambda>\tilde{\lambda}>0}\ \left\|\int_0^1(1-s)V^{\mathrm{T}}W(\theta(z,s))\,\mathrm{d}s(V(z-z_i))\otimes(V(z-z_i))\right\|_2\\&\quad\leqslant\tilde{\lambda}\|z\|_2;\ \exists_{\lambda>\tilde{\lambda}>0}\frac{1}{2}W\|z-z_i\|_2^2\leqslant\tilde{\lambda}\|z\|_2,\end{aligned} \tag{30}$$

for all $z\in D_i$, and $i=1,\ldots,(s-1)$, where $\theta(z,s)=x_i+s(Vz-x_i)$, $W=\sup_x\|W(x)\|_2$, and we used the fact that $\|V^{\mathrm{T}}\|_2=\|V\|_2=1$ since V is an orthonormal basis. Suppose now we consider a ball $B_{\epsilon_i}(z_i)$ for a given i and $D_i \subset B_{\epsilon_i}(z_i)$. This means that $w_i(z) \neq 0$ only if $\|z - z_i\|_2 < \epsilon_i$. If we impose that

$$\forall_{i=1,\ldots,(s-1)} \forall_{z \in D_i} \ \ \frac{1}{2} W \epsilon_i^2 \leqslant \tilde{\lambda} \|z\|_2,$$

for some $\lambda > \tilde{\lambda} > 0$ then condition (30) will be satisfied. Since $\|z\|_2 \geqslant \|z_i\|_2 - \epsilon_i \geqslant \|x_i\|_2 - \epsilon_i$ for every $i = 1, \ldots, (s-1)$ then we may write the following stronger condition

$$\forall_{i=1,\ldots,(s-1)} \ \ \epsilon_i^2 \leqslant \frac{2\tilde{\lambda}}{W} (\|x_i\|_2 - \epsilon_i).$$

Solving the above inequality for ϵ_i yields condition (24). Therefore, as we have shown condition (24) implies condition (23) which proves that the origin is an exponentially stable equilibrium point for TPWL system (21). □

One may note that, for any $i = 1, \ldots, (s-1)$, condition (24) implies that $\epsilon_i < \|x_i\|_2$. It is also clear that we should take $\tilde{\lambda}$ closest to λ in order to obtain the loosest bound for ϵ_i. Condition (24) also tells us that as λ grows (i.e. when the trajectories of the initial nonlinear system converge faster to the origin) we may take larger radii ϵ_i in the TPWL model, which is intuitively correct. Also, if f is smoother, i.e. if second order variations (represented by W) are smaller we may take larger radii.

Theorem 2 tells us that, provided f satisfies condition (19), we can select an 'optimal' linearized model provided the state z of the TPWL model is close enough to linearization points z_i or more precisely, provided z is in ball centered at z_i with radius ϵ_i, satisfying condition (24). Still, this condition is only a sufficient one and it is significantly stronger than condition (23) (with $x_i = V z_i$). Consequently, in practice we may use (23) instead of (24) in order to compute stability-preserving weights. This can be done on-the-fly during simulation of the TPWL reduced order model without significantly increasing the simulation cost. Yet another method of enforcing negative definiteness of the right hand side of (21) is adding artificial dissipative terms in form $-\gamma(z) z^{\mathrm{T}} z$, where $\gamma(z)$ is a nonnegative number, for those of z at which condition (23) is not satisfied.

4.2. L_p stability of TPWL models

The following fact links exponential stability with L_p stability:

Theorem 3. *Let us consider the following input-output system*:

$$\begin{cases} \dot{z} = \tilde{f}(z, u), \\ y = g(z, u). \end{cases}$$

Suppose $\tilde{f}$ is continuously differentiable, the Jacobian matrices $[\partial \tilde{f}/\partial z]$ and $[\partial \tilde{f}/\partial u]$ are bounded, and $g(x, u)$ satisfies the condition

$$\forall_{z,u} \quad \|g(z,u)\| \leqslant \eta_1\|z\| + \eta_2\|u\|, \tag{31}$$

for some nonnegative η_1, η_2. *If* $z = 0$ *is a globally exponentially stable equilibrium point of the unforced system* $\dot{z} = \tilde{f}(z, 0)$, *then the considered system is* L_p *stable for any initial condition* z_0.

