On the Invariance of Colin de Verdière's Graph Parameter Under Clique Sums

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Submitted by Willem H. Haemers

ABSTRACT

For any undirected graph G, let \( \mu(G) \) be the graph invariant introduced by Colin de Verdière. In this paper we study the behavior of \( \mu(G) \) under clique sums of

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0024-3795/95/$9.50

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SSDI 0024-3795(95)00160-S
1. INTRODUCTION

Colin de Verdière [2] (cf. [3]) introduced an interesting new invariant $\mu(G)$ for graphs $G$, based on algebraic and analytic properties of matrices associated with $G$. He showed that the invariant is monotone under taking minors and that $\mu(G) \leq 3$ if and only if $G$ is planar.

Colin de Verdière conjectured that $\gamma(G) \leq \mu(G) + 1$, where $\gamma(G)$ is the coloring number of $G$. This conjecture would follow from Hadwiger's conjecture [as $\mu(K_n) = n - 1$] and is true for $\mu(G) \leq 4$.

Graph $G$ is a clique sum of graphs $G_1$ and $G_2$ if $V_G = V_{G_1} \cup V_{G_2}$ and $E_G = E_{G_1} \cup E_{G_2}$, where $V_{G_1} \cap V_{G_2}$ is a clique both in $G_1$ and in $G_2$. Note that for the coloring number $\gamma$ one has that $\gamma(G) = \max\{\gamma(G_1), \gamma(G_2)\}$ if $G$ is a clique sum of $G_1$ and $G_2$. A similar relation holds for the size of the largest clique minor in a graph.

We therefore are interested in studying the behavior of $\mu(G)$ under clique sums (cf. also [4]). A critical example is the graph $K_{t+3} \setminus \Delta$ (the graph obtained from the complete graph $K_{t+3}$ by deleting the edges of a triangle). One has $\mu(K_{t+3} \setminus \Delta) = t + 1$ [since the star $K_{t+3} \setminus \Delta$ has $\mu(K_{t+3} \setminus \Delta) = 2$ and since adding a new vertex adjacent to all existing vertices increases $\mu$ by 1].

However, $K_{t+3} \setminus \Delta$ is a clique sum of $K_{t+1}$ and $K_{t+2} \setminus e$ (the graph obtained from $K_{t+3}$ by deleting an edge), with common clique of size $t$. Both $K_{t+1}$ and $K_{t+2} \setminus e$ have $\mu = t$. So, generally one does not have that, for fixed $t$, the property $\mu(G) \leq t$ is maintained under clique sums. Similarly, $K_{t+3} \setminus \Delta$ is a clique sum of two copies of $K_{t+2} \setminus e$, with common clique of size $t + 1$.

These examples where $\mu$ increases by taking a clique sum are in a sense the only cases: We show that if $G$ is a clique sum of $G_1$ and $G_2$, with common clique $S$, then $\mu(G) > t := \max\{\mu(G_1), \mu(G_2)\}$ if and only if $t > 0$ and either $|S| = t$ and $G - S$ has three components, the contraction of which makes with $S$ a $K_{t+3} \setminus \Delta$, or $|S| = t + 1$ and $G - S$ has two components, the contraction of which makes with $S$ a $K_{t+3} \setminus \Delta$. Moreover, if $\mu(G) > t$, then $\mu(G) = t + 1$ and $\mu(G_1) = \mu(G_2) = t$.

So $\mu(G) = \max\{\mu(G_1), \mu(G_2)\}$ if and only if $G$ does not contain $K_{t+3} \setminus \Delta$ as a minor.

In Section 2 we give the definition of Colin de Verdière's invariant, including an alternative linear algebraic characterization of the "strong Arnold
hypothesis.” In Section 3 we prove a lemma, while in Section 4 we derive our main characterization.

If \( M \) is a matrix, then \( M_K \) denotes the submatrix of \( M \) induced by the row and column indices in \( K \). Similarly, if \( x \) is a vector, then \( x_K \) denotes the subvector of \( x \) induced by the indices in \( K \). We denote the \( i \)th eigenvalue (from below) of \( M \) by \( \lambda_i(M) \).

2. COLIN DE VERDIÈRE’S INVARIANT

We describe Colin de Verdière’s invariant. Important is a certain general position assumption for matrices called the strong Arnold hypothesis. We here formulate it and give an equivalent linear algebraic characterization.

