Presenting Locale Pullback
Via Directed Complete Posets

Extended Abstract

C.F. Townsend

Pure Mathematics
Open University
Milton Keynes, UK

Abstract

A summary is given of the main results in "Presenting locale pullback via directed complete posets". Rather than modeling frames (i.e. the open set lattices of locales) as ring objects in the category of suplattices (following Joyal and Tiernery), frames are here seen to be order internal distributive lattices in the category of directed complete posets (dcpos). The presentations for directed complete posets are stable under (the inverse images of) geometric morphisms and this provides a left adjoint to the direct image functor applied to dcpos. This adjunction specializes to order internal distributive lattices, thereby describing locale pullback along any geometric morphism. Re-stating the situation in terms of dcpos allows a new description of triquotient assignments to be given (generalizing Plewe's work on localic triquotient surjections, and following the work of Vickers). This description allows the pullback stability results for proper and open locale maps to be recovered. A further new application is given, describing directed join preserving maps between frames in terms of certain natural transformations indexed by geometric morphisms over a base topos.

Keywords: Topos theory, locale theory, change of base, lattice theory, generators and relations, triquotient, proper maps, open maps, dcpos

1 Locale pullback as a left adjoint

The direct image part of a geometric morphism preserves suplattices (that is, complete posets) and so defines a functor from suplattices, internal in one topos to suplattices internal to the codomain topos. Joyal and Tierney in [2] show that this functor has a left adjoint. This seemingly highly technical observation has important implications since it specializes to frames (complete

Heyting algebras) and so provides a description of the pullback of locales along a geometric morphism. (Recall that pullback can be described as a right adjoint and that the category of locales is opposite to frames.)

The trick of this result is to use suplattice presentations. The presentations (as formal objects) are stable under the inverse image of geometric morphisms and so this defines a functor in the opposite direction to the direct image functor. That this is left adjoint amounts to checking that under the bijection defined by the adjunction of the geometric morphism, maps which satisfy R correspond to maps which satisfy f^*R where R is the set of relations in the presentation and f is the geometric morphism. So the case where R is empty is immediate since the power set on a set of generators forms the free suplattice (and $f^*\phi = \phi$).

This result can be extended to the directed complete partial orders (posets with joins for all directed subsets). The same trick is used to prove this and so, firstly, work is needed to verify that dcpo presentations present (i.e. are well defined). This result appears to be folklore (e.g. [3]) and can be proved in an entirely constructive manner (and no natural numbers object is used). The left adjoint again describes pullback of locales along a geometric morphism since we are able to describe frames as particular objects in the category of dcpos. In fact this can be done in way that is not in keeping with Joyal and Tierney's view of frames as types of rings over suplattice tensor. The novel view can be taken that frames are internal distributive lattices in the ordered enriched category of dcpos. The internality required is a strong one in that the join and meet operations of the internal distributive lattices are required to be consistent with the given order enrichment. It is trivial to verify that dcpo tensor is the same thing as product and so internal distributive lattices are preserved by the left adjoint. This defines locale pullback.

2 Triquotient assignments

With the existence of this left adjoint a number of known results can be recovered: e.g. open and proper maps are pullback stable. These results can in fact be shown by looking at locale maps with triquotient assignments (types of dcpo homomorphisms); these locale maps generalise both proper and open maps in a natural way, essentially by forgetting all the finitary data in the definition of proper and open. The definition of a triquotient assignment is:

Definition 2.1 (Adapting Vickers, "The double powerlocale and triquotient maps of locales", unpublished note.) A locale map $p : Z \to Y$ has a triquotient assignment if there exists $p_\# : \Omega Z \to \Omega Y$ a dcpo homomorphism such that

$$p_\#(c_1 \wedge [c_2 \vee \Omega p(b)]) = [p_\# c_1 \wedge b] \vee p_\#(c_1 \wedge c_2)$$

$\forall c_1, c_2 \in \Omega Z$ and $\forall b \in \Omega Y$.

These assignments are not in any way unique and the definition is much weaker than the usual definition of triquotient (e.g. [4]) since a locale map with a triquotient assignment need not be surjective, whereas, "triquotient" in the literature invariably means a surjective map.

The ability to pullback dcpo homomorphisms allows a description of the triquotient assignments in terms of dcpo homomorphisms internally in the topos of sheaves over the codomain locale. The triquotient assignments on a locale map $g : Y \to Z$ are exactly the dcpo homomorphisms

$$\Omega_{SZ}(Y_g) \to \Omega_{SZ}$$

internally in SZ, where $\Omega_{SZ}(Y_g)$ is the frame corresponding to the locale map $g : Y \to Z$ under Joyal and Tierney's well known corrsepondence between the slice over Z and the category of frames internal to the topos of sheaves over Z.

