Some degree bounds for the circumference of graphs
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Abstract

Let $C$ be a longest cycle in a connected graph $G$ and $L(G)$ the length of the longest path in $G$. Suppose $L(G - V(C)) \geq k - 1$, where $k \in \{3, 4, 5\}$. It is known that $c(G) = |C| \geq (k + 1)\delta - (k - 1)(k + 1)$ if $G$ is $(k + 1)$-connected, and $n = |V(G)| \geq (k + 1)\delta - k(k - 1)$ if $G$ is $k$-connected. In this paper the exceptional classes for these estimates, when the connectivity is reduced by one, are essentially determined.

Keywords: Circumference; Connectivity; Exceptional classes

1. Introduction

We consider only finite, undirected graphs $G = (V, E)$ of order $n$. The independence number and the connectivity of $G$ are denoted by $\alpha := \alpha(G)$ and $\kappa(G)$, respectively. For $x \geq k \geq 1$ let $\sigma_x = \min\{d(u_1) + \cdots + d(u_k) : \{u_1, \ldots, u_k\} \text{ is an independent set in } G\}$. Let $L(G)$ be the length of a longest path in $G$, and let $\delta = \sigma_1$. The length of a longest cycle in $G$ is called the circumference of $G$ and denoted by $c(G)$. A cycle $C$ in $G$ is called a $D_2$-cycle, if all components of $G - C := G - V(C)$ have fewer than $\lambda$ vertices.

The following result of Dirac [5] is well-known.

E-mail address: jung@math.tu-berlin.de (H. Jung).
Theorem 1 (Dirac [5]). Let $G$ be a 2-connected graph with minimum degree $\delta$. Then $G$ has a hamiltonian cycle or $c(G) \geq 2\delta$.

An extension of Dirac’s Theorem was given by Jung [12].

Theorem 2 (Jung [12]). Let $C$ be a longest cycle in a graph $G$ and $H$ a component of $G - C$ such that $L(H) \geq k - 1$ ($k \in \{1, 2, 3, 4, 5\}$). There exists a vertex $v$ in $H$ such that

(i) $|C| \geq (k + 1)d(v) - (k - 1)(k + 1)$, if $G$ is $(k + 1)$-connected;
(ii) $n \geq (k + 1)d(v) - k(k - 1) + 1$, if $G$ is $k$-connected.

Our goal in this paper is to extend Theorem 2 for $k \in \{3, 4, 5\}$ to graphs with the connectivity relaxed by one. We essentially determine the exceptional classes of graphs for the estimate $c(G) \geq (k + 1)\delta - c_k'$ for $k$-connected graphs $G$ with $k \in \{3, 4, 5\}$. Moreover, we essentially determine the corresponding “splitting structure” for $(k - 1)$-connected graphs with $n \geq (k + 1)\delta - c_k'$, where $c_k$ and $c_k'$ are constants ($k \in \{3, 4, 5\}$). Our main result is the following Theorem 3, the classes $\mathcal{G}_k$ and $\mathcal{G}_{k-1}'$ are defined below.

Theorem 3. Let $C$ be a longest cycle in a graph $G$ and let $L(G - C) \geq k - 1$, where $k \in \{3, 4, 5\}$. There exist non-adjacent vertices $v$ and $w$ in $G$ such that

(i) $|C| \geq (k - 1)d(v) + 2d(w) - (k - 1)(k + 1)$, if $G$ is $k$-connected and $G \notin \mathcal{G}_k$;
(ii) $n \geq (k - 1)d(v) + 2d(w) - k(k - 1) + 1$, if $G$ is $(k - 1)$-connected and $G \notin \mathcal{G}_{k-1}'$.

Obviously, $L(G - C) < 2$ just means that $C$ is a $D_3$-cycle. Thus, we obtain the following Corollary 1.

Corollary 1. If $G$ is a 2-connected graph with $n \leq 2\sigma_2 - 6$, then either every longest cycle of $G$ is a $D_3$-cycle or $G \in \mathcal{G}_2'$.

A graph $G$ is called 3-cyclable, if any three vertices of $G$ lie on a common cycle. Since the graphs in $\mathcal{G}_2'$ are not 3-cyclable we obtain the following:

Corollary 2. If $G$ is a 3-cyclable graph on $n \leq 2\sigma_2 - 6$ vertices, then every longest cycle is a $D_3$-cycle.

The $\delta$-version of Theorem 3(ii) with $k = 3$ was announced by Jung in the workshop on hamiltonian graph theory at the University of Twente in 1992. In 1995, a proof was given by Brandt (see [3]).

Before giving the definitions of the classes $\mathcal{G}_k$ and $\mathcal{G}_{k-1}'$ we list some known results, which are related to Theorem 3(ii) with $k = 3$.

Theorem 4 (Nash-Williams [17]). If $G$ is a 2-connected graph with $n \leq 3\delta - 2$ and $x \leq \delta$, then $G$ is hamiltonian.

The following Theorem 5 is implicit in Nash-Williams’ proof of Theorem 4.
Theorem 5. If $G$ is a 2-connected graph with $n \leq 3\delta - 2$, then every longest cycle of $G$ is a $D_2$-cycle.

From the definition of the class $\mathcal{G}'_2$ it will become clear that Theorem 6 of Veldman is a consequence of Corollary 1. As already noted by Veldman in [20], if $G$ has a $D_3$-cycle, then either $G$ is hamiltonian or $\alpha \geq \delta$. Hence Theorem 7 of Veldman is a consequence of Theorem 6.

Theorem 6 (Veldman [20]). If $G$ is a 2-connected graph with $n \leq 4\delta - 6$, then $G$ contains a $D_3$-cycle or $G \in \mathcal{G}'_2$.

Theorem 7 (Veldman [20]). If $G$ is a 2-connected graph with $n \leq 4\delta - 6$ and $\alpha \leq \delta - 1$, then $G$ is hamiltonian or $G \in \mathcal{G}'_2$.

Theorem 7 was extended by Trommel [18]. He obtained Theorem 8. As noted by Trommel in [18], the proof of Theorem 8 can be considerably shortened by using Theorem 3(ii) ($k = 3$).

Theorem 8 (Trommel [18]). If $G$ is a 2-connected graph with $n \leq 4\delta - 6$, then $G$ contains a cycle of length at least $\min\{n, n + 2\delta - 2\alpha - 2\}$ or $G \in \mathcal{G}'_2$.

Now we give the definitions of the exceptional classes $\mathcal{G}_k$ and $\mathcal{G}'_{k-1}$.

For a subgraph $H$ of $G$ let $N(H)$ denote the set of all vertices in $G - V(H)$ that are adjacent to some vertex in $H$. A connected subgraph $H$ of $G$ is called normally linked in $G$, if $|H| = 1$ or $|(N(x) \cup N(y)) \cap H| \geq 2$ for any distinct vertices $x, y$ of $N(H)$. We call $H$ strongly linked in $G$, if moreover $H$ is hamilton-connected.

Let $C$ be a cycle in a 2-connected graph $G$ and let $S$ be a non-empty subset of $V(C)$. We say that $S$ splits $C$, if $C - S$ has $|S|$ components $C_1, \ldots, C_{|S|}$ and each $V(C_i)$ spans a component of $G - S$. If $S_1, S_2$ split $C$ and $|S_1| = \kappa(G)$, then clearly $S_1 \subseteq S_2$. By definition a graph $G$ belongs to the class $\mathcal{G}_k(\mathcal{G}'_1)$, if there exists a set $S \subseteq V(G)$ of cardinality $k := \kappa(G) \geq 3$ which splits every longest cycle in $G$ and all components of $G - S$ are strongly linked in $G$ (and $\omega(G - S) = \kappa(G) + 1$), where $\omega(G - S)$ is the number of components of $G - S$. As just noted the set $S$ in this definition is uniquely determined.

The class of graphs which are not 3-cyclable was characterized by Watkins and Mesner (see [21]). Classes $\mathcal{H}_{1,1}, \mathcal{H}_{1,3}$ and $\mathcal{H}_{3,3}$ are subclasses of the three corresponding classes in that characterization. We say that $G$ is in $\mathcal{H}_{1,1}$, if there is a 2-vertex cut $S = \{x_1, x_2\}$ in $G$ such that $\omega(G - S) = 3$ and all three components of $G - S$ are strongly linked in $G$. By definition $G$ is in $\mathcal{H}_{1,3}$, if there exist vertex-disjoint connected graphs $G_1, G_2, G_3$ and a 4-element set $S = \{x_1, x_2, x_3, y\}$ in $G$ such that $G - S = G_1 \cup G_2 \cup G_3$ and $G_1, G_2, G_3$ are strongly linked in $G$, furthermore $N(G_i) = \{x_i, y\}$ ($i = 1, 2, 3$) and $\{x_1, x_2, x_3\}$ spans a triangle in $G$. By definition $G$ is in $\mathcal{H}_{3,3}$, if there exist vertex-disjoint connected graphs $G_1, G_2, G_3$ and a 6-element set $S = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ in $G$ such that $G - S = G_1 \cup G_2 \cup G_3$ and $G_1, G_2, G_3$ are strongly linked in $G$, furthermore $N(G_i) = \{x_i, y_i\}$ ($i = 1, 2, 3$) and both $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ span triangles in $G$. Set $\mathcal{G}'_2 = \mathcal{H}_{1,1} \cup \mathcal{H}_{1,3} \cup \mathcal{H}_{3,3}$. It is easy to see that the set $S$ in the definition of $\mathcal{G}'_2$ is uniquely determined. The graphs in $\mathcal{G}'_2$ are depicted in Fig. 1.
On the way to the proof of Theorem 3 we encounter classes of graphs, in which better bounds are available. In most cases these bounds are sharp. In the last section we tie these intermediate results together to finish the proof of Theorem 3. We end the last section with some miscellaneous comments.