Let \( M = (m_{i,j}) \) be a symmetric \( n \times n \) matrix. Let \( R(M) \) be the set of all symmetric \( n \times n \) matrices \( A \) with \( \text{rank}(A) = \text{rank}(M) \). Let \( S(M) \) be the set of all symmetric \( n \times n \) matrices \( A = (a_{i,j}) \) such that \( a_{i,j} = 0 \) whenever \( i \neq j \) and \( m_{i,j} = 0 \).

The matrix \( M \) is said to fulfill the strong Arnold hypothesis (SAH) if \( R(M) \) intersects \( S(M) \) at \( M \) “transversally”; that is, if the tangent space of \( R(M) \) at \( M \) and the tangent space of \( S(M) \) at \( M \) together span the space of all symmetric \( n \times n \) matrices. In other words, if the intersection of the normal spaces at \( M \) of \( R(M) \) and \( S(M) \) only consists of the all-zero matrix.

The tangent space of \( R(M) \) at \( M \) consists of all symmetric \( n \times n \) matrices \( N \) such that \( x^TNx = 0 \) for each \( x \in \ker(M) \). Thus the normal space of \( R(M) \) at \( M \) is equal to the space generated by all matrices \( xx^T \) with \( x \in \ker(M) \). This space is equal to the space of all symmetric \( n \times n \) matrices \( X \) satisfying \( MX = 0 \). Trivially, the normal space of \( S(M) \) at \( M \) consists of all symmetric \( n \times n \) matrices \( X = (x_{i,j}) \) such that \( x_{i,j} = 0 \) whenever \( i = j \) or \( m_{i,j} \neq 0 \). Therefore, the SAH is equivalent to:

\[
\text{there is no nonzero symmetric } n \times n \text{ matrix } X = (x_{i,j}) \text{ such that } MX = 0 \text{ and such that } x_{i,j} = 0 \text{ whenever } i = j \text{ or } m_{i,j} \neq 0. \tag{1}
\]

Now Colin de Verdière’s invariant \( \mu(G) \) is defined as follows. Let \( G \) be an undirected graph, which throughout this paper we assume without loss of generality to have vertex set \( \{1, \ldots, n\} \). Then \( \mu(G) \) is the largest corank of any symmetric \( n \times n \) matrix \( M = (m_{i,j}) \) satisfying:

\[
M \text{ has exactly one negative eigenvalue (of multiplicity 1), and for all } i, j \text{ with } i \neq j, m_{i,j} < 0 \text{ if } i \text{ and } j \text{ are adjacent, and } m_{i,j} = 0 \text{ otherwise}. \tag{2}
\]
and such that $M$ fulfills the SAH. [The corank $\text{corank}(M)$ of a matrix $M$ is the dimension of its kernel.]

It turns out, as proved in [2], that if $G'$ is a minor of $G$, then $\mu(G') \leq \mu(G)$. (In proving this, the SAH is essential.) So for each fixed $t$, the class of graphs $G$ satisfying $\mu(G) \leq t$ is closed under taking minors. Hence, by the theorem of Robertson and Seymour [6] there is a finite collection of "forbidden minors" for such a class of graphs.

Colin de Verdière [2] showed that the graphs $G$ satisfying $\mu(G) \leq 1$ are exactly the paths, those satisfying $\mu(G) \leq 2$ are exactly the outerplanar graphs, and those satisfying $\mu(G) \leq 3$ are exactly the planar graphs. If $\mu(G) \leq 4$, then $G$ is linklessly embeddable, since each graph $G$ in the complete class of forbidden minors found by Robertson, Seymour, and Thomas [7] has $\mu(G) > 4$ (cf. Bacher and Colin de Verdière [1]). In fact, Robertson, Seymour, and Thomas [8] conjecture that also the reverse implication holds.

3. A LEMMA

The following lemma gives us some tools:

**Lemma.** Let $G = (V, E)$ be a graph and let $M$ be a matrix satisfying (2). Let $S \subseteq V$ and let $C_1, \ldots, C_m$ be the components of $G - S$. Then:

(i) If $\lambda_i(M_{C_j}) < 0$, then $\lambda_i(M_{C_j}) > 0$ for all $j \neq 1$.

(ii) If $\lambda_1(M_{C}) = 0$, then there are at least $\text{corank}(M) - |S| + 2$ components $C_i$ with $\lambda_1(M_{C_i}) = 0$.

(iii) If $M$ fulfills the SAH, then there are at most three components $C_i$ with $\lambda_1(M_{C_i}) = 0$.