Proof of the above theorem may be found in [8].

Let us now consider the following forced TPWL system:

$$\begin{cases} \dot{z} = \sum_{i=0}^{s-1} w_i(z)[V^{\mathrm{T}} f(x_i) + V^{\mathrm{T}} A_i V z - V^{\mathrm{T}} A_i x_i))] + V^{\mathrm{T}} B u, \\ y = C^{\mathrm{T}} V z. \end{cases} \tag{32}$$

Applying Theorems 2 and 3 with $h(z, u) = C^{\mathrm{T}} V z$, and

$$\tilde{f}(z,u) = \sum_{i=0}^{s-1} w_i(z)[V^{\mathrm{T}} f(x_i) + V^{\mathrm{T}} A_i V z - V^{\mathrm{T}} A_i x_i))] + V^{\mathrm{T}} B u$$

immediately yields the following corollary:

Corollary 1. *Suppose* $f : R^N \to R^N$ *is Lipschitz continuous and differentiable,* $f(0) = 0$, *and* f *satisfies condition*

$$\exists_{\lambda>0} \forall_x \quad x^{\mathrm{T}} f(x) \leqslant -\lambda x^{\mathrm{T}} x.$$

Suppose also, that all linearization points x_i *can be represented exactly in basis orthonormal* V, *i.e.* $x_i = V z_i$ *for* $i = 1, \ldots, (s-1)$. *Then system* (4) *is* L_p *stable for any initial condition* x_0, *if for all* $i = 1, \ldots, (s-1)$, $w_i(z) = 0$ *for* $z \in (R^q \backslash B_{\epsilon_i}(z_i))$, *where*

$$\epsilon_i \leqslant \left(\sqrt{\frac{\tilde{\lambda}^2}{W^2} + \frac{2\tilde{\lambda}}{W}\|x_i\|_2} - \frac{\tilde{\lambda}}{W} \right), \tag{33}$$

$\tilde{\lambda}$ *is any positive constant such that* $\tilde{\lambda} < \lambda$, $W = \sup_x \|W(x)\|_2$, $W(x)$ *is the Hessian of* f *evaluated at* x, *and* $B_{\epsilon_i}(z_i) = \{z \in R^q : \|z - z_i\| \leqslant \epsilon_i\}$.

5. Improved selection of linearization points

The a posteriori error estimation procedure described in the Section 3.1 may be used not only to assess errors of simulation with an existing TPWL reduced order model, but also to help one select a more optimal collection of linearization points from the training trajectory, during extraction of the reduced model. In a basic approach, during the 'training simulation' subsequent linearization points are selected using a simple geometric criterion: if the current state is 'far enough' from the previous linearization point, i.e. if

$$\left(\frac{\|x - x_i\|}{\|x_i\|}\right) > \delta,$$

where $\delta > 0$ is fixed, then $x_{i+1} = x$, i.e. x becomes the next linearization point. Instead of this geometric criterion one may use a measure based on error estimates (which use information on the nonlinear system at hand) to select a collection of linearization points. Below we present a procedure which uses fast approximate simulation and a posteriori error estimates derived in Section 3.1 to select a collection of linearization points $x_0, \ldots, x_{s-1}$:

(1) Generate a linearized model about the initial state x_0; Set $i = 0$ and the initial error estimate $e(0) = 0$.
(2) Simulate the linearized reduced order system generated at state x_i while

$$\left(\frac{\|e(t_k)\|_2}{\|Vz\|_2}\right) < \epsilon,$$

where

$$\begin{aligned} e(t_k) = \frac{1}{2\lambda} \sup_{\tau \in [t_{k-1}, t_k]} &\left[\sup_{x \in T(Vz(t_{k-1}), Vz(k_i))} \|W(x)\|_2 \, \|Vz(\tau) - x_i\|_2^2 \right. \\ &\left. + \|(I - VV^{\mathrm{T}})(A_i z(\tau) + Bu(\tau))\|_2 \right] (1 - \exp(-\lambda(t_k - t_{k-1}))) \\ &+ e(t_{k-1}) \exp(-\lambda(t_k - t_{k-1})), \end{aligned} \tag{34}$$

t_k and t_{k-1} are the current and previous timesteps, respectively, $z(t_k)$ is the current reduced order state, and $T(\alpha, \beta)$ denotes a trajectory of the simulated system between states α and β.
(3) Generate a new linearized model about $x_{i+1} = Vz(t_k)$, Set $e(t_k) = 0$, $i := i + 1$.
(4) If $i < (s - 1)$ return to step (2).