2. Preliminaries

For a path $P$ with end vertices $x$ and $y$ we write $P = P[x, y]$ and call $P$ an $(x, y)$-path. The undirected graph corresponding to $P$ will occasionally be identified with $P$. Given a cycle $C$ with a fixed cyclic orientation and vertices $x, y \in V(C)$, we use $C[x, y], C[x, y]$ and $C(x, y)$ to denote the corresponding subpaths of $C$. A path $Q$, which has its end vertices on $C$ and is openly disjoint from $C$, is called a $C$-chord. For $x \in V(C)$ let $x^+$ and $x^−$ denote the successor and the predecessor of $x$, respectively, according to the given orientation of $C$. For $Z \subseteq V(C)$ we set $Z^+ = \{z^+ | z \in Z\}$ and $Z^- = \{z^- | z \in Z\}$. As usual we call a non-trivial connected graph separable, if it has a cut vertex. For terminology and notation not defined here see [2].

For $H, K \subseteq V$ we use the abbreviation $N_K(H) = N(H) \cap K$. In particular, $N_K(v) = N(v) \cap K$. We set $d_K(v) := |N_K(v)|$ for $v \in V(G)$. For edge-disjoint subgraphs $H, K$ of $G$ let $e(H; K)$ be the number of edges between $H$ and $K$. If $H = \{v_1, \ldots, v_s\}$, we write $e(v_1, \ldots, v_s; K)$ instead of $e(\{v_1, \ldots, v_s\}; K)$. For vertices $a$ and $b$ in a connected graph $G$, we denote by $L_G(a, b)$ the length of a longest $(a, b)$-path in $G$. If $G$ is not separable and $|G| \geq 2$, we set $D(G) = \min\{L_G(a, b) : a, b \in V(G), a \neq b\}$. For $|G| = 1$ we set $D(G) = 0$.

Now we supply some preliminary results.

Let $K_4^−$ be the graph obtained from $K_4$ by deleting one edge. The following lemma is due to Jung.

Lemma 1 (Jung [12]). Let $H$ be a 2-connected graph. There exist distinct vertices $v_1, v_2$ and $v_3$ in $H$ such that

(i) $D(H) \geq d_H(v_i)$ for $i = 1, 2$ and $L_H(v_1, v_2) \geq d_H(v_3)$;
(ii) $D(H) \geq d_H(v_3) - 1$ with strict inequality unless $H = K_4^−$. 
We also need the following result of Enomoto.

**Theorem 9** (Enomoto [6]). Let \( H \) be a 3-connected graph which is not hamilton-connected. There exist non-adjacent vertices \( v_1, v_2 \) in \( H \) such that \( D(H) \geq d_H(v_1) + d_H(v_2) - 2 \).

A standard tool for estimating the length of a longest cycle \( C \) is the following “Chord-Lemma”, which is an easy consequence of the fact that \( C \) is a longest cycle in \( G \).

**Lemma 2 (Chord-Lemma).** Let \( C \) be a longest cycle in \( G \) with a fixed cyclic orientation.

(i) If \( Q \) is a \( C \)-chord with end vertices \( x, x' \) on \( C \), then \( |C(x, x')| \geq |Q| - 2 \);

(ii) if \( Q \) and \( R \) are disjoint crossing \( C \)-chords in \( G \) with end vertices \( x, x' \) and \( y, y' \), respectively, on \( C \), then \( |C(x, y)| + |C(x', y')| \geq |Q| + |R| - 4 \).

In the following lemma we consider a 2-connected component \( H \) of \( G - C \) with small \( D(H) \), where a longest cycle in \( G \).

**Lemma 3.** Let \( C \) be a longest cycle in the 2-connected graph \( G \), and let \( H \) be a 2-connected component of \( G - C \) such that \( L(H) > D(H) \) and \( D(H) \leq 3 \). There exist non-adjacent vertices \( v \) and \( w \) in \( H \) such that

(i) \( |C| \geq 3d(v) + 2d(w) - 10 \), if \( D(H) = 2 \) and \( H \neq K_4^- \);

(ii) \( |C| \geq 4d(v) + 2d(w) - 18 \), if \( |H| > 4 \).

**Proof.** Determine vertices \( a, b \) in \( V(H) \) such that \( D(H) = L_H(a, b) \). Label \( N(H) = \{x_1, \ldots, x_s\} \) in order around \( C \) and set \( x_{s+1} = x_1 \).

**Case 1: \( D(H) = 2 \).**

Clearly, \( V(H) - \{a, b\} \) is an independent set and hence \( d_H(v) = 2 \) for all \( v \in V(H) - \{a, b\} \). First, assume \( |H| \geq 5 \) and pick distinct vertices \( v_1, v_2, v_3 \) in \( V(H) - \{a, b\} \). Observe that \( L_H(v_p, v_q) = 4 \) for \( 1 \leq p < q \leq 3 \). If \( e(v_1, v_2, v_3; x_i, x_{i+1}) \geq 3 \), then the Chord-Lemma yields \( |C(x_i, x_{i+1})| \geq 6 \). Since \( |C(x_i, x_{i+1})| \geq 2 \) for all \( x_i \in N(H) \) we have \( |C(x_i, x_{i+1})| \geq e(v_1, v_2, v_3; x_i, x_{i+1}) \) for all \( x_i \in N(H) \). Summing this over \( N(H) \) we obtain \( |C| \geq 2d_C(v_1) + 2d_C(v_2) + 2d_C(v_3) \geq 2d(v_1) + 2d(v_2) + 2d(v_3) - 12 \). Thus, it remains the subcase when \( H \) is a 4-cycle. We then can choose \( v_1, v_2 \) and \( b \) in \( H \) such that \( v_1, v_2 \) are not adjacent and \( d(b) \geq d(v_1) \). It readily follows by an application of the Chord-Lemma that \( C(x_i, x_{i+1}) \geq e(v_1, v_2; x_i, x_{i+1}) \) and \( e(b; x_{i+1}) \) for all \( x_i \). Summation of these inequalities yields \( |C| \geq 2d(v_1) + 2d(v_2) + d(b) - 10 \geq 3d(v_1) + 2d(v_2) - 10 \). This settles Case 1.

**Case 2: \( D(H) = 3 \).**

It is easy to see that the components \( T_1, \ldots, T_r \) of \( H - \{a, b\} \) are trees. Furthermore, \( L(T_p) \leq 2 \) for \( 1 \leq p \leq r \). Let \( |T_1| > \cdots > |T_r| \). Note that \( |T_1| \geq 2 \) since \( D(H) = 3 \). Pick distinct end vertices \( v_1, v_2 \) of \( T_1 \) and an end vertex \( v_3 \) of \( T_2 \). Observe that \( L_H(v_1, v_2) \geq 4 \) and \( L_H(v_p, v_q) \geq 3 \) for \( p = 1, 2 \). Now consider \( x_i \in N(H) \) such that \( e(v_1, v_2, v_3; x_i, x_{i+1}) > 2 \). By the Chord-Lemma we obtain \( |C(x_i, x_{i+1})| \geq 5 \) with strict inequality unless \( e(v_1, v_2; x_i, x_{i+1}) \leq 2 \). This shows that \( |C(x_i, x_{i+1})| \geq e(v_1, v_2, v_3; x_i, x_{i+1}) \) holds for all \( x_i \). Taking the sum we obtain \( |C| \geq 2d_C(v_1) + 2d_C(v_2) + 2d_C(v_3) \geq 2d(v_1) + 2d(v_2) + 2d(v_3) - 18 \).
Lemma 4. Let $C$ be a longest cycle in the 2-connected graph $G$, and let $H$ be a separable component of $G - C$. Suppose $L(B) \geq 3$ for some end block $B$ of $H$. There exist non-adjacent vertices $v$ and $w$ in $H$ such that

(i) $|C| \geq d(v) + d(w) + 2$ and $|C \cup H| \geq 2d(v) + 2d(w) - 4$;
(ii) $|C| \geq 2d(v) + 2d(w) - 4$ and $|C \cup H| \geq 3d(v) + 2d(w) - 4$, if $G$ is 3-connected;
(iii) $|C| \geq 4d(v) + 2d(w) - 14$, if $G$ is 4-connected.

Proof. Let $B^*$ be another end block of $H$. Let $c$ and $c^*$ denote the unique cut vertices of $H$ in $B$ and $B^*$, respectively. We label $N_C(B - c) \cup N_C(B^* - c^*) = \{z_1, \ldots, z_r\}$ around $C$.

Using Lemma 1 we determine a vertex $w \in B^* - c^*$ such that $D^* := D(B^*) \geq d_H(w)$. We abbreviate $D := D(B)$ and $\hat{D} := \max\{3, D\}$. Next, we determine vertices $v_1$ and $v_2$ according to the following:

Claim 1. There exist distinct vertices $v_1, v_2 \in B - c$ such that $L_H(v_1, c) \geq 3$ (h = 1, 2), $D \geq d_H(v_1)$ and $\hat{D} \geq d_H(v_2)$.

This is an immediate consequence of Lemma 1, if $D \geq 3$. Next assume $D = 2$ and pick vertices $a, b$ in $B$ such that $L_H(a, b) = 2$. As noted above $N(H) - \{a, b\}$ is an independent set. Hence in the proof of Claim 1 it remains the case when $|B| = 4$ and $B \neq K_4$. In the subcase, when $B$ is a cycle, let $v_1, v_2$ be the two neighbors of $c$ in $B$. In the remaining subcase we have $B = K_4$ and may assume $d_B(c) = 2$. In this event, we pick $v_1, v_2 \in B - c$ such that $d_H(v_1) = 2 = d_H(v_2) - 1$. This settles Claim 1.

In the case, when $r = 1$, we have $\kappa(G) = 2$ and $N(H) - \{z_1\} \neq \emptyset$. By the Chord-Lemma $|C| \geq (D + 2) + (D^* + 2)$, consequently $|C| \geq d(v_1) + d(v_2) + 2$ and $|C \cup H| \geq 2D + 2D^* + 5 \geq 2d(v_1) + 2d(v_2) + 1$.