**Proof.** If (i) does not hold, we may assume that $\lambda_1(M_{C_1}) < 0$ and $\lambda_1(M_{C_2}) \leq 0$. Let $z, x_1,$ and $x_2$ be the eigenvectors belonging to the smallest eigenvalues of $M$, $M_{C_1}$, and $M_{C_2}$, respectively. By the Perron-Frobenius theorem we may assume that $z, x_1, x_2 > 0$ and by scaling that $z^T x_1 = z^T x_2$.

Define $y \in \mathbb{R}^n$ by $y_i := (x_1)_i$ if $i \in C_1$, $y_i := -(x_2)_i$ if $i \in C_2$, and $y_i := 0$ if $i \notin C_1 \cup C_2$. Then $z^T y = z^T x_1 - z^T x_2 = 0$ and $y^T M y = x^T M_{C_1} x_1 + x^T M_{C_2} x_2 < 0$. However, $z^T y = 0$ and $y^T M y < 0$ imply that $\lambda_2(M) < 0$, contradicting (2).
To see (ii), if \( \lambda_i(M_{c_i}) = 0 \), then by (i), \( \lambda_i(M_{c_i}) \geq 0 \) for all \( i \); that is, \( M_{c_i} \) is positive semidefinite for each \( i \). Let \( D \) be the vector space of all vectors \( x \in \ker(M) \) with \( x_s = 0 \) for all \( s \in S \). Then:

for each vector \( x \in D \) and each component \( C_i \) of \( G - S \),

\[
\begin{align*}
x_{C_i} &= 0, \quad x_{C_i} > 0 \text{ or } x_{C_i} < 0; \text{ if moreover } \lambda_i(M_{c_i}) > 0, \text{ then } \quad (3) \\
x_{C_i} &= 0.
\end{align*}
\]

Indeed, if \( x \in D \), then \( M_{c_i}x_{C_i} = 0 \). Hence if \( x_{C_i} \neq 0 \) (as \( M_{c_i} \) is positive semidefinite), \( \lambda_i(M_{c_i}) = 0 \) and \( x_{C_i} \) is an eigenvector belonging to \( \lambda_i(M_{c_i}) \), and hence (by the Perron-Frobenius theorem) \( x_{C_i} > 0 \) or \( x_{C_i} < 0 \).

Let \( m' \) be the number of components \( C_i \) with \( \lambda_i(M_{c_i}) = 0 \). By (3), \( \dim(D) \leq m' - 1 \) (since each nonzero \( x \in D \) has both positive and negative components, as it is orthogonal to \( z \)).

Since \( \lambda_i(M_{c_i}) = 0 \), there exists a vector \( w > 0 \) such that \( M_{c_i}w = 0 \). Let \( F \) be the vector space of all vectors \( x_s \) with \( x \in \ker(M) \). Suppose that \( \dim(F) = |S| \). Let \( j \) be a vertex in \( S \) adjacent to \( C_i \). Then there is a vector \( y \in \ker(M) \) with \( y_j = -1 \) and \( y_i = 0 \) if \( i \in S \setminus \{j\} \). Let \( u \) be the \( j \)-th column of \( M \). So \( u_{c_i} = M_{c_i}y_{C_i} \). Since \( u_{c_i} \leq 0 \) and \( u_{c_i} \neq 0 \), we have

\[
0 > u_{c_i}^T w = y_{C_i}^T M_{c_i} w = 0, \text{ a contradiction.}
\]

Hence \( \dim(F) \leq |S| - 1 \), and so

\[
m' - 1 \geq \dim(D) = \text{corank}(M) - \dim(F) \geq \text{corank}(M) - |S| + 1.
\]

If (iii) does not hold, we may assume that \( \lambda_i(M_{c_i}) = 0 \), for \( i = 1, \ldots, 4 \). Let \( x_i \) be an eigenvector belonging to the smallest eigenvalue of \( M_{c_i} \), for \( i = 1, \ldots, 4 \). Let \( z \) be the eigenvector belonging to smallest eigenvalue of \( M \).

