Application of active piezoelectric patches in controlling the dynamic response of a thin rectangular plate under a moving mass

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A B S T R A C T

The governing differential equation of motion for an undamped thin rectangular plate with a number of bonded piezoelectric patches on its surface and arbitrary boundary conditions is derived using Hamilton’s principle. A moving mass traveling on an arbitrary trajectory acts as an external excitation for the system. The effect of the moving mass inertia is considered using all the out-of-plane translational acceleration components. The method of eigenfunction expansion is used to transform the equation of motion into a number of coupled ordinary differential equations. A classical closed-loop optimal control algorithm is employed to suppress the dynamic response of the system, determining the required voltage of each piezoactuator at any time interval. In a numerical example for a simply supported square plate under two different loading paths, the effect of the mass velocity and mass weight of the moving load on the dynamic behavior of the uncontrolled system is investigated. The results show that, depending on the path of the moving mass, the inertia effect is very important, causing different behaviors of the system. In addition, the number of vibrational modes involved in determining the dynamic response of the system is crucial. The inertia effect is more important for an orbiting mass loading case compared to the case in which the moving mass is traversing the plate on a straight line. A number of equally spaced piezo patches are used on the lower surface of the plate to control the displacement of the center point of the plate. The implemented control mechanism proves to be very efficient in suppressing the near resonant dynamic response of the system, requiring fairly low levels of voltage for each patch. Increasing the area of the employed piezo patches would reduce the required maximum voltage for controlling the response of the system.

1. Introduction

Application of smart materials to improve structural behavior is a rapidly developing interdisciplinary area with advanced technology that embraces the fields of materials and structures, sensor and actuator systems and information processing and control. Crawley and de Luis (1987) developed the first analytical model for embedded or surface bonded piezoelectric elements to be used as sensors and actuators. Due to the electro-mechanical properties of the piezo materials, the electrical energy generated by the external strains can be flowed to a shunt circuit connected to the electrodes of piezo patches with the ability of dissipating part of this energy. Based on this phenomenon, many researchers have focused their studies on the capacity of passive damping generation in the shunted piezoelectric elements (e.g., Hagood and von Flotow, 1991; Park, 2003).

A rich literature survey is available on the shape control of structures, especially through the application of piezoelectric materials (Irschik, 2002). Moreover, in an extensive study, Song et al. (2006) have introduced a wide range of applications of piezoceramic materials in vibration suppression of civil structures. In their work, the applicability of piezoceramic patches or stacks in controlling the response of relatively large cantilever beams, trusses, frames and cable-stayed structures were shown using some experimental studies.

The full bonding between the main structure and the piezo actuators is one of the basic assumptions that should be met for proper performance of the piezo materials. Even small debonding of piezo actuators from the host structure can threaten the stability of the implemented closed-loop control system (Sun and Tong, 2003).

On the other hand, the dynamic behavior of the structures under the influence of moving loads has attracted the attention of many researchers in the last few decades. There are clearly many problems in which the load inertia is not negligible and can significantly alter the dynamic behavior of the system (e.g., Rao, 2000;
Nikkhoo et al., 2007). The dynamic stability of different types of the beams and plates under the effect of moving loads can be challenged due to the role of the inertia effect and the high velocity of the moving load (Kononov and de Borst, 2002; Verichev and Metrikine, 2003). The problem of a moving mass on a Kirchhoff plate has been considered by a number of researchers, all recognizing the importance of the load inertia (Cifuentes and Lalapet, 1992; Gbadeyan and Oni, 1995). A complete formulation of a moving mass on a Kirchhoff plate is provided by Fryba (1999).

However, in the majority of the studies available in the literature, only the vertical translational component of the moving mass acceleration is considered in the full term formulation of the problem, neglecting the other convective acceleration terms. As Nikhoo et al. (2007) have shown for an Euler–Bernoulli beam, ignoring the convective terms in the formulation could lead to a remarkable error for mass velocities greater than a so-called critical velocity. In plate problems with more convective terms, this could be even more important.

Application of active structural control in reducing the dynamic response of continuous or discrete flexible structures has received much attention in recent years (Meirovitch, 1997). Many control algorithms and control strategies have been proposed by researchers in this regard (Tadjbakhsh and Rofooei, 1992; Rofooei and Tadjbaksh, 1993; Rofooei and Monajemi-Nejad, 2005; Monajemi-Nejad and Rofooei, 2007). Sung (2002) has studied the active control of a simply supported Euler–Bernoulli beam under a moving mass using piezo materials as actuators. Similarly, Qiu et al. (2007) have investigated the application of piezoelectric ceramics as sensors and actuators for vibration suppression of a smart flexible clamped cantilever plate analytically and experimentally.