In the rest of this proof we assume $r \geq 2$.

We call a segment $C[z_i, z_{i+1}]$ useful, if $z_i \in N(B - c)$ and $z_{i+1} \in N(B^* - c^*)$ or vice versa. For such a segment we have $|C(z_i, z_{i+1})| \geq D + D^* + 2$ by the Chord-Lemma. Let $m$ denote the number of useful segments and let $m_1$ denote the number of segments $C[z_i, z_{i+1}]$ such that $|C(z_i, z_{i+1})| \geq D + 2$ and $z_i, z_{i+1} \notin N(B^* - c^*)$. Note that $m \geq m_1$ since $r \geq 2$.

In the following we use the abbreviations $e_i = e(v_1, w; z_i, z_{i+1})$, $f_i = e(v_1, w; z_i, z_{i+1}) + e(v_2; z_i)$ and $g_i = e(v_1, v_2, w; z_i, z_{i+1})$.

Claim 2. $|C(z_i, z_{i+1})| \geq g_i$ for all segments $C[z_i, z_{i+1}]$.

If $z_i \in N(v_1) \cup N(v_2)$ and $z_{i+1} \in N(B^* - c^*)$, then $|C(z_i, z_{i+1})| \geq \hat{D} + D^* + 2$ by the Chord-Lemma. Now assume $z_i, z_{i+1} \notin N(B^* - c^*)$ and $z_i \notin N(v_1) \cup N(v_2)$. We may moreover assume $z_{i+1} \in N(v_1) \cup N(v_2)$ and $z_i \notin N(B^* - c^*)$. Then $g_i \leq 3$. If $|N_H(z_i) \cup N_H(z_{i+1})| \geq 2$, clearly $|C(z_i, z_{i+1})| \geq 3$. This settles Claim 2.

Claim 3. If $C[z_i, z_{i+1}]$ is a useful segment, then $|C(z_i, z_{i+1})| \geq \hat{D} + D^* - 2 + \max\{e_i, f_i - 1, g_i - 2\}$. If $|C(z_i, z_{i+1})| \geq D + 2$ and $z_i, z_{i+1} \notin N(B^* - c^*)$, then $|C(z_i, z_{i+1})| \geq D - 1 + \max\{f_i, g_i - 1\}$. 
If \(|C(z_i, z_{i+1})| \geq D + D* + 2\) and \(e_i \geq 3\), then moreover \(|C(z_i, z_{i+1})| \geq L_H(v_h, c) + D* + 2\) for \(h = 1\) or 2. If \(e_i = 2\), then \(f_i \leq 3\) and \(g_i \leq 4\). This settles the first part of Claim 3. The second part holds by definition.

**Claim 4.** If \(G\) is 4-connected, then \(m + m_1 \geq 4\) or \(|N_C(B - c) \cup N_C(B^* - c^*)| = 3\).

There exist three disjoint edges from \(B - c\) to vertices \(x_1, x_2\) and \(x_3\) on \(C\). Let \(x_2 \in C(x_1, x_3)\). If \(N_C(B^* - c^*) \subseteq \{x_1, x_2, x_3\}\), then clearly \(m \geq 4\) or \(N_C(B^* - c^*) = \{x_1, x_2, x_3\}\). Now assume \(N(B^* - c^*) \cap C(x_1, x_2) \neq \emptyset\). Then \(C[x_1, x_2]\) contains at least two useful segments. We thus may in addition assume \(N(B^* - c^*) \cap C(x_2, x_1) = \emptyset\). If \(x_2 \in N(B^* - c^*)\) (or \(x_1 \in N(B^* - c^*)\)), then \([x_2, x_3]\) (or \([x_3, x_1]\)) contains a useful segment. This settles Claim 4.

We now are ready for the proof of the stipulated estimates. Observe that \(\sum e_i = 2d_C(v_1) + 2d_C(w)\). Since \(m \geq 2\) we obtain \(|C| \geq 2d_C(v_1) + 2d_C(w) + 2(\hat{D} + D - 2) \geq 2d(v_1) + 2d(w) - 4\). Similarly,

\[|C \cup H| \geq 2d_C(v_1) + 2d_C(v_2) + 2d_C(w) + 2(\hat{D} + D - 3) + |H| \geq 2d(v_1) + d(v_2) + 2d(w) - 5 + D^*.

Hence it remains the case when \(G\) is 4-connected. Next assume \(m + m_1 \leq 3\). By Claim 4 we have \(N_C(B - c) = N_C(B^* - c^*) = \{z_1, z_2, z_3\}\), consequently \(\kappa(G) = 4\) and \(|N(H) - \{z_1, z_2, z_3\}| \neq \emptyset\). By the Chord-Lemma we obtain

\[|C| \geq 2(D + D^* + 2) + 2(D + 2) \geq 2d_H(v_1) + 2(d_H(v_2) - 1) + 2d_H(w) + 8 \geq 2d(v_1) + 2d(v_2) + 2d(w) + 8 - 18.

Finally, let \(m + m_1 \geq 4\). By Claim 3 we obtain

\[|C| \geq 2d_C(v_1) + 2d_C(v_2) + 2d_C(w) + m(\hat{D} + D^* - 2) + m_1(D - 2) \geq 2d_C(v_1) + 2d_C(v_2) + 2d_C(w) + 2\hat{D} + 2D + 2D^* - 14 \geq 2d(v_1) + 2d(v_2) + 2d(w) - 14. \quad \square

The following result is a variant of Theorem 3.4 in [12]. The proof is along similar lines but considerably shorter.

**Theorem 10.** Let \(C\) be a longest cycle in \(G\), and let \(H\) be a separable component of \(G - C\). There exist non-adjacent vertices \(v\) and \(w\) in \(H\) such that

(i) \(|C| \geq d(v) + d(w) + 2 \text{ and } |C \cup H| \geq 2d(v) + 2d(w) - 4\), if \(G\) is 2-connected;
(ii) \(|C| \geq 2d(v) + 2d(w) - 4 \text{ and } |C \cup H| \geq 3d(v) + 2d(w) - 4\), if \(G\) is 3-connected and \(|L(H)| \geq 3\);
(iii) \(|C| \geq 4d(v) + 2d(w) - 14\), if \(G\) is 4-connected and \(|L(H)| \geq 4\).

**Proof.** By the previous lemma we may assume \(|L(B)| \leq 2\) (that is \(|B| \leq 3\) for all end blocks \(B\) of \(H\). Again we consider distinct end blocks \(B\) and \(B^*\) of \(H\). We use the notation of the
previous proof. In particular let $V(B) = \{v_1, v_2, c\}$, if $|B| = 3$, and otherwise $V(B) = \{v_1, c\}$. Let $w$ be a vertex of $B^* - c^*$.

In the proof of Lemma 4(i) the condition $L(B) \geq 3$ was not used. Thus, we are left with the proof of (ii) and (iii). Now let $G$ be 3-connected and $L(H) \geq 3$.

Case 1: $|B| = 3$.

First assume $L_H(c, c^*) + |B^*| - 1 \geq 3$. Then $L_H(v_h, w) \geq 4$ for $h = 1, 2$. As in the preceding proof it readily follows (by the Chord-Lemma) that $|C(z_i, z_{i+1})| \geq e(v_1, v_2, w; z_i, z_{i+1})$ for all segments $C[z_i, z_{i+1}]$. By summation we obtain $|C| \geq 2d_C(v_1) + 2d_C(v_2) + 2d_C(w) \geq 2d(v_1) + 2d(v_2) + 2d(w) - 12$.

Also $|C| \geq 2d_C(v_1) + d_C(v_2) + 2d_C(w) \geq 2d(v_1) + d(v_2) + 2d(w) - 10$. Moreover, $|H| \geq 5$ or $d_C(w) = 1$, consequently $|C \cup H| \geq 2d(v_1) + d(v_2) + 2d(w) - 4$. This settles Case 1.

In the rest of this proof we assume that the end blocks of $H$ have only two vertices. We abbreviate $V_0 = V(H) - V(B \cup B^*)$.

Case 2: $V_0 = \emptyset$ or $d_H(v_2) \leq 3$ for some vertex $v_2$ of $V_0$.

If $V_0 = \emptyset$, then $|H| = 4 = L(H) + 1$. In this event again, $|C| \geq 2d_C(v_1) + 2d_C(c) + 2d_C(w) \geq 2d(v_1) + 2d(c) + 2d(w) - 8$. In the remaining subcase we have $L_H(v_1, w) \geq 4$ and $|C(z_i, z_{i+1})| \geq e(v_1, v_2, w; z_i, z_{i+1})$ for all segments $C[z_i, z_{i+1}]$. Now $|C| \geq 2d_C(v_1) + 2d_C(v_2) + 2d_C(w) \geq 2d(v_1) + 2d(v_2) + 2d(w) - 10$. Furthermore, $|H| \geq 6$ or $d_H(v_2) \leq 2$. Hence again $|C \cup H| \geq 2d(v_1) + 2d(v_2) + 2d(w) - 4$. This settles Case 2.

Case 3: $V_0 \neq \emptyset$ and $d_H(v) \geq 4$ for all $v \in V_0$.