We may assume that \( z, x_1, \ldots, x_4 > 0 \) and that \( z_{C_i}^T x_1 = z_{C_2}^T x_2 \) and \( z_{C_3}^T x_3 = z_{C_4}^T x_4 \). Define the vectors \( y_1 \) and \( y_2 \) by \( (y_1)_1 := (x_1)_1 \) if \( i \in C_1 \), \( (y_1)_1 := -(x_2)_1 \) if \( i \in C_2 \), and \( (y_1)_i := 0 \) if \( i \notin C_1 \cup C_2 \), and \( (y_2)_1 := (x_3)_1 \) if \( i \in C_3 \), \( (y_2)_1 := -(x_4)_1 \) if \( i \in C_4 \), and \( (y_2)_i := 0 \) if \( i \notin C_3 \cup C_4 \). Then \( z^T y_1 = z_{C_1}^T x_1 - z_{C_2}^T x_2 = 0 \) and \( z^T y_2 = z_{C_3}^T x_3 - z_{C_4}^T x_4 = 0 \). Since \( y_1^T M y_1 = x_1^T M_{c_i} x_1 + x_2^T M_{c_i} x_2 = 0 \) and similarly \( y_2^T M y_2 = 0 \), the vectors \( y_1 \) and \( y_2 \) belong to \( \ker(M) \).

Define \( X := y_1 y_2^T + y_2 y_1^T \). Then \( x_{i,j} \neq 0 \) implies \( i \in C_1 \cup C_2 \) and \( j \in C_3 \cup C_4 \) or conversely. As \( MX = 0 \), this contradicts the SAH.

\[\blacksquare\]
4. CLIQUE SUMS OF GRAPHS

Now let $G$ be a clique sum of $G_1$ and $G_2$. Let $S := V_{G_1} \cap V_{G_2}$ and $t := \max\{\mu(G_1), \mu(G_2)\}$. For any $U \subseteq V_G$, let $N(U)$ denote the set of vertices in $V_G \setminus U$ that are adjacent to at least one vertex in $U$.

**Theorem.** If $\mu(G) > t$, then $\mu(G) = t + 1$ and we can contract two or three components of $G - S$ so that the contracted vertices together with $S$ form a $K_{t+3} \setminus \Delta$.

**Proof.** We apply induction on $|V_G| + |S|$. Let $M$ be a matrix satisfying (2) and fulfilling the SAH, with corank equal to $\mu(G)$. We first show that $\lambda_i(M_C) \geq 0$ for each component $C$ of $G - S$. Suppose $\lambda_1(M_C) < 0$. Hence by (i) of the lemma, $\lambda_1(M_{C'}) > 0$ for each other component $C'$. Let $G'$ be the subgraph of $G$ induced by $C \cup S$; so $G'$ is a subgraph of $G_1$ or $G_2$. Let $L$ be the union of the other components, so $\lambda_1(M_L) > 0$. We write

$$M = \begin{pmatrix} M_C & U_C & 0 \\ U_C^T & \Pi & U_L \\ 0 & U_L^T & M_L \end{pmatrix}. \quad (5)$$

Let

$$A := \begin{pmatrix} I & 0 & 0 \\ 0 & I & -U_L M_L^{-1} \\ 0 & 0 & I \end{pmatrix}. \quad (6)$$

Then by Sylvester's law of inertia (cf. [5, Section 5.5]), the spectrum of the matrix

$$AMA^T = \begin{pmatrix} M_C & U_C & 0 \\ U_C^T & \Pi - U_L M_L^{-1} U_L^T & 0 \\ 0 & 0 & M_L \end{pmatrix} \quad (7)$$

has the same signature as the spectrum of $M$; that is, $AMA^T$ has exactly one negative eigenvalue and has the same corank as $M$. Let $\Pi' = \Pi - U_L M_L^{-1} U_L^T$. 

As $M_L$ is positive definite, the matrix

$$M' := \begin{pmatrix} M_C & U_C \\ U_C^T & \Pi \end{pmatrix} \quad (8)$$

has exactly one negative eigenvalue and has the same corank as $M$. Since $(M_L)_{i,j} \leq 0$ if $i \neq j$, we know that $(M_L^{-1})_{i,j} \geq 0$ for all $i, j$. Indeed, for any symmetric positive-definite matrix $D$, if each off-diagonal entry of $D$ is nonpositive, then each entry of $D^{-1}$ is nonnegative. This can be seen directly, and also follows from the theory of “$M$-matrices” (cf. [5, Section 15.2]): Without loss of generality, each diagonal entry of $D$ is at most 1. Let $B := I - D$. So $B \geq 0$ and the largest eigenvalue of $B$ is equal to $1 - \lambda_1(D) < 1$. Hence $D^{-1} = I + B + B^2 + B^3 + \cdots \succeq 0$ (cf. Theorem 2 in Section 15.2 of [5]).

Hence, $(\Pi_i')_{i,j} \leq 0$ for each $i$ and $j$ with $i \neq j$. Thus $M'$ satisfies (2) with respect to $G'$.