The main contribution of the current study rests on the application of piezoelectric materials as smart actuators to suppress the response of thin rectangular plates excited by the means of a moving mass traveling along arbitrary paths. In this regard, the governing differential equation of motion of a Kirchhoff plate bonded to a limited number of square piezo patches on its lower surface and arbitrary boundary conditions of the plate is derived using Hamilton’s principle. The piezo patch thickness is presumably small enough to maintain a uniform and constant electrical displacement over its thickness. Observing all the out-of-plane translational acceleration components of the moving mass, the eigenfunction expansion method is used to transform the derived governing differential equation into a number of coupled ordinary differential equations (Nikkhoo et al., 2007). A classical linear optimal closed-loop control algorithm with displacement–velocity feedback is used to determine the required voltages for each piezo patch.

A numerical example for a square simply supported plate under two different load paths is considered. Parametric studies are carried out to investigate the effect of the velocity and weight of the moving mass on the dynamic behavior of the uncontrolled system. Furthermore, the required number of plate vibrational modes in capturing the true response of the system is studied. A number of uniformly distributed piezo patches are used on the lower surface of the plate to reduce its dynamic response to any desired level.

2. Problem formulation

A uniform undamped thin (Kirchhoff) rectangular plate with arbitrary boundary conditions is considered. The mass per unit area of the plate, \( \rho_{\text{plate}} \), and its bending stiffness \( D_{\text{plate}} = \frac{E_{\text{plate}} h_{\text{plate}}^{3}}{12(1-\nu^{2})} \) in which \( E \), \( h_{\text{plate}} \), and \( \nu \), are plate’s modulus of elasticity, thickness and Poisson’s ratio, respectively, are assumed to be constant. Also, \( w(x,y,t) \) denotes the deflection of the mid-plane of the plate at any point and at any time \( t \). On the other hand, following Lee and Kim (2000) as well as Park (2003) analogy for the one dimensional problems, the constitutive equations for a homogeneous isotropic piezoelectric element in case of plane-stress problems can be written as:

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
2\sigma_{12}
\end{bmatrix}
= \begin{bmatrix}
\frac{E_{r}^{p}}{1-v_{e}^{2}} & \frac{G_{r}^{p}}{1-v_{e}^{2}} & 0 & -h_{31} \\
\frac{G_{r}^{p}}{1-v_{e}^{2}} & \frac{E_{r}^{p}}{1-v_{e}^{2}} & 0 & -h_{32} \\
-h_{31} & -h_{32} & 1 & 0
\end{bmatrix}
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
2\varepsilon_{12}
\end{bmatrix}
\]

(1)

where \( \sigma_{ij} \) and \( \varepsilon_{ij} \) are the stress and strain components, respectively. \( E_{r} \) and \( D_{r} \) denote the components of electrical field and electrical displacement, correspondingly. The superscripts \( (E) \) and \( (T) \) indicate that the material properties were measured under constant electrical and tension fields, respectively. Therefore, \( E_{r}^{p} \) and \( G_{r}^{p} \) are the elastic and shear moduli of the piezo materials measured under constant electrical field. Moreover, \( h_{ij} \) is the piezoelectric constant, \( \beta_{ij}^{p} \) interprets the dielectric constant measured under a constant tension field, and \( v_{e} \) denotes the appropriate Poisson’s ratio (IEEE Std, 1987). Eq. (1) indicates the existence of two equal in-plane strains \( e_{11} \) and \( e_{22} \) for any applied voltages along the polling axis of the piezo patch. The piezo patch thickness is presumably small enough to maintain a uniform and constant electric displacement over its thickness. It is also assumed that full bonding between the plate and the piezo patches is maintained during the control process.

\( m \) piezoelectric rectangular patches with the same material properties are considered to be bonded on the lower surface of the thin rectangular plate, as shown in Fig. 1, with the \( z \)-axis directed upward. The location of each patch could be specified by its corner coordinates \( (x_{2s}, y_{2s}, z_{s}) \) and \( (y_{2s}, x_{2s}, z_{s}) \) where:

\[
a_{s} = x_{2s} - x_{2s-1}, \quad b_{s} = y_{2s} - y_{2s-1}, \quad s = 1, 2, \ldots, m.
\]

(2)

The dimensions of each piezo patch along the \( x \) and \( y \) axes are specified by \( a_{s} \) and \( b_{s} \), respectively, as shown in Fig. 1. The kinetic energy of the plate and the piezo patches are the following:

\[
K = K_{\text{plate}} + K_{\text{piezo}}
\]

(3)

where

\[
K_{\text{plate}} = \frac{1}{2} \int_{A} \rho_{\text{plate}} w^{2} \, dA
\]

(4)
and

\[ K_{\text{piezo}} = \frac{1}{2} \sum_{i=1}^{m} \int_{A} \mu_{\text{piezo}} W^{2} \Delta H_{i}(x, y) \, dA \]  

(5)

In the above equations, \( A \) represents the area of the plate and \( (\cdot) \) represents the derivatives with respect to time. Also:

\[ \Delta H_{i}(x, y) = [H(x - x_{21} - 1) - H(x - x_{21})][H(y - y_{21} - 1) - H(y - y_{21})] \]  

(6)

where \( H \) is the Heaviside step function. On the other hand, assuming infinitesimal strains, the strain energy of plate and the piezoelectro patches become:

\[ U = U_{\text{plate}} + U_{\text{piezo}} \]  