One may also develop a similar procedure if exact, full-order nonlinear simulation is performed to find the training trajectory.

We have tested the above procedure again for the example of a transmission line circuit model with quadratic nonlinearity, introduced in the previous section. First, we used a simple procedure for selecting linearization points, based solely on distances between the points in the state space, and obtained a TPWL model of order $q = 27$ with $s = 15$ linearizations. Then, we applied the above procedure based on error estimates to select linearization points. We have taken $\epsilon = 0.18$ to obtain a TPWL model (of order $q = 27$) again with 15 linearizations. In order to compare the quality of both TPWL models we tested them for a step input voltage with amplitude 1, and computed the relative error. The results of this comparison are shown in Fig. 7. One may note that, on average, the relative error for a TPWL model obtained with the

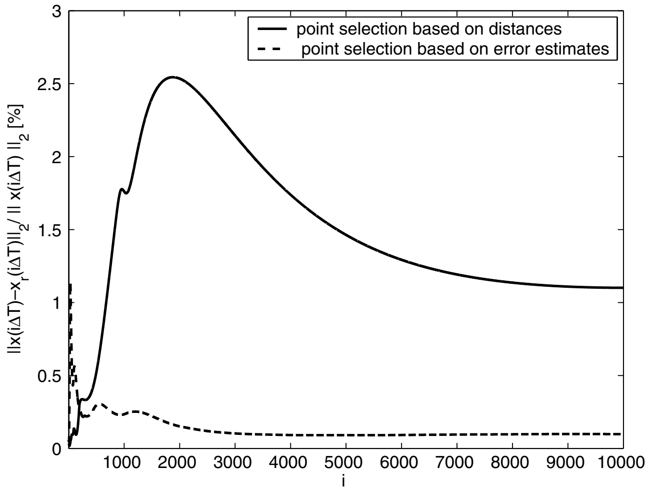

Fig. 7. Comparison of relative errors for two different reduced order TPWL models. x and x_r are vectors of states computed with full nonlinear, and reduced order TPWL models, respectively.

procedure discussed above in this section is significantly lower than the error for the TPWL model obtained with a simple algorithm of selecting linearization points. As one would expect, taking into account the impact of system's nonlinearity, results in a better selection of linearization points.

A different set of guidelines for selecting subsequent linearization points follows from Theorem 2, which gives a condition to be satisfied by the weights, sufficient to ensure stability of a TPWL reduced order system. The discussed theorem defines state-space regions around linearization points x_i (or, more precisely, balls with radii ϵ_i, centered at x_i) which are admissible support regions for the corresponding weighting functions w_i. Note, that if distances between x_i become large enough, then those regions may become disjoint (cf. Fig. 6), which means that in order to enforce stability we may need to set w_0 to 1 in the regions located between the linearization points. This in turn may adversely affect accuracy of a given TPWL model. In order to avoid such situation one may select subsequent linearization points in such a way that e.g. $\|x_{i-1} - x_i\|_2 < (\epsilon_{i-1} + \epsilon_i)$.

6. Preserving passivity with TPWL models

A problem related to input-output stability of reduced order TPWL models is that of preserving passivity of the original system by a TPWL model. This problem plays a crucial role, not only in the context of controlling the approximation error, but more importantly, in the context of using MOR to model interconnected subsystems (e.g. subsystems in a feedback loop). For instance, if we replace an initial passive subsystem (i.e. a subsystem which does not generate energy) of a larger system with a non-passive reduced model, simulation of the entire system may lead to non-physical solutions. Then, the reduced order model may act as an artificial energy source. In order to avoid such situation a number of approaches to extract stable and passive reduced order models have been proposed for linear systems [3,10]. Below we present a short passivity analysis for the case of TPWL reduced order models.