Now $B$ and $B^*$ are the only end blocks of $H$. We pick a path $Q$ in $H$ with end vertices $v_1$ and $w$. Since $B$ and $B^*$ are the only end blocks of $H$, it follows that all cut vertices of $H$ are on $Q$. Furthermore, each block of $H$ other than $B, B^*$ contains exactly two cut vertices of $H$. At least one of them, say $B_2$, has more than two vertices. Let $c_2, c_2'$ be the two cut vertices of $H$ in $B_2$, where $c_2'$ is on $Q(c_2, c^*)$. Since the vertices in $B_2 - c_2 - c_2'$ have degree at least four in $B_2 - c_2 - c_2'$, we obtain $D(B_2 - c_2 - c_2') \geq 3$. Using Lemma 1((i) and (ii)) we determine a vertex $v_2$ in $B_2 - c_2 - c_2'$ such that $D(B_2) \geq d_H(v_2)$. We label $N_C(v_1) \cup N_C(w) \cup N_C(B_2 - c_2 - c_2') = \{y_1, \ldots, y_k\}$ around $C$. Again $|C(y_1, y_{i+1})| \geq e(v_1, v_2, w; y_i, y_{i+1})$ for all segments $C[y_i, y_{i+1}]$. Since $G$ is 3-connected, there exist distinct vertices $y_p \in N(v_1)$ and $y_q \in N(w)$. We can choose $y_p$ so that in addition $y_p$ and $y_{p+1}$ have distinct neighbors in $\{v_1, w\} \cup (B_2 - c_2 - c_2')$. Then $|C(y_p, y_{p+1})| \geq D(B_2) + 3$ by the Chord-Lemma. This shows that $|C(y_1, y_{i+1})| \geq D(B_2) + 3$ for at least two segments $C[y_i, y_{i+1}]$ on $C$. Thus, $|C| \geq 2d_C(v_1) + 2d_C(v_2) + 2d_C(w) + 2D(B_2) - 3 \geq 2d(v_1) + 2d(v_2) + 2d(w) - 10$. This settles Case 3. □

Theorem 10 clearly covers the case of Theorem 3, when $L(H) \geq k - 1$ for some separable component $H$ of $G - C$. In the following Sections 3 and 4 we investigate two main cases pertaining to the proof of Theorem 3.

3. $H$ is not normally linked in $G$ or $\omega(G - C) \geq 2$

In this section we consider a longest cycle in a 2-connected graph $G$. We first investigate the situation, when two subsequent vertices on $C$ have neighbors outside $C$. 

Lemma 5. Let $H$ be a 2-connected component of $G - C$ such that $L(H) \geq k - 1$ ($k \in \{3, 4, 5\}$). Let $x$ be an element of $N(H)$ and $K$ be a component of $G - C$ such that $x^+ \in N(K)$. There exist vertices $v$ in $H$ and $w$ in $K$ such that

(i) $|C| \geq (k - 1)d(w) + 2d(w) - k(k - 2)$, if $G$ is $k$-connected;

(ii) $|C \cup H \cup K| \geq (k - 1)d(w) + 2d(w) - k(k - 3)$, if $G$ is $(k - 1)$-connected.

Proof. For separable $K$ and end blocks $B$ of $K$ let $c_B$ denote the unique cut vertex of $K$ in $V(B)$. If $K$ is separable, then we use Lemma 1 to determine an end block $B$ of $K$ and a vertex $w$ of $B - c_B$ such that $D(B) \geq d_B(w) = d_K(w)$ and $N_K(x^+) \neq \{w\}$. If $K$ is not separable, then we set $B = K$ and use Lemma 1 to determine a vertex $w$ of $K$ such that $D(K) \geq d_K(w)$.

If in addition $|K| \geq 2$, then we can choose $w$ so that $N(x^+) \neq \{w\}$. Abbreviate $D^* = D(B)$.

In the case, when $H \neq K^-$, we use Lemma 1(i) and (ii)) to determine distinct vertices $v_1$ and $v_2$ in $H$ such that $D := D(H) \geq d_H(v_h)$ ($h = 1, 2$) and $N_H(x) \neq \{v_h\}$ ($h = 1, 2$).

In the case, when $H = K^-$, we set $D = 3$ and choose distinct vertices $v_1$ and $v_2$ in $H$ such that $D(H) \geq d_H(v_1)$ and $N_H(x) \neq \{v_h\}$ for $h = 1, 2$. If $N_H(x) = \{v_0\}$, then we set $X = \{x\} \cup N_C(H - v_0)$. If $|N_H(x)| \geq 2$, then we set $X = N(H)$. If $|K| \geq 2$ and $N_K(x^+) = \{w_0\}$, then we set $Y = \{x^+\} \cup N_C(K - w_0)$, otherwise $Y = N(K)$. In the case, when $K$ is separable, we set $W = V(B - c_B)$, and otherwise $W = V(K)$.

We label $X = \{x_1, \ldots, x_i\}$ in order around $C$ so that $x_1 = x$, where the subscripts are taken modulo $s$. For $1 \leq i \leq s$ let $t_i = |Y \cap C(x_i, x_{i+1})|$. Furthermore, if $Y \cap C(x_i, x_{i+1}) \neq \emptyset$, then we denote by $z_i$ the first element of $Y$ on $C(x_i, x_{i+1})$.

Let $\mathcal{S}$ denote the set of segments $C[x_i, x_{i+1}]$ such that $x_i \neq x_1$ and $Y \cap C(x_i, x_{i+1}) \neq \emptyset$.

Let $\mathcal{S}_0$ denote the set of segments $C[x_i, x_{i+1}]$ such that $|N_H(x_i) \cup N_H(x_{i+1})| \geq 2$ and $Y \cap C(x_i, x_{i+1}) = \emptyset$. Abbreviate $|\mathcal{S}| = m$, $|\mathcal{S}_0| = m_0$ and $m^* = m + \sum_{i=1}^s m_i$.

Claim 1. $m^* \geq 1$ with strict inequality if $\kappa(G) \geq 3$.

Suppose $m^* \leq 1$. First assume $Y \cap C(x_j, x_{j+1}) \neq \emptyset$ for some $x_j \in X - \{x_1\}$. Then $m = 1$ and $\sum_{i=1}^s m_i = 0$. By construction $Y \cap C(x_j, x_{j+1}) \subseteq C(x_j, x_{j+1})$ and all edges from $W$ to $C(x_j, x_{j+1})$ have a common end vertex in $W \cup C(x_j, x_{j+1})$. Also all edges from $W$ to $C(x_1, x_2)$ have a common end vertex in $W \cup C(x_1, x_2)$. Hence $\kappa(G) = 2$. Now let $Y \subseteq C[x_1, x_2]$. Clearly $m_1 \geq 1$ and all edges from $W$ to $C(x_1, x_2) \cup C(x_1, x_2)$ have a common end vertex in $W \cup C(x_1, x_2)$. This settles Claim 1.

Claim 2. Let $G$ be $p$-connected and $p \leq |H|$. Then $m + m_0 \geq p - 1$.

By Menger’s Theorem there exist $p$ distinct edges from $H$ to $C[x_2, x_1]$. This readily yields Claim 2.

In the following we use the parameter $l$, where $l \in \{0, 1, 2\}$, and abbreviate $e_l^{(i)} = e(v_1; x_i, x_{i+1}) + 2e(w; C(x_i, x_{i+1})) + le(v_2; x_i, x_{i+1})$ for $x_i \in X$.

Claim 3. If $C[x_i, x_{i+1}] \in \mathcal{S}$, then $|C(x_i, x_{i+1})| = e_l^{(i)} - 1 + D + (m_i + 1)D^* + \gamma_i^{(l)}$ with $\gamma_i^{(l)} \geq 0$ defined by this equation.
In the case, when $H \neq K_4^-$, this is an immediate consequence of $|C(x_i, z_i)| \geq D(H) + D^* + 2$. If $H = K_4^-$ and $z_i \in N(v_1)$, then the claim follows from the fact that $L_H(v_1, v) \geq 3$ for all $v \in V(H) - \{v_1\}$.

**Claim 4.** If $C[x_i, x_{i+1}] \in \mathcal{S}_0$, then $|C(x_i, x_{i+1})| = e_i^{(l)} - l + D + \gamma_i^{(l)}$ with $\gamma_i^{(l)} \geq 0$ defined by this equation.

Indeed $|C(x_i, x_{i+1})| \geq D(H) + 2$. If $H = K_4^-$, then again $|C(x_i, x_{i+1})| \geq D + 2$ or $x_i \notin N(v_1)$.

**Claim 5.** $|C(x_1, x_2)| = e_1^{(l)} - l - 3 + m_1 D^* + \gamma_1^{(l)}$ with $\gamma_1^{(l)} \geq 0$ defined by this equation.

This is an immediate consequence of $|C(x_1, x_2)| \geq 2t_1 - 1 + m_1 D^*$.

**Claim 6.** $\gamma_1^{(l)} + \gamma_2^{(l)} \geq 2$.

By construction $C[x_i, x_{i+1}] = C[x_i, x_1] \in \mathcal{S} \cup \mathcal{S}_0$ and $x_1 \notin N(K)$. Hence $\gamma_i^{(l)} \geq 2$, if $C[x_i, x_1] \in \mathcal{S}_0$ or $x_i \notin N(w)$. Now let $x_i \in N(w)$. Then $|K| = 1$ by construction and the Chord-Lemma. Again by the Chord-Lemma we have $|C(z_i', x_2)| \geq D(H) + 3$, where $z_i'$ is the last element of $N(w)$ on $C(x_1, x_2)$. This settles Claim 6.

For the remaining segments $C[x_i, x_{i+1}]$ we have $Y \cap C(x_i, x_{i+1}) = \emptyset$ and $|N_H(x_i) \cup N_H(x_{i+1})| = 1$, consequently $e_i^{(l)} \leq 2$ by construction and the Chord-Lemma.

By Claim 1, summation of the preceding estimates (with $l = 0$) yields $|C| \geq 2d_C(v_1) + 2d_C(w) + D + D^* - 2$. Hence $|C \cup H \cup K| \geq 2d_C(v_1) + 2d_C(w) + 2D + 2D^* \geq 2d(v_1) + 2d(w)$.

If $\kappa(G) \geq 3$, then $|C| \geq 2d_C(v_1) + 2d_C(w) + 2D + 2D^* - 3 \geq 2d(v_1) + 2d(w) - 3$. If moreover $L(H) \geq 3$, then summation of the preceding estimates with $l = 1$ gives rise to $|C \cup H \cup K| \geq 2d_C(v_1) + 2d_C(w) + d_C(v_2) + 3D + 2D^* - 4 \geq 2d(v_1) + 2d(w) + d(v_2) - 4$.