(7)

where

\[ U_{\text{plate}} = \frac{1}{2} \int_{A} D_{\text{plate}} \left[ W_{x}^{2} + W_{y}^{2} + 2t_{p}w_{x}W_{xy} + 2(1 - v)W_{y}^{2} \right] \, dA \]  

and

\[ U_{\text{piezo}} = \frac{1}{2} \int_{V} \left[ \varepsilon^{T} \sigma + \mathbf{E}^{T} \mathbf{D} \mathbf{E} \right] \, dV \]

\[ = \frac{1}{2} \sum_{i=1}^{m} \int_{A} \left\{ \text{D}_{\text{piezo}} \left[ W_{x}^{2} + W_{y}^{2} + 2t_{p}w_{x}W_{xy} + 2(1 - v_{p})W_{y}^{2} \right] + \text{h}_{\text{piezo}} \text{D}_{i} \left[ \text{h}_{\text{piezo}} + \text{D}_{\text{plate}} \right] \left[ W_{x} + W_{y} \right] \right\} \, dA \]

(9)

The parameter \( D_{\text{piezo}} = \frac{1}{2} \text{h}_{\text{piezo}}^{2} \text{D}_{i} \) is the bending stiffness of each piezoelectro patch, \( h_{\text{piezo}} \) the patch thickness and \( z = (h_{\text{piezo}} + h_{\text{plate}}) \). The virtual work done by external forces is:

\[ \delta W = \int_{f} \left( \delta U - \delta K + \delta W \right) \, dt = 0 \]  

(10)

Finally, applying the Hamilton’s principle:

\[ \delta H = \int_{t_{1}}^{t_{2}} \left( \delta U - \delta K + \delta W \right) \, dt = 0 \]  

(11)

leads to the following constitutive equation of motion of the system and its related natural and geometric boundary conditions:

\[ D_{\text{plate}} \nabla^{4} W + \mu_{\text{plate}} \frac{\partial^{2} W}{\partial t^{2}} + \sum_{i=1}^{m} D_{\text{piezo}} \left\{ \nabla^{4} W + \frac{\text{D}_{\text{piezo}}}{\text{D}_{\text{plate}}} \right\} \Delta H_{i}(x, y) + 2 \left[ W_{xx} + W_{yy} \right] \Delta H_{x}(x, y) + 2 \left[ W_{xy} + W_{yx} \right] \Delta H_{y}(x, y) + 2 \left( (1 - v_{p})W_{xy} \right) \Delta H_{x,y}(x, y) + 2 \left( (1 + v_{p})W_{xy} \right) \Delta H_{x,y}(x, y) = f(x, y, t) \]

\[ - \frac{1}{2} \sum_{i=1}^{m} \left[ h_{3i} \text{D}_{i} \right] z \nabla^{2} \Delta H_{i}(x, y) \]  

(12)

In these equations, it is assumed that the piezoelectro patch thickness ratio is small so that one could preserve the position of the neutral plane of the host plate unchanged in the presence of the piezoelectric patches with acceptable accuracy. Furthermore, due to the smaller modulus of elasticity of the piezoelectric materials in comparison to the host plate in most of the practical cases, there would be even less change in the position of the neutral plane of the combined system with respect to the host plate alone.

Based on the equation presented by Crawley and de Luis (1987) relating the induced strain in piezoelectric patches caused by the applied voltage to the mechanical strains (similar to a thermo-elastic material behavior), the following equation can be obtained:

\[ (D_{3})_{i} = \frac{E_{3i} d_{3i} V_{i}(t)}{(1 - v_{p})h_{\text{piezo}}} \]  

(13)

in which \( V(t) \) is the applied voltage, and \( d_{31} \) refers to the piezoelectric material constant (IEEE Std, 1987). Finally, it can be assumed that the bending stiffness and inertial effects of the piezoelectric patches are negligible in comparison to those of the host plate. In addition, considering the same dimensions for the piezoelectric patches, the equation of motion reduces to:

\[ D_{\text{plate}} \nabla^{4} W + \mu_{\text{plate}} \frac{\partial^{2} W}{\partial t^{2}} = f(x, y, t) - Z \sum_{i=1}^{m} \left[ V_{i}(t) \nabla^{2} \Delta H_{i}(x, y) \right] \]  

(14)

where \( Z = \frac{d_{3i} d_{3i}}{\varepsilon_{31} \varepsilon_{0} \varepsilon_{0}} \).