We define system (1) as passive if there exists a continuously differentiable positive semidefinite function $V : R^N \to R$ such that

$$\forall_{x\in R^N}\forall_{u\in R^M}\ \ u^{\mathrm{T}}y \geqslant \frac{\partial V(x)}{\partial x}\tilde{f}(x,u) + \delta y^{\mathrm{T}}y,$$

for some $\delta \geqslant 0$, where $\tilde{f}(x,u) = f(x) + Bu$. If $\delta > 0$ then we call system (1) output strictly passive. The following fact holds:

Fact 1. *Suppose we consider a nonlinear dynamical system*

$$\begin{cases} \dot{x} = f(x) + Bu, \\ y = C^{\mathrm{T}}x. \end{cases} \tag{35}$$

Then, the above system is output strictly passive provided $C = B$ and

$$\exists_{\lambda>0}\forall_x\ \ x^{\mathrm{T}}f(x) \leqslant -\lambda x^{\mathrm{T}}x.$$

Proof. We consider a simple quadratic function $V(x) = x^{\mathrm{T}}x/2$. This function is clearly positive semidefinite. We compute:

$$\begin{aligned} u^{\mathrm{T}}y - \frac{\partial V(x)}{\partial x}(f(x) + Bu) &= y^{\mathrm{T}}u - x^{\mathrm{T}}f(x) - x^{\mathrm{T}}Bu \\ &= y^{\mathrm{T}}u - x^{\mathrm{T}}f(x) - x^{\mathrm{T}}Cu = -x^{\mathrm{T}}f(x) \geqslant \lambda x^{\mathrm{T}}x \geqslant 0. \end{aligned}$$

Therefore, system (35) is passive. Furthermore, since $y^{\mathrm{T}}y = x^{\mathrm{T}}CC^{\mathrm{T}}x \leqslant \|CC^{\mathrm{T}}\|_2 x^{\mathrm{T}}x$, then from the above we obtain:

$$u^{\mathrm{T}}y \geqslant \frac{\partial V(x)}{\partial x}(f(x) + Bu) + \lambda x^{\mathrm{T}}x \geqslant \frac{\partial V(x)}{\partial x}(f(x) + Bu) + \frac{\lambda}{\|CC^{\mathrm{T}}\|_2}y^{\mathrm{T}}y.$$

Consequently, system (35) is output strictly passive. □

Fact 1 immediately implies the following corollary:

Corollary 2. *Suppose assumptions of Corollary* 1 *are satisfied. Then the reduced order trajectory piecewise-linear system* (32) *is output strictly passive provided $C = B$.*

This corollary states that if nonlinear function f is negative definite and $C = B$ (which implies passivity of the initial nonlinear system), then this passivity is preserved by a TPWL reduced order system, provided weighting functions w_i satisfy appropriate conditions, as described in Corollary 1.

7. Additional computational examples

Trajectory Piecewise-Linear (TPWL) model order reduction strategy has been successfully applied to modeling various dynamical systems demonstrating strongly nonlinear behavior, or composed of strongly nonlinear components. It has been used to effectively and efficiently perform time-domain, transient simulations, as well as periodic steady state simulations for a number of devices or systems, such as nonlinear transmission line models, operational amplifier circuits, or micro-electro-mechanical systems [14,15]. In this section we present performance of the discussed MOR method for two additional examples of strongly nonlinear systems, based on actual designs taken from the electrical engineering domain.

We first consider an example of a micromachined switch shown in Fig. 8. The switch consists of a polysilicon beam suspended over a silicon substrate, controlled by a voltage source inserted between the beam and the substrate. Beam dynamics are modeled by Euler's equation, and damping caused by air layer between the beam and the substrate is modeled by the nonlinear Reynolds equation. Discretization of those equations leads to a nonlinear dynamical system of order $N = 880$.

In order to obtain an inexpensive approximation of this large dynamical system we have generated a TPWL reduced order system of order $q = 26$ with $s = 26$ linearization points located on a training trajectory corresponding to a step 9-V input voltage. The obtained TPWL reduced order model has been tested first for the same 9-V step input voltage, and then for a cosinusoidal input voltage $u(t) = (7\cos(4\pi t))^2$ with a 7-V amplitude. Figs. 9 and 10 present results of the simulations for the two cases.