Now let $\kappa(G) \geq 4$. If $L(H) \geq 3$, then by the preceding claims with $l = 1$ we have $|C| \geq 2d_C(v_1) + 2d_C(w) + d_C(v_2) + 3D + 2D^* - 8 \geq 2d(v_1) + 2d(w) + d(v_2) - 8$.

If $L(H) \geq 4$, then as above (with $l = 2$) we obtain $|C \cup H \cup K| \geq 2d_C(v_1) + 2d_C(w) + 2d_C(v_2) + 4D + 3D^* - 10 \geq 2d(v_1) + 2d(w) + 2d(v_2) - 10$.

If finally $\kappa(G) \geq 5$ and $L(H) \geq 4$, then $|C| \geq 2d(v_1) + 2d(v_2) + 2d(w) - 15$. This completes the proof of Lemma 5. \qed

**Lemma 6.** Let $H$ be a 2-connected component of $G - C$, which is not normally linked in $G$. Let $L(H) \geq k - 1$ ($k \in \{3, 4, 5\}$). There exist non-adjacent vertices $u$ and $w$ in $G$ such that

(i) $|C| \geq (k - 1)d(u) + 2d(w) - (k - 1)^2$, if $G$ is $k$-connected;
(ii) $n \geq (k - 1)d(u) + 2d(w) - (k - 1)(k - 2) + 1$, if $G$ is $(k - 1)$-connected.

**Proof.** Using the assumption that $H$ is not normally linked in $G$ we determine distinct elements $z_1, z_2$ of $N(H)$ such that $N_H(z_1) \cup N_H(z_2) = \{y\}$. Label $X = \{z_1, z_2\} \cup N_C(H - y) = \{x_1, \ldots, x_t\}$ according to the given orientation on $C$. Let $\{z_1, z_2\} = \{x_p, x_q\}$.

By Lemma 5 it remains the case, when $d_C(x_i^+) = d_C(x_i^-)$ for all $x_i \in X$. If $D(H) \geq 4$ or $L(H) = D(H)$, then we use Lemma 1((i) and (ii)) to determine
distinct vertices $v_1$ and $v_2$ in $H - y$ such that \( D := D(H) \geq d_H(v_h) \) \((h = 1, 2)\). Otherwise we determine $v_1$ and $v_2$ according to the following:

**Claim 1.** If $L(H) > D(H)$ and $D(H) \leq 3$, then there exist vertices $v_1$ and $v_2$ in $H - y$ such that \( D(H) \geq d_H(v_1) \), $\max(D(H), 3) \geq d_H(v_2)$ and $L_H(v_1, y) \geq D(H) + 1$.

Pick vertices $b$ and $c$ in $H$ such that $D(H) = L_H(b, c)$. As already noted in the proof of Lemma 3, the graph $H - b - c$ has at least two components and those are trees. At least one of the trees, say $T$, has more than $D(H) - 2$ vertices. If $y = b$, then we can determine distinct vertices $v_1$ and $v_2$ in $H - y - c$ with degree at most $D(H) - 2$ in $H - b - c$. If in addition $D(H) = 3$, then we can choose the pair so that $L_H(v_1, y) \geq 4$. Thus, in the proof of Claim 1 it remains the case, when $L_H(y, c) \geq D(H) + 1$ for all vertices $c$ of $H - y$. If in addition $D(H) = 2$, then we choose $v_1$ in $H - b - c - y$ and a vertex $v_2$ in $H - y - v_1$ with minimum $d_H(v_2)$. If finally $D(H) = 3$, then we choose a vertex $v_1$ with degree 1 in $T - y$ and a vertex $v_2$ in $H - y - v_1$ with minimum $d_H(v_2)$. Clearly, $D(H) \geq d_H(v_h)$ for $h = 1, 2$. This settles Claim 1.

In the case, when $L(H) > D(H)$ and $D(H) \leq 3$, we set $D := L_H(v_1, y) \geq D(H) + 1$. Hence we have always $D \geq 3$ or $D = L(H) = D(H) = 2$.

For $1 \leq i \leq s$ let $u_i$ be the first vertex on $C(x_i, x_{i+1})$ in $N(x_p^+) \cup N(x_q^+) \cup \{x_i+1\}$. Let $\mathcal{S}$ denote the set of segments $C(x_i, x_{i+1})$ such that $x_i \notin \{x_p, x_q\}$, and furthermore $u_i \in N(x_p^+) \cup N(x_q^+)$ or $|N_H(x_i) \cup N_H(x_{i+1})| \geq 2$. Abbreviate $|\mathcal{S}| = m$.

As in the proof of Lemma 5 (Claim 2) we obtain

**Claim 2.** If $G$ is $r$-connected and $r \leq |H|$, then $m \geq r - 1$.

As in the proof of the previous lemma, we use the parameter $l$, where $l \in \{0, 1, 2\}$, and abbreviate $e_i^{(l)} = e(x_p^+, x_q^+; C(x_i, x_{i+1})) + 2e(v_1; x_i) + le(v_2; x_i)$ for $x_i \in X$.

We need the following standard estimate:

$$|C[u_i, x_{i+1}]| \geq e(x_p^+, x_q^+; C(x_i, x_{i+1})) - 1. \quad (1)$$

Consider $x_i \in C[x_p, x_q]$. For any $u \in N(x_q^+) \cap C(x_i, x_{i+1})$ we have $u \neq x_q^+$ and $u^- \notin N(x_p^+)$ by the Chord-Lemma since $C$ is a longest cycle. This readily yields (1).

Using (1) we next present estimates for $|C(x_i, x_{i+1})|$.

**Claim 3.** We have $|C(x_i, x_{i+1})| \geq e_i^{(l)}$ for $C[x_i, x_{i+1}] \notin \mathcal{S}$. Let $C[x_i, x_{i+1}] \in \mathcal{S}$. If $v_1 \in N(x_i)$ and $u_i \in N(x_p^+) \cup N(x_q^+)$, then $|C(x_i, x_{i+1})| \geq e_i^{(l)} + L_H(v_1, y) - l - 2 = e_i^{(l)} + D - l - 2$; if $v_1 \notin N(x_i)$ or $u_i \notin N(x_p^+) \cup N(x_q^+)$, then $|C(x_i, x_{i+1})| \geq e_i^{(l)} + D(H) - l$.

The first part of Claim 3 follows from (1) and the definition of $e_i^{(l)}$. Let $C[x_i, x_{i+1}] \in \mathcal{S}$. In the case, when $v_1 \in N(x_i)$ and $u_i \in N(x_p^+) \cup N(x_q^+)$ we have $|C(x_i, u_i)| \geq L_H(v_1, y) + 1$ by the Chord-Lemma, consequently the postulated estimate by (1) and the choice of $v_1, v_2$. Hence Claim 3 is true.

Summation of the estimates in Claims 3 with $l = 0$ gives rise to $|C| \geq d_C(x_p^+) + d_C(x_q^+) + 2d_C(v_1) + D - 2$, and consequently $n \geq d(x_p^+) + d(x_q^+) + 2d(v_1) - 1$. 

If \( \kappa(G) \geq 3 \), then as above (with \( l = 0 \)) we have \( |C| \geq d(x_p^+) + d(x_q^+) + 2d_c(v_1) + 2D - 4 \geq d(x_p^+) + d(x_q^+) + 2d(v_1) - 4 \).

Similarly, in the case when \( D(H) \geq 3 \), summation of the estimates of Claim 3 (with \( l = 1, 2 \)) yields the postulated results in Lemma 6.

If finally \( \kappa(G) \geq 4 \) and \( L(H) \geq 4 = D(H) + 2 \), then \( |C| \geq 2d(w_1) + 2d(w_2) + 2d(w_3) - 12 \) for some independent vertices \( w_1, w_2, w_3 \) in \( H \), as shown in the proof of Lemma 3 (case 1).

\[ \square \]

**Lemma 7.** Let \( H \) be a 2-connected component of \( G - C \) such that \( D(H) \leq 3 \) and \( D(H) < L(H) \). Let \( L(H) \geq k - 1 \) \((k \in \{3, 4, 5\}) \). There exist non-adjacent vertices \( u \) and \( w \) in \( G \) such that

1. \( |C| \geq (k - 1)d(u) + 2d(w) - k(k - 2) - 3 \), if \( G \) is \( k \)-connected;
2. \( n \geq (k - 1)d(u) + 2d(w) - k(k - 3) - 3 \), if \( G \) is \((k - 1)\)-connected.

**Proof.** By Lemma 6 it remains the case when \( H \) is normally linked in \( H \). We label \( N(H) = \{x_1, \ldots, x_s\} \) as usual.

**Case 1:** \( D(H) = 2 \).

In the subcase, when \( |H| \geq 5 \), we determine vertices \( v_1, v_2 \) of degree 2 in \( H \) (cf. Lemma 3). As shown above we have \( |C(x_i, x_{i+1})| \geq e(v_1, v_2; x_i, x_{i+1}) + 2 \) for all \( x_i \). By summation therefore \( |C| \geq 2d_c(v_1) + 2d_c(v_2) + 2s \). This inequality clearly yields the stipulated estimates. Now let \( |H| = 4 \) and hence \( L(H) = 3 \). In the case when \( H \) is the 4-cycle we determine \( v_1, v_2 \) and \( b \) as in the proof of Lemma 3, that is \( d(b) \geq d(v_1) \geq d(v_2) \) and deduce \( |C(x_i, x_{i+1})| \geq e(v_1, v_2; x_i, x_{i+1}) + e(b; x_i) \) for all \( x_i \). Again \( |C| \geq 2d_c(v_1) + 2d_c(v_2) + d_c(b) \). Furthermore, \( |C| \geq 2d_c(v_1) + 2d_c(v_2) + 2s \). For if \( N_C(b) = \{x_j\} \), then \( |C(x_j, x_j)| \geq e(v_1, v_2; x_j, x_{j+1}) + e(b; x_{j-1}) + 1 \). Hence the lemma. In the final subcase, when \( H = K_4^+ \), let \( v_1, v_2 \) be the two vertices of degree 2 in \( H \). Observe that by the Chord-Lemma we have \( |C(x_i, x_{i+1})| \geq L(H) + 2 \), if \( x_i \) is adjacent to \( v_1 \) or \( v_2 \). Hence \( |C(x_i, x_{i+1})| \geq e(v_1, v_2; x_i, x_{i+1}) + 1 \) for all \( x_i \). By summation we obtain \( |C| \geq 2d(v_1) + 2d(v_2) + s - 8 \geq 3d(v_1) + 2d(v_2) - 10 \). This settles Case 1.