3. Moving mass excitation

Assume a mass \( M \), with an inertia relatively large compared to that of the main plate, is traveling on an arbitrary trajectory on the plate surface. Using the Dirac-delta function \( \delta \), the excitation term \( f(x, y, t) \) on the right-hand side of Eq. (14) becomes:

\[ f(x, y, t) = -M \left( g + \frac{d^{2}w_{0}(t)}{dt^{2}} \right) \delta(x - x_{0}(t)) \delta(y - y_{0}(t)) \]  

(15)

where \( g \) is the acceleration of gravity. The vertical displacement of the moving mass is shown by \( w_{0}(t) \), while \( x_{0}(t) \) and \( y_{0}(t) \) describe the path of the moving mass. Considering all the out-of-plane translational acceleration components of the moving mass, and observing the full contact condition between the moving mass and plate, Eq. (15) can be expanded as:

\[ f(x, y, t) = -M \left( g + \frac{d^{2}w_{0}(t)}{dt^{2}} \right) \delta(x - x_{0}(t)) \delta(y - y_{0}(t)) \]

\[ = -Mg - M \left( \frac{\partial^{2}w}{\partial t^{2}} \right) + x_{0}(t) \delta(x - x_{0}(t)) \delta(y - y_{0}(t)) + y_{0}(t) \delta(x - x_{0}(t)) \delta(y - y_{0}(t)) \]

\[ \delta(x - x_{0}(t)) \delta(y - y_{0}(t)) \]

The eigenfunction expansion method can be used to solve Eqs. (14) and (16). Consider the free vibration response of the plate:

\[ w(x, y, t) = \sum_{i=1}^{n} \phi_{i}(x, y) e^{i\omega_{i}t} \]  

(17)

where \( n \) is the number of vibrational modes of the plate considered, \( \phi_{n}(x, y) \) and \( \omega_{n} \) describe the modal shape and the natural frequency of the \( n \)th mode, respectively, and \( i = \sqrt{-1} \). Substituting Eq. (17) into the left-hand side of Eq. (14) results in:

\[ D_{\text{plate}} \nabla^{4} W(x, y, t) = M_{\text{plate}} \rho_{\text{plate}} \omega_{n}^{2} W(x, y, t) \]  

(18)

For the case of forced vibration, Eq. (17) changes:

\[ w(x, y, t) = \sum_{i=1}^{n} \phi_{i}(x, y) Q_{i}(t) \]  

(19)

where \( Q_{i}(t) \) is the time-dependent modal amplitude of the plate. Now, it is assumed that the left-hand side of Eq. (14) is equivalent to \( \sum_{i=1}^{n} \phi_{i}(x, y) A_{i}(t) \), and its right-hand side can be replaced by \( \sum_{i=1}^{n} \phi_{i}(x, y) B_{i}(t) \), in which \( A_{i}(t) \) and \( B_{i}(t) \) are any time-dependent amplitude parameters. According to Eqs. (16), (18) and (19), one can obtain:

\[ D_{\text{plate}} \sum_{i=1}^{n} \left\{ M_{\text{plate}} \left[ \bar{Q}_{i}(t) + \omega_{n}^{2} Q_{i}(t) \right] \right\} \phi_{i}(x, y) = \sum_{i=1}^{n} \phi_{i}(x, y) A_{i}(t) \]  

(20)
Multiplying both sides of Eqs. (20) and (21) by $\psi_j(x,y)$ and integrating over the surface of the plate leads to:

$$A_j(t) = \tilde{Q}_j(t) + \sigma_j^2 \tilde{Q}_j(t)$$

(22)

Also,

$$B_j(t) = \left( \frac{M}{\mu_{\text{plate}}} \right) \left( -g - \sum_{j=1}^{n} \psi_j(x_0, y_0) \tilde{Q}_j(t) \right) \left( \frac{\psi_j(x_0, y_0)}{\bar{A}} \right)$$

$$+ \left[ \tilde{x}_0(t) \psi_{jx}(x, y) + \tilde{y}_0(t) \psi_{jy}(x, y) \right] \tilde{Q}_j(t) + \left[ \tilde{x}_0(t) \psi_{jx}(x, y) + \tilde{y}_0(t) \psi_{jy}(x, y) \right]$$

$$+ \tilde{y}_0(t) \psi_{jx}(x, y) + 2\tilde{x}_0(t) \psi_{jy}(x, y) + \tilde{x}_0(t) \psi_{jx}(x, y)$$

$$+ \tilde{y}_0(t) \psi_{jx}(x, y) + \tilde{y}_0(t) \psi_{jy}(x, y) \right] \tilde{Q}_j(t)$$

$$+ \left[ \tilde{x}_0(t) \psi_{jx}(x, y) + \tilde{y}_0(t) \psi_{jy}(x, y) \right] \tilde{Q}_j(t)_{x=x_0(t), y=y_0(t)}$$

$$\delta(x-x_0(t)) \delta(y-y_0(t)) - Z \sum_{i=1}^{m} V_i(t) \nabla^2 \Delta H_i(x, y)$$

(21)

Thus:

$$U(t) = A(t)X(t) + B(t)U(t) + F(t)$$

(31)

where $Q_0$ and $Q_1$ are the appropriate initial conditions, while $V_i(t), \ldots, V_m(t)$ represent the required voltages of the piezo patches. Even though the original system is assumed to be undamped, as Eq. (25) shows, there is an induced damping constant term $C(t)$ present in the modal equation of motions due to the effect of the moving mass. The induced damping constant could take negative, zero, or positive values during an external excitation, depending on the different parameters involved. State-space representation of Eq. (25) becomes:

$$X(t) = A(t)X(t) + B(t)U(t) + F(t)$$

(31)

where

$$X(t) = \left[ \begin{array}{c} Q(t) \\ \dot{Q}(t) \end{array} \right], \quad A(t) = \left[ \begin{array}{c} 0 & -M^{-1}K \\ -M^{-1}C \end{array} \right], \quad B(t) = \left[ \begin{array}{c} 0 \\ -M^{-1}D \end{array} \right]$$