The graphs clearly show that in the considered cases, TPWL models provide very accurate approximations for the initial nonlinear system. They also demonstrate that TPWL models outperform linear and quadratic reduced order models based on single-state polynomial expansions of the nonlinear function f (cf. [2,11] for details on those reduced order models). Fig. 9 shows that a strongly nonlinear pull-in effect (the beam is pulled in to the substrate), which is particularly important in applications, can be

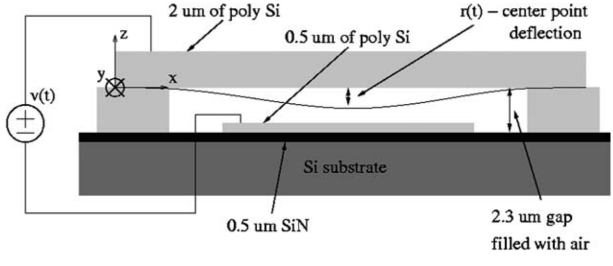

Fig. 8. Micromachined switch (following Hung et al. [7]).

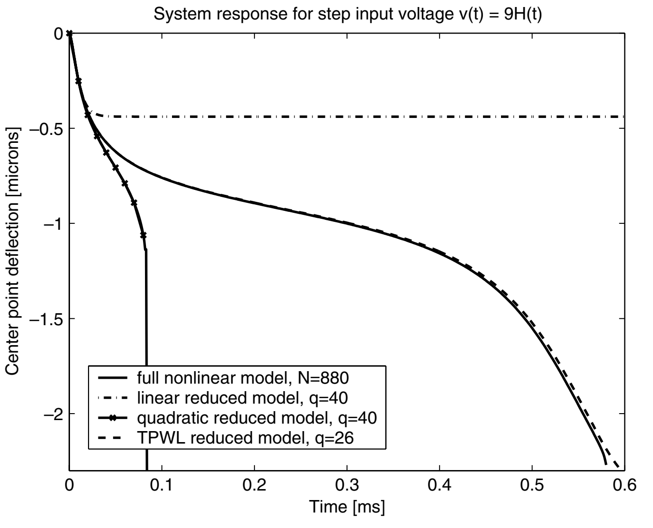

Fig. 9. Comparison of system response (micromachined switch example) computed with different MOR algorithms. The TPWL model was generated for the 9-V step input voltage.

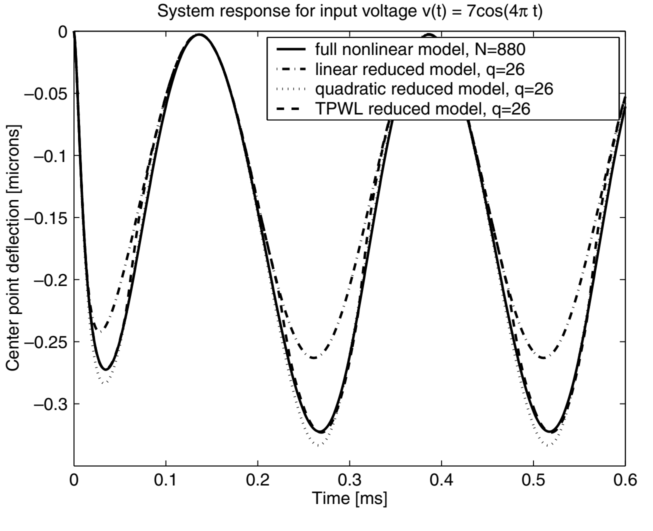

Fig. 10. Comparison of system response computed using the linear, quadratic and TPWL reduced order models. Input signal $u(t) = (7\cos(4\pi t))^2$.

correctly simulated by the discussed model. Also, Fig. 10 indicates that a TPWL model can be effectively applied even if the testing input (a cosinusoid with 7-V amplitude) is significantly different from the 'training' input (a step input with 9-V amplitude).