**Case 2:** \( D(H) = 3 \).

Let \( D(H) = L_H(a, b) \) and \( T \) a non-trivial tree-component of \( H - a - b \) as in the proof of Lemma 3. Choosing end vertices \( v_1 \) and \( v_2 \) in \( T \) we have \( L_H(v_1, v_2) \geq 4 \) and consequently \( |C(x_i, x_{i+1})| \geq e(v_1, v_2; x_i, x_{i+1}) + 2 \) for all \( x_i \). This in turn yields \( |C| \geq 2d_c(v_1) + 2d_c(v_2) + 2s \). Again the stipulated estimates follow readily. \[ \square \]

**Lemma 8.** Let \( H \) and \( K \) be distinct components of \( G - C \) such that \( \max\{L(H), L(K)\} \geq 4 \) \((k \in \{3, 4, 5\}) \). There exist non-adjacent vertices \( u \) and \( w \) in \( G \) such that

1. \( |C| \geq (k - 1)d(u) + 2d(w) - k(k - 1) \), if \( G \) is \( k \)-connected and \( N(H) \neq N(K) \);
2. \( n \geq (k - 1)d(u) + 2d(w) - (k - 1)^2 + 2 \), if \( G \) is \((k - 1)\)-connected.

**Proof.** Suppose \( L(H) \geq k - 1 \). By Theorem 10 we may assume that \( H \) is 2-connected. We use again the labelling \( N(H) = \{x_1, \ldots, x_s\} \) along \( C \) and pick \( v \in V(H) \) such that \( D := D(H) \geq d_H(v) \). In view of Lemmas 5 and 6, we may moreover assume that \( H \) is normally linked in \( G \) and \( N(x_{i}^+) \cup N(x_{i}^-) \subseteq V(C) \). By Lemma 7 we may assume \( D \geq k - 1 \). We write \( t = |N(K)| \).
If $K$ is not separable, then we set $D^* := D(K)$ and determine a vertex $w$ in $K$ such that $D^* \geq d_K(w)$. Otherwise we may assume $L(K) < k - 1$ and then set $D^* = 0$. In this event we determine distinct end blocks $B$ and $B^*$ of $K$ such that $|B| \leq |B^*|$. Let $c$ and $c^*$ denote the unique cut vertices of $K$ in $B$ and $B^*$, respectively. Since $L(K) < k - 1 \leq 4$ we have $|B| = 2$, say $V(B) = \{c, w\}$.

By the above construction, the following claim holds in all cases. □

**Claim.** We have

$$D^* + t \geq d(w) - 1$$

with strict inequality unless $K$ is separable and $N(K) = N_C(w)$.

Observe that the following estimate (3) implies (ii).

$$|C| \geq (s - 1)D + D^* + 2t.$$  \hspace{1cm} (3)

Clearly $|K| + t \geq d(w) + 1$, and strict inequality holds if $K$ is separable. If $|C| \geq (s - 1)D + D^* + 2t$, then by the above claim we have $|C \cup H \cup K| \geq sD + D^* + |K| + 2t + 1 \geq (k - 1)(D + s) + 2d(w) + 2 + (s - k + 1)D - (k - 1)s = (k - 1)d(v) + 2d(w) - (k - 1)^2 + 2 + (s - k + 1)(D - k + 1)$.

For the proof of (i), it suffices to show

$$|C| \geq (s - 1)D + 2d(w).$$  \hspace{1cm} (4)

Indeed, if we have (4), then $|C| \geq (k - 1)(D + s) + 2d(w) + (s - k)D - s(k - 1) \geq (k - 1)d(v) + 2d(w) - k(k - 1) + (s - k)(D - k + 1)$. For later use we note that (4) also implies $|C \cup H| \geq (k - 1)d(v) + 2d(w) - k(k - 2)$, if $s \geq k - 1$.

By the above claim, the following estimate (5) implies (4):

$$|C| \geq (s - 1)D + 2D^* + 2t + 2.$$  \hspace{1cm} (5)

In the following case analysis the somewhat weaker estimate

$$|C| \geq (s - 1)D + 2D^* + 2t,$$  \hspace{1cm} (6)

will come up. By our assumptions (6) implies (4) unless $K$ is separable and $N(K) = N_C(w)$.

A component $C(z, z')$ of $C = (N(H) \cup N(K))$ is called good with respect to $K$, if $z, z' \in N(K)$ and $|N_K(z) \cup N_K(z')| \geq 2$.

For $1 \leq i \leq s$ we abbreviate $|N(K) \cap C(x_i, x_{i+1})| = t_i$ and $|N(K) \cap C(x_i, x_{i+1})| = p_i$. Let $X = \{x_i \in N(H) : p_i > 0\}$. For $x_i \in X$ let $z_i$ and $z_i'$ denote, respectively, the first and the last element of $N(K)$ on $C(x_i, x_{i+1})$.

For $x_i \in N(H) - X$ we have $t_i \leq 1$, and hence

$$|C(x_i, x_{i+1})| \geq D + 2 \geq D + 2t_i.$$  \hspace{1cm} (7)

Obviously, for $x_i \in X$ we have

$$|C(z_i, z_i')| \geq 2t_i - 3.$$  \hspace{1cm} (8)

**Case 1:** $|X| \geq 2$. 

Consider distinct \( x_p, x_q \in X \). Let \( Q \) be a longest \((x_p, x_q)\)-path with inner vertices in \( H \) and let \( R \) be a longest \((z_p, z_q)\)-path with inner vertices in \( K \). By definition we have \(|Q| \geq 2D + 1\) and \(|R| \geq 2D^* + 1\). Since \( Q \cup R \cup (C - C(x_p, z_p) - C(x_q, z_q)) \) is a cycle we obtain \(|C(x_p, z_p) \cup C(x_q, z_q)| \geq D + D^* + 2\). Similarly, \(|C(z_p', x_p+1) \cup C(z_q', x_q+1)| \geq D + D^* + 2\). Hence by (8)

\[
|C(x_p, x_{p+1}) \cup C(x_q, x_{q+1})| \geq 2D + 2D^* + 6 + |C[z_p, z'_p]| + |C[z_q, z'_q]|
\geq 2D + 2D^* + 2t_p + 2t_q.
\]

Label \( X = \{x_{i_1}, \ldots, x_{i_m}\} \) in order around \( C \). Then

\[
\sum_{x_i \in X} |C(x_i, x_{i+1})| = \frac{1}{2} \sum_{x_i \in X} |C(x_i, x_{i+1})| \cup C(x_{i+1}, x_{i+1+1})| \geq \frac{1}{2} \sum_{x_i \in X} (2D + 2D^* + 2t_i + 2t_{i+1}) = |X|(D + D^*) + 2 \sum_{x_i \in X} t_i.
\]

Combination of the above estimate and (7) yields \(|C| \geq sD + 2D^* + 2t\). Since \( D \geq 2 \) the latter estimate settles Case 1.

Case 2: \(|X| = 1\).

We may assume \( X = \{x_1\} \). Then \( N(K) \subseteq C(x_1, x_2) \cup N(H) \).

Case 2.1: \( N(K) \cap C(x_2, x_1) \neq \emptyset \).

Let \( x_p \in N(K) \cap (N(H) - \{x_1, x_2\}) \). As in Case 1 we infer \(|C(x_1, z_1) \cup C(x_{p-1}, x_p)| \geq D + D^* + 2\) and \(|C(z'_1, x_2) \subseteq C(x_p, x_{p+1})| \geq D + D^* + 2\). Hence

\[
|C(x_1, x_2) \cup C(x_{p-1}, x_{p+1})| \geq 2D + 2D^* + 7 + |C[z_1, z'_1]| \geq 2D + 2D^* + 2t_1 + 4 \geq 2D + 2D^* + 2t_1 + 2t_p - 1 + 2t_p.
\]

Using (7) for all \( x_i \in N(H) - \{x_1, x_{p-1}, x_p\} \) we deduce \(|C| \geq (s - 1)D + 2D^* + 2t\). As noted above (see (3)) this estimate yields (ii). Now let \( G \) be 3-connected. By the preceding it remains the subcase, when \( K \) is separable and \( N(K) = N_C(w) \).

If \( N_C(B^* - c^*) \) contains an element of \( \{z_1, x_p\} \), then \(|C(x_1, z_1) \cup C(x_{p-1}, x_p)| \geq D + D(B^*) + 1 + 2\) and \(|C(z'_1, x_2) \cup C(x_p, x_{p+1})| \geq D + D(B^*) + 1 + 2\) by the Chord-Lemma, consequently (5). Recall that the estimate (5) yields (i).

By symmetry, it remains the subcase, when \( N_C(B^* - c^*) \) has no element in \( C(x_2, x_1) \cup \{z_1, z'_1\} \). Since \( G \) is 3-connected, we now have \( N_C(B^* - c^*) \cap C(x_1, x_2) = \emptyset \). Therefore, \( C[z_1, z'_1] \) contains at least two good segments with respect to \( K \), and consequently again (5).

Case 2.2: \( N(K) \subseteq C[x_1, x_2] \).

Since \( G \) is 3-connected, again \( C[x_1, x_2] \) contains at least two good segments with respect to \( K \), consequently (5). This settles Case 2.