The solution to Eq. (31) can be written as the following (Brogan, 1991):

$$X(t) = U(t)U^{-1}(t_0)X(t_0)$$

$$+ \int_{t_0}^{t} \left\{ U(t)U^{-1}(\tau)B(\tau)U(\tau) + F(\tau) \right\} d\tau$$

(32)

where $U(t)$ is the fundamental solution matrix and:

$$U(t) = A(t)U(t), U(t_0) = I_{2p}$$

(33)

Also:

$$X(t) = U(t)X(t_0)$$

(34)

A transfer matrix $\Phi(t)$ is used to obtain $U(t)$:

$$\Phi(t, \tau) = U(t)U^{-1}(\tau)$$

(35)

Thus:

$$\Phi(t, \tau) = \Phi(t, \tau)X(t)$$

(36)

An approximate solution can be used to obtain $\Phi$:

$$\Phi(t_{k+1}, t_k) = e^{\Phi(t)}$$

(37)

in which $\Delta t = t_{k+1} - t_k$ is an assumed time interval. If $A^{-1}(t_k)$ exists, Eq. (31) can be solved, leading to:

$$\Phi(t_{k+1}, t_k) = A_1(t_k)X(t_k) + B_1(t_k)U(t_k) + E_1(t_k)$$

(38)

where:

$$A_1(t_k) \equiv e^{\Phi(t_k)}$$

$$B_1(t_k) \equiv [A_1(t_k) - I]A^{-1}(t_k)B(t_k)$$

$$E_1(t_k) \equiv [A_1(t_k) - I]A^{-1}(t_k)F(t_k)$$

In this study, MATLAB is used to solve Eq. (38).

### 4. Control algorithm

A linear classical optimal control algorithm with displacement-velocity feedback is used to determine the required voltage for each piezoelectric patch. Thus, the following Riccati type matrix equation is considered (Soong, 1990):

$$PA - \frac{1}{2}PBR^{-1}BP + A^TQ + 2Q = 0$$

(39)

in which $Q_{m+1} > 0$ and $R_{m+m}$ are positive semi-definite and positive definite matrices, respectively. The parameters $P$ and $n$ represent the Riccati matrix and the number of controlled modes involved in the control process, respectively, while $m$ is the number of piezo actuators. These weighting matrices could be simply assumed to be $R = aI_{m+m}$ and $Q = bI_{m+m}$ in which $I$ is
the identity matrix and the coefficients $\alpha$ and $\beta$ are adjusted in a way to satisfy the control objectives. The resulting control gain matrix becomes:

$$G = -\frac{1}{2}R^{-1}B'P$$  \hspace{1cm} (40)

The control force vector can then be written as:

$$u(t) = GX(t)$$  \hspace{1cm} (41)

Substituting Eq. (41) into Eq. (31) leads to:

$$X(t) = (A + BG)X(t) + F(t), \quad X(0) = X_0$$  \hspace{1cm} (42)

Eq. (42) should be solved to determine the controlled response of the dynamic system. In addition, as Eq. (25) shows, the system matrix $A$ is a function of time, meaning that the control gain matrix components are continuously changing as long as the moving mass travels within the plate boundaries. It will change to a constant matrix for controlling the free vibration response of the system once the moving mass leaves the plate.

5. Numerical example

A simply supported $2.0 \, \text{m} \times 2.0 \, \text{m} \times 1.0 \, \text{cm}$ aluminum plate with a modulus of elasticity, $E = 7.1 \times 10^{10} \, \text{Pa}$, mass density, $\rho = 2700 \, \text{kg} \, \text{m}^{-3}$ and Poisson's ratio, $\nu = 0.33$, is considered. It is assumed that a number of square piezoelectric patches are used to control the dynamic response of the plate caused by a moving mass traveling on its surface. The mechanical and geometrical characteristics of the piezoelectric patches bonded to the lower surface of the plate are $E_p = 6.2 \times 10^{10} \, \text{Pa}$, $\nu_p = 0.3$, $d_{31} = -320 \times 10^{-12} \, \text{m/V}$, (PZT-5H) and $h_p = 0.10 \, \text{cm}$. Two trajectories are considered for the moving mass. The first trajectory is a straight line, while the...
second one is an orbiting path (Figs. 2a and b). In both cases, it is assumed that the plate is originally at rest. The modal shapes and natural frequencies of the plate according to Eq. (24) are:

$$\omega_n = \omega_0 = \pi^2 \left[ \left( \frac{i}{a} \right)^2 + \left( \frac{j}{b} \right)^2 \right] \sqrt{\frac{D_{\text{plate}}}{\mu_{\text{plate}}}}$$  