The matrix $M'$ also fulfills the SAH. To see this, let $X'$ be a symmetric matrix with $M'X' = 0$ and $(X')_{i,j} = 0$ if $i$ and $j$ are adjacent or if $i = j$. As $S$ is a clique, we can write

$$X' = \begin{pmatrix} X_C \\ Y^T \end{pmatrix}.$$ \quad (9)

Let $Z := -YU_L^T M_L^{-1}$ and

$$X := \begin{pmatrix} X'_C & Y & Z \\ Y^T & 0 & 0 \\ Z^T & 0 & 0 \end{pmatrix}. \quad (10)$$

Then $X$ is a symmetric matrix with $(X)_{i,j} = 0$ if $i$ and $j$ are adjacent or if $i = j$, and $MX = 0$. So $X = 0$ and hence $X' = 0$.

It follows that $\mu(G') \geq \text{corank}(M') = \text{corank}(M) = \mu(G) > t$, a contradiction, since $G'$ is a subgraph of $G_1$ or $G_2$.

So we have that $\lambda_1(M_C) \geq 0$ for each component $C$ of $G - S$. Suppose next that $N(C) \neq S$ for some component $C$ of $G - S$.

Assume that $C \subseteq V G_1$. Let $H_1$ be the graph induced by $C \cup N(C)$ and let $H_2$ be the graph induced by the union of all other components and $S$. So
G is also a clique sum of $H_1$ and $H_2$, with common clique $S' := N(C)$, and $H_2$ is a clique sum of $G_1 - C$ and $G_2$.

If $\mu(G) = \mu(H_2)$, then $\mu(H_2) > t' := \max(\mu(G_1 - C), \mu(G_2))$. As $|VH_2| + |S| < |VG| + |S|$, by induction we know that $\mu(H_2) = t' + 1$, and thus $\mu(G) = \mu(H_2) = t' + 1 \leq t + 1$. Thus $t' = t$ and $\mu(G) = t + 1$. Moreover, either $|S| = t + 1$ and $H_2 - S$ has two components $C', C''$ with $N(C') = N(C'')$ and $|N(C')| = t$, or $|S| = t$ and $H_2 - S$ has three components $C$ with $N(C') = S$, and the theorem follows.

If $\mu(G) > \mu(H_2)$, then $\mu(G) > t' := \max(\mu(H_1), \mu(H_2))$. As $|VG| + |S'| < |VG| + |S|$, we know that $\mu(G) = t' + 1$, implying $t' > t$, and that either $|S'| = t' + 1$ or $|S'| = t'$. However, $|S'| < |S| \leq t + 1 \leq t' + 1$, so $|S'| = t$ and $t' = t$. Moreover, $G - S'$ has three components $C'$ with $N(C') = S'$. This implies that $G - S$ has two components $C'$ with $N(C') = S'$, and the theorem follows.

So we may assume that $N(C) = S$ for each component $C$. If $|S| > t$, then $G_1$ would contain a $K_{t+2}$ minor, contradicting the fact that $\mu(G_1) \leq t$. So $|S| \leq t$. Since corank$(M) > |S|$, we have $\lambda_i(M_C) = 0$ for at least one component $C$ of $G - S$. Hence, by (ii) of the lemma, $G - S$ has at least corank$(M) - |S| + 2 = \mu(G) - |S| + 2 \geq 3$ components $C$ with $\lambda_i(M_C) = 0$, and by (iii) of the lemma, $\mu(G) - |S| + 2 \leq 3$, that is, $t \geq |S| \geq \mu(G) - 1 \geq t$.

We give as direct consequences the following corollaries:

**Corollary 1.** Let $G$ be a clique sum of $G_1$ and $G_2$ and let $S := VG_1 \cap VG_2$. Then $\mu(G) = \max(\mu(G_1), \mu(G_2))$ if $\mu(G_1) \neq \mu(G_2)$, or $|S| < \mu(G_1)$, or $|S| = \mu(G_1)$ and $G - S$ has at most two components $C$ with $N(C) = S$.

**Corollary 2.** Let $G$ be a clique sum of $G_1$ and $G_2$ and let $t = \max(\mu(G_1), \mu(G_2))$. Then $\mu(G) = t$ if and only if $G$ does not have a $K_{t+3} \setminus \Delta$-minor.

We thank the referee for carefully reading the paper and for helpful suggestions.

**References**


Received 8 February 1995; final manuscript accepted 16 February 1995