Case 3: \(|X| = 0\).

Now \( N(K) \subseteq N(H) \), consequently \( s > \kappa(G) \) or \( N(K) = N(H) \).
Theorem 11. Let C be a longest cycle in the connected graph G and let L(G − C) ≥ k − 1 (k ∈ {3, 4, 5}). There exist non-adjacent vertices u and w in G such that

(i) |C| ≥ (k − 1)d(u) + 2d(w) − k(k − 1), if G is k-connected and |N(G − C)| > k;
(ii) n ≥ (k − 1)d(u) + 2d(w) − k(k − 1) + 1, if G is (k − 1)-connected and |N(G − C)| ≥ k.

Proof. Let H be a component of G − C such that L(H) ≥ k − 1. By the earlier results we again are left with the case when H is 2-connected and normally linked in G, furthermore we may assume that D(H) ≥ k − 1. If |N(H)| = |N(G − C)|, then Lemma 8 yields the claim.

Now let |N(H)| = |N(G − C)| and N(H) = {x₁, . . . , xₜ} as usual. Pick distinct elements x₁, xₜ of N(H). As in the preceding proof we may assume that N(x₁) ∪ N(xₜ) ⊆ V(C). As shown in the proof of Lemma 6 (cf. (1)) we have |C[yᵢ, xᵢ₊₁]| ≥ e(x₁,xₜ); C(xᵢ, xᵢ₊₁) − 1, where yᵢ is the first element of N(x₁) ∪ N(xₜ) ∪ {xᵢ₊₁} on C(xᵢ, xᵢ₊₁). Since H is 2-connected and normally linked in G we have |C(xᵢ, yᵢ)| ≥ D + 1, again by the Chord-Lemma. Hence |C| ≥ d(x₁) + d(xₜ) + (s − 2)D = d(x₁) + d(xₜ) + (k − 1)(D + s) − (k − 1)(k + 1) + (s − k − 1)(D − k + 1). Moreover, we have |C ∪ H| ≥ d(x₁) + d(xₜ) + (s − 1)D + 1 = d(x₁) + d(xₜ) + (k − 1)(D + s) − k(k − 1) + 1 + (s − k)(D − k + 1).

Thus, Corollary 3 is proved. □

4. N(G − C) = N(H)

The main results in this section are

Theorem 12. Let C be a longest cycle in the 2-connected graph G such that N(G − C) does not split C and let L(G − C) ≥ k − 1 (k ∈ {3, 4, 5}). There exist non-adjacent vertices u and w in G such that

(i) |C| ≥ (k − 1)d(u) + 2d(w) − k(k − 1), if G is k-connected;
(ii) n ≥ (k − 1)d(u) + 2d(w) − k(k − 2), if G is (k − 1)-connected and |N(G − C)| ≥ 3.

Theorem 11. Let C be a longest cycle in the 2-connected graph G such that L(G − C) ≥ k − 1 (k ∈ {3, 4, 5}) and some component of G − C is not strongly linked in G. There exist non-
Proof. Without loss of generality we may assume that 

(i) $|C| \geq (k - 1)d(u) + 2d(w) - k(k - 1)$, if $G$ is $k$-connected;
(ii) $n \geq (k - 1)d(u) + 2d(w) - k(k - 2)$, if $G$ is $(k - 1)$-connected.

In the following let $C$ be a longest cycle in the 2-connected graph $G$ such that $L(G - C) \geq k - 1$ ($k \in \{3, 4, 5\}$). We choose a component $H$ of $G - C$ such that $L(H) \geq k - 1$ and label $N(H) = \{x_1, \ldots, x_s\}$ as usual around $C$. In view of Theorem 10 and Lemmas 6 and 7 we assume that $H$ is 2-connected, normally linked in $G$ and $D := D(H) \geq k - 1$. Again we determine a vertex $v$ in $H$ such that $D \geq d_H(v)$.

In the following lemmas we develop estimates of the form

$$|C| \geq d(u_1) + d(u_2) + (s - 1)D,$$

where $u_1, u_2$ are vertices in $C - N(H)$.

Remark 1 below will be used in the proofs to come. Observe that this remark was shown and used already in the last section (cf. (4) in Lemma 8).

**Remark 1.** Let $u_1, u_2$ be vertices in $C - N(H)$ such that (9) holds:

(i) If $G$ is $k$-connected, then $|C| \geq d(u_1) + d(u_2) + (k - 1)d(v) - k(k - 1)$;
(ii) If $G$ is $(k - 1)$-connected, then $n \geq d(u_1) + d(u_2) + (k - 1)d(v) - k(k - 2)$.

In the proofs of this section we use the following variant of the Chord-Lemma. We omit the straightforward proof.

**Proposition 1.** Let $z$ and $u$ be vertices on $C$.

(i) If $z$ and $u$ are distinct neighbors of respectively $x_i^+$ and $x_{i+1}^-$ on $C[x_i, x_{i+1}]$, then $|C(z, u)| \geq D + 1$ and $|C(u, z)| \geq D + 1$.
(ii) If $x_i^u$ and $x_q^z$ are crossing edges, then $|C(u, z)| \geq D + 1$ and $|C(z, u)| \geq D + 1$.

The estimates in this section will depend on the presence (or non-presence) of certain edges with end vertices in $C - N(H)$.

We call a component $C[u, w]$ of $C - N(H)$ a special segment of $C$, if $N(u) \cap C(u, w) \subseteq C(u, y)$ and $N(w) \cap C[u, w] \subseteq C[y, w]$ for some $y \in C(u, w)$.

**Lemma 9.** Let some component of $C - N(H)$ be special. There exist non-adjacent vertices $u$ and $w$ in $G$ such that

(i) $|C| \geq (k - 1)d(u) + 2d(w) - k(k - 1)$, if $G$ is $k$-connected;
(ii) $n \geq (k - 1)d(u) + 2d(w) - k(k - 2)$, if $G$ is $(k - 1)$-connected.

**Proof.** Without loss of generality we may assume that $C(x_1, x_2)$ is special. Let $y$ be the last neighbor of $x_i^+$ on $C(x_1, x_2)$ and $y'$ the first neighbor of $x_2^-$ on $C(x_1, x_2)$. For $1 \leq i \leq s$ we abbreviate $t_i = |N(x_i^+) \cap N(x_2^-) \cap C(x_i, x_{i+1})|$. Then $t_1 \leq 1$ by hypothesis.
For \( C(x_1, x_2) \) we use the representation

\[ |C(x_1, x_2)| = e(x_1^+, x_2^-; C(x_1, x_2)) + 1 + z_1, \]  

(10)

for some \( z_1 \) defined by this equation.

For \( 2 \leq i \leq s \) let

\[ |C(x_i, x_{i+1})| = e(x_i^+, x_{i+1}^-; C(x_i, x_{i+1})) + 1 + D + z_i, \]  

(11)

for some \( z_i \) defined by this equation.

Obviously, \( z_1 \geq |C(y, y')| + 1 - t_1 \geq 0 \). It is not difficult, by applying Proposition 1(i), to show that \( z_i \geq t_i D \) for \( 2 \leq i \leq s \).

Combination of (10) and (11) yields

\[ |C| = d(x_1^+) + d(x_2^-) + (s - 1)D + \sum_{i=0}^{s} z_i, \]

where \( z_0 = |N(H) - N(x_1^+)| + |N(H) - N(x_2^-)| \). Hence (9). By Remark 1 this settles Lemma 9. □

**Lemma 10.** Let \( N(x_p^+) \cap C(x_q, x_{q+1}^-) \neq \emptyset \) for some distinct \( x_p, x_q \in N(H) \). There exist non-adjacent vertices \( u \) and \( v \) in \( G \) such that

(i) \( |C| \geq (k - 1)d(u) + 2d(w) - k(k - 1) \), if \( G \) is \( k \)-connected;

(ii) \( n \geq (k - 1)d(u) + 2d(w) - k(k - 2) \), if \( G \) is \((k - 1)\)-connected.

**Proof.** For \( x_i \in N(H) - \{x_p, x_q\} \) we use the representation

\[ |C(x_i, x_{i+1})| = e(x_i^+, x_{i+1}^+; C(x_i, x_{i+1})) + D + \varepsilon^{(i)}_{pq}, \]

and for \( x_i \in \{x_p, x_q\} \) we use the representation

\[ |C(x_i, x_{i+1})| = e(x_i^+, x_{i+1}^-; C(x_i, x_{i+1})) + \varepsilon^{(i)}_{pq}, \]

for some \( \varepsilon^{(i)}_{pq} \geq 0 \) defined by these equations, respectively. Clearly,

\[ |C| = d(x_p^+) + d(x_q^+) + (s - 2)D + \sum_{i=1}^{s} \varepsilon^{(i)}_{pq}. \]  

(12)

By Remark 1 it suffices to show \( \sum_{i=1}^{s} \varepsilon^{(i)}_{pq} \geq D \).

The following Claim 1 follows readily by an application of Proposition 1(ii).

**Claim 1.** \( \varepsilon^{(i)}_{pq} \geq |N(x_p^+) \cap N(x_q^-) \cap C(x_i, x_{i+1})| - 1 \) \( D \). Furthermore \( \varepsilon^{(i)}_{pq} \geq 1 \), if \( |N(x_p^+) \cap N(x_q^+ \cap C(x_i, x_{i+1})| = 0 \).

**Claim 2.** \( \varepsilon^{(q)}_{pq} \geq D \).
By Claim 1 we may assume \(|N(x^+_p) \cap N(x^+_q) \cap C(x_q, x_{q+1})| \leq 1\). Let \(z\) and \(z'\) be the first and last elements of \(N(x^+_p)\) on \((x_q, x_{q+1})\), respectively.