$$\phi_n(x,y) = \phi_0(x,y) = \frac{2}{\sqrt{\mu_{\text{plate}} ab}} \sin \left( \frac{i\pi x}{a} \right) \sin \left( \frac{j\pi y}{b} \right)$$

in which \(i, j = 1, 2, \ldots, n\), and \(a = b = 2.0\, \text{m}\). Also, the number of vibrational modes involved in the numerical analyses is equal to \(n^2\). For each trajectory, the importance of the moving mass inertia and the effect of higher vibrational modes on the response of the system are investigated. Then, using uniformly distributed arrays of piezo patches on the lower surface of the plate, the deflection of the center point of the plate is controlled. As it was mentioned earlier, it is assumed that the response of the plate remains geometrically linear under the applied loading cases. For remainder of the work, the parameters \(D_{\text{plate}}\) and \(\mu_{\text{plate}}\) are replaced with \(D\) and \(\mu\) for simplicity.

5.1. Linear path

In this case, the moving mass is assumed to travel along a linear path over the plate with different velocities and mass weights. It is the most encountered loading case for the moving mass problem traveling on various structural systems such as slab type bridges or the deck of aircraft carriers (Humar and Kashif, 1995). Under the moving mass excitation, the dynamic response of the plate includes a forced vibration part followed by a transient free vibration once the moving mass reaches the plate boundaries. As an example, a linear path is defined by the following (Fig. 2a):

$$x_0(t) = a/2$$
$$y_0(t) = vt$$

Fig. 4. The time history of the center point response of the plate for the moving load and the moving mass cases. (—— moving mass; ----- moving load) \((M = 0.3ab, v = 0.2v)\).
In which $v$ is the velocity of the moving load. Fig. 3 shows the effect of mass weight, mass velocity and the number of vibrational modes on the dynamic amplification factor (DAF) of the center point of the plate. The DAF is defined as the ratio of absolute maximum dynamic deflection of the plate to its maximum static response at the center point. The static deflection of the center point under a concentrated mass $M$, applied at the same point is equal to $A_{\text{static}} = \frac{0.01160}{v}$ (Timoshenko and Woinowsky-Kreiger, 1959). The static deflection of the center of the plate under its own weight is negligible in comparison to that of the concentrated load. In this figure, the horizontal axis shows the moving mass velocity divided by the parameter $\frac{v}{v_0}$, where $T_1$ is the first vibrational period of the plate. The velocity $v$ has been used in a previous work to compare the dynamic response of an Euler-Bernoulli beam under moving load and moving mass excitations (Nikhgoo et al., 2007).

As it is seen for the moving load case, the DAF of the center point remains almost the same for different moving mass weights and numbers of involved modes. However, for the case with moving mass, the inertia effect becomes more important as the moving mass velocity increases. Obviously, increasing the mass ratio, $M/\mu ab$, makes the inertial effect of the moving mass more dominant. Contrary to the moving load case, the contribution of the higher modes has considerable influence on the dynamic behavior of the plate, especially for high values of moving mass weight and velocity. Therefore, simplifying the problem of a moving mass as a moving load case alone would lead to nonconservative results except for small moving mass weights and velocities. As determined by Eqs. (28), (43) and (44), for the moving mass case, the mass velocity should be limited to $v = \frac{\pi \sqrt{D/M}}{T_1}$ in order to ensure the stability of the dynamic system. This limitation is recognized in Fig. 3. Additionally, increasing the number of modes to 49 did not improve
the results appreciably with respect to the case with 25 modes, while the computation time was noticeably increased.

Figs. 4 and 5 show the time history of the normalized displacement as well as the normalized velocity and acceleration of the plate center point under a moving load and moving mass, considering the first nine modes of the plate. As these figures show, the parameter time is normalized by $T_1$, while the center point displacement of plate is normalized by the static deflection $D_{\text{static}}$ defined earlier. In addition, the velocity and acceleration of the center point of the plate are normalized by $D_{\text{static}}/T_1$ and $D_{\text{static}}/T_1^2$, respectively. Using these normalized figures, we can determine the maximum response parameters for any plate with the same fundamental vibration period.

As Fig. 4 indicates, for $M = 0.3l_{ab}$ and $v = 0.2v_0$, the effect of moving mass inertia is shown to be of considerable importance. Increasing the mass weight and velocity of the moving mass to $M = 0.5l_{ab}$ and $v = 0.5v_0$, which is an indication of a severe loading, leads to totally a different dynamic response of the plate (Fig. 5).

Having identified the importance of the moving mass inertia, the applicability of the control mechanism in reducing the dynamic response of the plate is evaluated. Using nine square piezo patches as shown in Fig. 2a, a full state displacement–velocity feedback is
employed to limit the response of the system. The weighting matrices $\mathbf{R}$ and $\mathbf{Q}$ are selected such that the DAF of the center point of the plate is reduced by half.