These dcpo homomorphisms (internal in SZ) are stable under change of base, allowing the pullback stability result for maps with triquotient assignment to go through. This specializes to pullback stability for proper and open maps since the extra finitary data involved is preserved by the relevant adjunctions. The Beck-Chevalley conditions can be verified and therefore the result specializes further to triquotient, proper and open surjections. The pullback stability results of [4], [9] and [2] respectively are thus recovered in a uniform manner (though the related results on descent remain as further work).

3 Topos theoretic application

The main new observation that is available with this "dcpo technology" is the following result which is a mild extension of a recent result of Townsend and Vickers [7]. It can be shown that the dcpo maps between frames are exactly the natural transformations between certain functors indexed by geometric morphisms. The functors are

$$\Lambda^{\Omega_{\mathcal{E}} W} : (\mathbf{Top}/\mathcal{E})^{op} \to SET$$
$$(h : \mathcal{E}' \to \mathcal{E}) \longmapsto \mathbf{Top}(\mathcal{E}' \times_{\mathcal{E}} W, \mathbb{S})$$

for any frame $\Omega_{\mathcal{E}} W$, corresponding to a locale W in $\mathcal{E}$. Here $\mathbb{S}$ is the (topos of sheaves corresponding to the) Sierpiński locale. SET is some background category of possibly large sets.

4   Proper Open Duality

By clearly separating out the infinitary directed join structure of frames from the finitary (distributive lattice) structure it is hoped that further light is shed on the parallel that exists between proper and open (e.g. [6]) since the basis of that parallel appears to be that the finitary structure is dualised whilst the infinitary structure remains fixed. Thus a single pullback stability result is available (that of locale maps with triquotient assignments) which specializes (dually) to the known pullback stability results for proper and open maps.

The importance of the Townsend/Vickers result appears to be that it offers an external description of that part of the theory of locales which is fixed under this duality. This has possible application to an axiomatization of the notion of continuity. By further extending to geometric morphisms this work may therefore offer insight into the parallel between proper and open in topos theory.

5   Further work

An invisible motivation for the work has been an attempt to answer the question of what the topos theoretic analogue to the upper power locale should be (see B4.5 of [1] and [5])? More broadly, and indeed more simply: What is the topos theoretic version of a dcpo presentation for a Grothendieck topos? It is hoped that the structure of the results about dcpo homomorphisms will be applicable in the topos theoretic context (that is, with toposes in the place of locales). We therefore end with a

Conjecture 5.1 *There is a 1-1 correspondence between filtered cocontinuous functors between Grothendieck toposes, $\mathcal{E}, \mathcal{E}'$ and natural transformations $\Lambda^{\mathcal{E}} \dot{\to} \Lambda^{\mathcal{E}'}$, where in this context*

$$\Lambda^{\mathcal{E}} : (\mathbf{BTop}/Set)^{op} \to SET$$
$$(h : \mathcal{F} \to Set) \longmapsto \mathbf{BTop}/Set(\mathcal{F} \times_{Set} \mathcal{E}, [Set])$$

and $[Set]$ is the object classifier.

References

[1] Johnstone, P.T. *Sketches of an elephant: A topos theory compendium.* Vols 1, 2, Oxford Logic Guides **43**, **44**, Oxford Science Publications, 2002.

[2] Joyal, A. and Tierney, M. An Extension of the Galois Theory of Grothendieck, *Memoirs of the American Mathematical Society* **309**, 1984.

[3] Markowsky, George "Categories of chain-complete posets". *Theoret. Comput. Sci.* **4** (1977), 125-135.

[4] Plewe, T. "Localic triquotient maps are effective descent maps", Math. Proc. Cambridge Philos. Soc. **122** (1997), 17-43.

[5] Karazeris, P. *Categorical Domain Theory: Scott topology, powercategories, coherent categories.* Theory and Applications of Categories, Vol. **9**, No. 6, (2001), 106-120.

[6] Townsend, C.F. *Preframe Techniques in Constructive Locale Theory*, Phd Thesis, 1996, Imperial College, London.

[7] Townsend, C.F. and Vickers, S.J. *A Universal Characterization of the Double Power Locale.* To appear (2002).

[8] Vickers, S.J. "The double powerlocale and exponentiation: a case study in geometric logic". To appear.

[9] Vermeulen, J.J.C. *Proper maps of locales.* Journal of Pure and Applied Algebra **92** (1994), 79-107.