Another example we considered is an operational amplifier circuit with differential input and output, and consisting of 70 MOSFET transistors, 13 resistors and 9 linear capacitors connected to 51 circuit nodes. Nodal analysis [19] yields a nonlinear model of the device in form (1), with voltages at the circuit nodes defining a state vector. Also, the operational amplifier is modeled as a multiple input system with eight inputs: 1) the differential input with signals v_{in1} and v_{in2}, 2) the auxiliary inputs v_{cmmrst}, v_{gnd}, v_{intn}, v_{intp}, v_{rst}, and v_{cmmin}.

In order to generate the reduced order TPWL models we applied the following set of training inputs:

$$v_{in1}(t) = \begin{cases} 0 & \text{if } t < 290, \\ 12.5 \times 10^{-3}(t-290)/10 & \text{if } 290 \leqslant t < 300, \\ 12.5 \times 10^{-3} & \text{if } t \geqslant 300, \end{cases}$$

where time t is in nanoseconds, $v_{in2} = -v_{in1}$ (cf. Fig. 11), and auxiliary input signals shown in Fig. 12.

We obtained a TPWL reduced model of order $q = 34$ (with 29 linearization points and 8 inputs). The model was then tested for the following piecewise-linear input (cf. Fig. 11):

$$v_{in1}(t) = \begin{cases} 0 & \text{if } t < 290, \\ 11.5 \times 10^{-3}(t-290)/110 & \text{if } 290 \leqslant t < 400, \\ 11.5 \times 10^{-3} & \text{if } t \geqslant 400, \end{cases}$$

where time t is in nanoseconds, and $v_{in2} = -v_{in1}$. Fig. 13 shows a comparison of the transients computed with full nonlinear circuit simulator and with the reduced

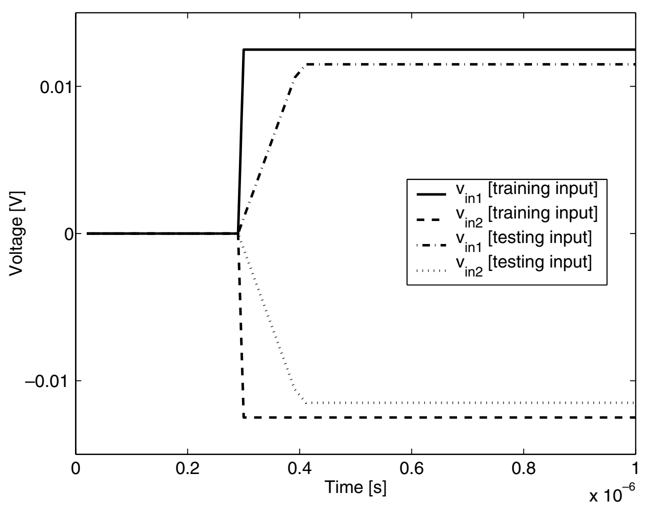

Fig. 11. Operational amplifier input signals v_{in1} and v_{in2}.

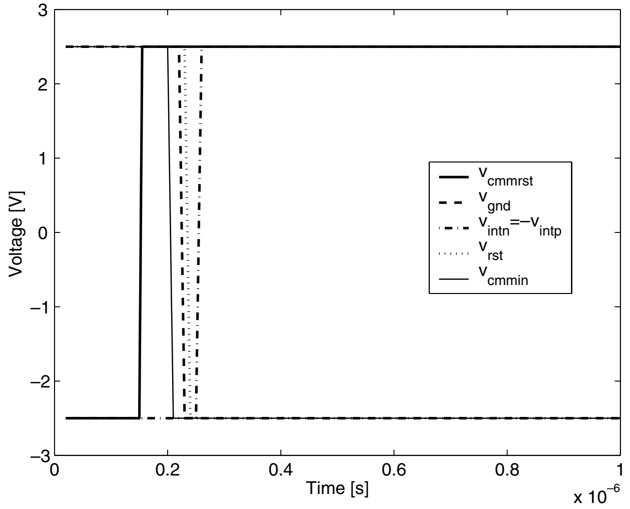

Fig. 12. Auxiliary inputs for the operational amplifier circuit.