In Figs. 6 and 7, the time histories of the uncontrolled and controlled dynamic response of the center point, for $M = 0.3\mu ab$, $\nu = 0.2\nu'$ and $M = 0.5\mu ab$, $\nu = 0.5\nu'$, respectively, are illustrated. Having controlled the first nine modes of the plate, the center point response of the plate is reduced to the target displacement within a short time. The required control voltages for the different piezo patches are shown in Figs. 8 and 9. In addition, two different sizes of piezo patches, i.e., $15 \times 15 \times 0.1$ cm and $20 \times 20 \times 0.1$ cm, are considered to evaluate the size effect on the performance of the control system. As expected, the maximum required voltage for the case with larger patches was 43% less than the maximum required voltage for the smaller patches. More work could be done in selecting the weighting matrices in an optimal manner to further improve the efficiency of the control system.

5.2. Orbiting path

A more complex loading pattern is an orbiting trajectory that can be described by the following equations (Fig. 1):

$$x_0(t) = r\cos(\omega t) + a/2$$
$$y_0(t) = r\sin(\omega t) + b/2$$

(46)
where \( r \) is the radius and \( \omega' \) is the angular frequency of the orbiting mass (Fig. 2b). This type of loading is of considerable interest in mechanical engineering, specifically in the analysis of rotating machinery (Cifuentes and Lalapet, 1992). Fig. 10 shows the variation of different parameters involved in the maximum deflection of the center point at resonant state, considering 1, 9 and 25 vibrational modes of the plate. In this figure, the variation of the resonance orbiting frequency (\( \omega' \)) is presented versus the feasible ranges of relative mass (\( M/l_{ab} \)) and relative radius (\( r/a \)) quantities of the orbiting mass.

In the case of moving load for any values of mass (\( M \)) and radius (\( r \)), there exists a unique orbiting frequency equal to 0.25\( \omega_1 \), with \( \omega_1 \) being the first natural frequency anti-censorship that causes the resonant state for the given square plate, as shown in Fig. 10. The resonance orbiting frequency would be different for non-square plates. For the case with moving mass, an increase in the relative mass (\( M/l_{ab} \)) reduces the resonance orbiting frequency for any relative radius (\( r/a \)) of the orbiting mass. Moreover, as the relative mass and relative radius increase, the rate of change of the resonance orbiting frequency becomes more sensitive to the number of vibrational modes considered. That could be related to the dominance of the first mode of the plate on the dynamic response for small relative masses and radii.

In addition, the resonance frequency approaches 0.25\( \omega_1 \) as the relative mass ratio becomes very small. As Fig. 10 shows, for the case of \( r/a = 0.1 \) and \( M/l_{ab} < 0.1 \), the resonance frequency of the

![Fig. 12. Dynamic amplification factor of the center point of the plate under a moving mass, considering the first 100T, seconds of excitation (\( r/a = 0.25 \); —— 1 mode; —— 9 modes; —— 25 modes).](image)

![Fig. 13. Dynamic amplification factor of the center point of a plate for a moving mass and moving load excitations (\( r/a = 0.25 \); —— moving load; —— moving mass, \( M = 0.05l_{ab} \); —— moving mass, \( M = 0.25l_{ab} \)).](image)
system does not exist for 9 or 25 modes. This is due to the fact that the resonant state is not distinct for this condition, as can be observed for the case $M/l_{a b} = 0.05$ in Fig. 11. This can be related to the effect of the induced damping to the system caused by the moving mass, as Eq. (28) indicates. In other words, the peak response of the system caused by the small weight and radius of orbiting mass is largely reduced due to the induced damping to the system.

The importance of mass inertia could be better demonstrated by considering the response of the system for different mass weights and orbiting radii as well as the number of considered vibrational modes. Figs. 11 and 12 show the variation of the DAF of the center point versus the variation of the excitation frequency for different mass weights and numbers of vibrational modes. The DAF of the center point of the plate is determined in the first $100T_1$ seconds of the related excitation. In these figures, we observe the effect of higher modes in shifting the resonance frequency and increasing the DAF values with respect to case with one mode, especially for larger values of mass weight and orbiting radius.

Equally important is the role of mass inertia on the dynamic behavior of the plate. As shown in Fig. 13, the DAF values for the moving mass and the moving load cases are compared considering the first 25 vibrational modes for $M/l_{a b} = 0.05, 0.25$ and a relative radius $r/a = 0.25$. One should notice that the DAF values for the moving load case for both relative mass ratios ($M/l_{a b} = 0.05, 0.25$) are the same. On the other hand, while the resonance fre-
frequencies are totally different for the moving mass and moving load cases, the DAF values for the moving load excitation are not generally a function of mass weight or even the number of vibrational modes considered.

Furthermore, the time history of the normalized deflection of the center point for a relative mass $M/l_{ab} = 0.25$, relative radius $r/a = 0.25$, and the related resonance orbiting frequency $\omega' \approx 0.1968\omega_1$ (according to Fig. 10) is shown in Fig. 14. This figure indicates that for the case of a moving mass, the resonant response of the plate depends on the number of vibrational modes considered. The results of a similar study for the case of a moving load and the related resonance excitation frequency $\omega' = 0.25\omega_1$ is shown in Fig. 15, keeping other parameters as before. Unlike the case with a moving mass, considering a greater number of modes does not change the resonant response of the system significantly.

In the next step, the feasibility of the piezo patches in controlling the response of the system under the orbiting mass is investigated. In this case, 16 uniformly distributed piezo patches are considered as shown in Fig. 2b, using two different sizes of piezo patches, i.e., $10 \times 10 \times 0.1$ cm and $15 \times 15 \times 0.1$ cm. Considering $M = 0.25\mu ab$, $r = 0.25a$ and $\omega' = 0.1968\omega_1$, which relates to a resonant state, a full state displacement–velocity feedback control system is employed to reduce the normalized dynamic deflection of

![Graph showing controlled and uncontrolled normalized deflections of the center point of the plate for moving mass case.](image)

*Fig. 16.* Controlled and uncontrolled normalized deflections of the center point of the plate for moving mass case (— uncontrolled; --- controlled) ($M = 0.25\mu ab$, $r/a = 0.25$, $\omega' = 0.1968\omega_1$, 16 modes).

![Deflection profiles of the controlled and uncontrolled plate under a moving mass at three different time instants.](image)

*Fig. 17.* Deflection profiles of the controlled and uncontrolled plate under a moving mass at three different time instants ($M = 0.25\mu ab$, $r/a = 0.25$, $\omega' = 0.1968\omega_1$, 16 modes).
the center point to almost 10% of that for an uncontrolled system. The response reduction is assumed to take place within the first four cycles of orbiting mass, controlling the first 16 vibrational modes of the system.

The time-history of the uncontrolled and controlled normalized dynamic deflection of the center point is shown in Fig. 16. The deflection profiles of the uncontrolled and controlled plate at three different time instants are shown in Fig. 17, showing the performance of the control system in suppressing the response of the system. Again the required voltages for a number of piezo patches are presented in Figs. 18 and 19. It should be noticed that, depending on the piezo patch dimension, different weighting matrices $R$ and $Q$ are used to achieve the controlled objectives. As in the previous case, larger piezo patches require less maximum voltage compared smaller piezo patches. As already mentioned, the control gains in this case should be calculated at each time step using Eqs. (39) and (40). Figs. 20 and 21 show the phase portraits of the uncontrolled and controlled center point responses, respectively, after the first, second and fifth cycle of external excitation. As expected, the phase portrait of the uncontrolled system is growing with time, while it remains completely bounded for the controlled system. Finally, utilizing fairly low levels of voltage which is easily achievable, the response of the system could be reduced remarkably.

![Fig. 18. The required voltage for piezo patches 1-4 ($M = 0.25\mu m, r/a = 0.25, \omega/\omega_{1} = 0.1968$, 16 modes, —— 10 $\times$ 10 $\times$ 0.1 cm, ... 15 $\times$ 15 $\times$ 0.1 cm).](image1)

![Fig. 19. The required voltage for piezo patches 9-12 ($M = 0.25\mu m, r/a = 0.25, \omega/\omega_{1} = 0.1968$, 16 modes, —— 10 $\times$ 10 $\times$ 0.1 cm, ... 15 $\times$ 15 $\times$ 0.1 cm).](image2)
6. Conclusions

The constitutive equation of motion for a thin rectangular plate with a number of piezo patches bonded on its surface under the excitation of a moving mass was derived, observing all the out-of-plane translational acceleration components of the moving mass. A classical optimal control algorithm was employed to control the dynamic response of the system under external excitations. To demonstrate the efficiency of the control mechanism, a simply supported square plate under the excitation of a moving mass was considered using two different loading paths. Assuming a straight path, the importance of the mass inertia especially for high moving mass velocities was verified.

Moreover, for large values of moving mass weight and velocity, inclusion of higher vibrational modes is necessary for capturing the true response of the system. In addition, simplifying the problem as a moving load case would lead to nonconservative results except for small moving mass weights and velocities. For the orbiting path case, it was shown that any increase in the mass weight would reduce the resonance orbiting frequency of the system for any radius of the orbiting mass. Furthermore, as the relative mass and relative radius increases, the rate of change of the resonance orbiting frequency becomes more sensitive to the number of vibrational modes considered. The effect of higher modes in shifting the resonance frequency and increasing the DAF values with respect to the one mode case was shown to

![Fig. 20. The phase portrait of the center point of the uncontrolled plate under the excitation of a moving mass (M = 0.25μab, r/a = 0.25, ω' = 0.1968ω1, 16 modes).](image1)

![Fig. 21. The phase portrait of the center point of the controlled plate under the excitation of a moving mass (M = 0.25μab, r/a = 0.25, ω' = 0.1968ω1, 16 modes).](image2)
be significant, especially for larger values of mass weight and orbiting radius. On the other hand, the DAF values for the moving load excitation were generally shown to be independent of the mass weight or the number of vibrational modes considered. The efficiency of the control algorithm was investigated using a number of square piezo patches that were uniformly distributed on the lower surface of the plate. It was observed that the response of the system, especially those near the resonant states, was reduced to the desired target displacement with fairly low levels of applied voltages. Finally, increasing the area of the employed piezo patches would reduce the maximum required voltage for controlling the response of the system.

References