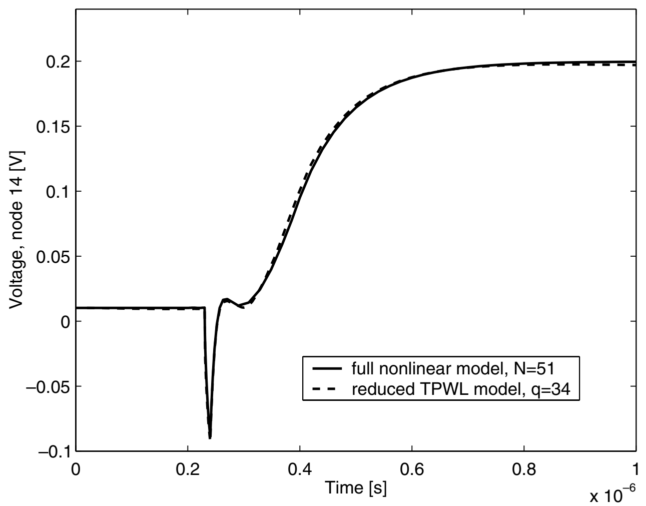

Fig. 13. Comparison of the output voltages for the operational amplifier example, computed with a reduced order TPWL model and full order nonlinear solver.

order TPWL models for one (of the two) output nodes of the amplifier. One may note excellent agreement of the output signals for both cases, which indicates that suitable reduced order TPWL models of the original systems have been constructed. It is important to point out that not only do the TPWL models have a lower order than the original system, but also they are much easier to use. Since a TPWL model consists of a weighted combination of linear models, the time-stepping is very straightforward. In a simplified backward Euler time-stepping scheme we compute the weights w_i (cf. Eq. (4)) e.g. using the previous state of the system or a predictor of the next state and then, assuming that these weights are fixed, we find the state at the next timestep by performing only a single Newton update (i.e. by solving a low order linear system of equations). In a more sophisticated time stepping scheme, one can account for derivatives of $w_i(x)$'s, which is also straightforward, since the weights are simple scalar functions. In a regular simulator, if using backward Euler scheme, finding the next state requires computation of a number of Newton updates for the full order nonlinear system, which is considerably more complex.

8. Conclusions

In this paper we have discussed a recently developed Trajectory Piecewise-Linear approach toward Model Order Reduction of nonlinear dynamical systems. As illustrated with a few computational examples, the proposed MOR strategy provides accurate reduced models for the simulated nonlinear systems. It has also been found to outperform MOR methods based on single-point polynomial expansions or bilinearization, as well as generally provide larger speedups in computation time, as compared to MOR algorithms based on Karhunen–Loève expansion (cf. also [1,2,7,15]).

We have also presented a procedure for a posteriori estimating the error in the solutions computed with TPWL models, for the case of negative monotone nonlinearities. A numerical experiment described confirms that the derived estimate can be effectively used to assess quality of the solutions. It has also been shown that the estimates can be applied during extraction of the reduced order models, to obtain an improved distribution of the linearization points, and consequently—more accurate models.

Apart from considering a posteriori error estimates we have also obtained a priori conditions for boundedness of the error. Then, by drawing an analogy with linear negative definite systems, we derived a set of conditions sufficient to guarantee input-output stability and passivity of TPWL reduced order models for nonlinear systems, with negative definite nonlinearities.

In this paper we have attempted to analyze properties of TPWL reduced models without imposing any significant constraints on the projection basis used. This approach has lead us to restricting the class of initial nonlinear systems for which e.g. stability or passivity can be preserved with TPWL reduced models. One may try to take an opposite approach, by considering only some restricted classes of projection bases (e.g. generated with balancing transformations), and analyzing properties of the resulting TPWL models (for initial nonlinear systems which do *not* satisfy the negative definiteness constraint).

It should also be noted that in this paper we have considered examples of very damped systems (except for the shock modeling problem). It seems that thanks to this property those systems can be effectively reduced using a single training trajectory. One of key questions is therefore identifying classes of systems which can be reduced using linearizations about just a few trajectories. A converse problem concerns selecting a training input (or a set of training inputs) which would provide the largest scope of applicability of the extracted TPWL model for a given system.

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