If \(N(x^+_q) \cap C(z, x_{q+1}) \neq \emptyset\), then there exists a segment \(C(y, y') \subseteq C(z, x_{q+1})\) such that \(y \in N(x^+_p)\) and \((N(x^+_p) \cup N(x^+_q)) \cap C(y, y') = \{y'\}\). As \(|N(x^+_p) \cap N(x^+_q) \cap C(x_q, x_{q+1})| \leq 1\) we obtain \(e_{pq} \geq |(y, y')| - 1 \geq D + 1 - 1 = D\).

Now suppose \(N(x^+_q) \cap C(z, x_{q+1}) = \emptyset\). By Lemma 9 we may assume that \(C(x_q, x_{q+1})\) is not special. Then there exists a segment \(C[z_1, z_2] \subseteq C(x^+_q, z]\) such that \(z_1 \in N(x^+_q)\), \(z_2 \in N(x^+_q) \cap C(z_1, z)\) and \((N(x^+_q) \cup N(x^+_q)) \cap C(z_1, z_2) = \emptyset\). Let \(Q\) be a longest \((x_p, x_q)\)-path with inner vertices in \(V(H)\). Then \(|Q| \geq D + 3\). Since \(Q \cup C[x^+_q, z_1] \cup x^+_q \cup C(z_2, x_q] \cup z_1 x^+_{q+1} \cup C[x^+_{q+1}, x_p]\) is a cycle which contains all vertices of \(C - (C(z_1, z_2) \cup C(z', x^+_{q+1}))\) and at least \(D + 1\) vertices in \(V(H)\) we have \(|C(z_1, z_2) \cup C(z', x^+_{q+1})| \geq D + 1\), and therefore \(e_{pq} \geq D\), consequently (9), and this completes the lemma. \(\square\)

**Lemma 11.** Let some edge \(e = y_p y_q\) join the distinct segments \(C(x^+_p, x^-_{p+1})\) and \(C(x^+_q, x^-_{q+1})\). There exist non-adjacent vertices \(u\) and \(w\) in \(G\) such that

(i) \(|C| \geq (k - 1)d(u) + 2d(w) - k(k - 1)\), if \(G\) is \(k\)-connected;

(ii) \(n \geq (k - 1)d(u) + 2d(w) - k(k - 2)\), if \(G\) is \((k - 1)\)-connected.

**Proof.** In view of Lemma 9 we assume that no segment of \(C - N(H)\) is special. We continue the meaning of \(e_{ij}^{(k)}\) as introduced in the proof of Lemma 10. In view of Lemma 10 we may moreover assume \(N(x^+_i) \subseteq C(x_i, x_{i+1}) \cup N(H) \cup N^-(H)\) and \(N(x^-_{i+1}) \subseteq C(x_i, x_{i+1}) \cup N(H) \cup N^+(H)\) for all \(x_i \in N(H)\).

Since \(C(x_p, x_{p+1})\) is not special, we have either \(N(x^+_p) \cap C(y_p, x_{p+1}) \neq \emptyset\) or \(N(x^-_{p+1}) \cap C(x_p, y_p) \neq \emptyset\), say the former. Let \(y'_p\) be the first neighbor of \(x^+_p\) on \(C(y_p, x_{p+1})\). Let \(Q\) be a longest \((x_p, x_q)\)-path with inner vertices in \(H\). Then \(|Q| \geq D + 3\). In view of (12) and Remark 1 it suffices to show \(\sum_{i=1}^{3} e_{pq}^{(i)} \geq D\).

**Case 1:** \(N(x^+_{q-1}) \cap C(y_q, x_{q+1}) \neq \emptyset\).

Let \(y'_q\) be the first element of \(N(x^+_{q-1})\) on \(C(y_q, x_{q+1})\). Using \(Q\) and edges \(e, x^+_p y'_p\) and \(x^+_q y'_q\) we can construct a cycle which contains all vertices of \(Q \cup (C - C(y_p, y'_p) \cup C(y_q, y'_q))\).

Hence \(|C(y_p, y'_p) \cup C(y_q, y'_q)| \geq D + 1\). This implies \(e_{pq}^{(p)} + e_{pq}^{(q)} \geq D + 1\).

**Case 2:** \(N(x^+_{q-1}) \cap C(y_q, x_{q+1}) = \emptyset\).

Since \(C(x_q, x_{q+1})\) is not special, there exists a segment \(C[z_q, z'_q] \subseteq C[x^+_{q-1}, y_q]\) such that \(C(z_q, z'_q) \cap N(x^+_p) \cup N(x^+_q) = \emptyset\) and \(z_q \in N(x^-_{q-1})\), \(z'_q \in N(x^+_q)\). Then, as in Case 1, one can construct a cycle which contains all vertices of \(Q \cup (C - C(z_q, z'_q)) \cup C(y_q, x^-_{q+1}) \cup C(y_p, y'_p)\). Since \(C\) is a longest cycle and \((N(x^+_p) \cup N(x^+_q)) \cap (C(z_q, z'_q) \cup C(y_q, x^-_{q+1}) \cup C(y_p, y'_p)) = \emptyset\) we have

\[e_{pq}^{(p)} + e_{pq}^{(q)} \geq |C(z_q, z'_q) \cup C(y_q, x^-_{q+1}) \cup C(y_p, y'_p)| \geq D + 1.\] \(\square\)
Lemma 12. Let some edge join distinct components of \( C - N(H) \). There exist non-adjacent vertices \( u \) and \( w \) in \( G \) such that

(i) \( |C| \geq (k - 1)d(u) + 2d(w) - k(k - 1) \), if \( G \) is \( k \)-connected;
(ii) \( n \geq (k - 1)d(u) + 2d(w) - k(k - 2) \), if \( G \) is \((k-1)\)-connected and \( |N(H)| \geq 3 \).

Proof. By Lemma 8 and the preceding two lemmas it remains the case when all \( C \)-chords between distinct components of \( C - N(H) \) are edges of the form \( x_i^+x_{i+1}^- \) such that \( p \neq q \) and \( (N(x_p) \cup N(x_{q+1})) \cap (C(x_p^+, x_{q+1}^-) - N(H)) \neq \emptyset \).

Without loss of generality, we may assume \( N(x_p) \cap C(x_j, x_{j+1}) \neq \emptyset \) for some \( x_j \in C[x_p, x_q] \cap N(H) \). Consider a neighbor \( y \) of \( x_p \) on \( C(x_j, x_{j+1}) \). Set \( \gamma_{pq} = 0 \), if \( p = q + 1 \), and \( \gamma_{pq} = 1 \), if \( p \notin \{ q, q + 1 \} \).

First assume \( x_j \neq x_q \) and let \( z \) be the last neighbor of \( y^- \) on \( C[y, x_{j+1}] \). We can embed the path \( C[y^+, y^-] \cup C[y, z] \cup C[y, z] \cup C[y, z] \cup C[y, z] \) into a cycle which contains all vertices of \( C - C(z, x_{j+1}) \) and at least \( D + 1 \) vertices of \( H \). Hence \( |C(z, x_{j+1})| \geq D + 1 \), and \( |C(x_j, x_{j+1})| \geq d(y^-) + D + 2 - s \). Since \( N(x_i^+) \subseteq C(q, q+1) \cup N(H) \) and \( x_q^+ \notin N(x_p) \cup N(x_{q+1}) \) we have \( |C(x_q, x_{q+1})| \geq d(x_q^+) + 3 + \gamma_{pq} - s \). Hence we have \( |C| \geq d(y^-) + d(x_q^+) + (s - 1)D + 1 + \gamma_{pq} \).

Next let \( x_j = x_q \) and let \( z \) be the first neighbor of \( y^+ \) on \( C(x_q, y) \). Then a similar argument as above yields \( |C| \geq d(x_q^-) + d(x_q^-) + (s - 1)D + 1 + \gamma_{pq} \) and hence (9). By Remark 1 this settles Case 1.

In view of the preceding case we assume \( (N(x_i) \cup N(x_{j+1})) \cap (C(x_i^+, x_{j+1}^-) - N(H)) = \emptyset \), whenever \( x_i^+x_{i+1}^- \in E(G) \) for \( x_i, x_{i+1} \in N(H) \) with \( i \neq j \).

Case 2: There exists an edge \( x_p^+x_{q+1}^- \) such that \( p \notin \{ q, q + 1 \} \) and \( (N(x_p) \cup N(x_{q+1})) \cap (C(x_p^+, x_{q+1}^-) - N(H)) = \emptyset \).

If \( x_{p+1}^-x_r^+ \in E(G) \) for some \( r \neq p \), then we set \( S = C(x_p^+, x_{p+1}^-) \), otherwise set \( S = C(x_p^+, x_{p+1}^-) \). By our assumption in both cases we have \( |N(S)| < s \). Hence \( s - 1 \geq \kappa(G) \).

As noted in Case 1 we have \( |C(x_p, x_{p+1})| \geq d(x_p^-) + 4 - s \) and \( |C(x_q, x_{q+1})| \geq d(x_q^+) + 4 - s \). Thus,

\[
|C| \geq (s - 2)(D + 2) + d(x_p^-) + d(x_q^+) + 8 - 2s
= (k - 1)(D + s) + d(x_p^-) + d(x_q^+) - s(k - 1) + 4 + (s - k - 1)D
\geq (k - 1)d(v) + d(x_p^-) + d(x_q^+) - (k + 1)(k - 1) + 4.
\]

Similarly we have \( |C \cup H| \geq (k - 1)d(v) + d(x_p^-) + d(x_q^+) - k(k - 1) + 5 \). This settles Case 2.

Now we may assume that all edges between distinct components of \( C - N(H) \) are of the form \( x_i^+x_i^- \).

Case 3: \( x_p^+x_p^- \in E(G) \) and \( N(x_p) \cap (C(x_p^+, x_p^-) - N(H)) = \emptyset \) for some \( x_p \in N(H) \).

If \( x_p^+x_p^- \notin E(G) \), then by the above assumption we have a 2-vertex cut, and hence \( s - 1 \geq \kappa(G) = 2 \). In this event the estimates in (ii) can easily be deduced. Now suppose
$x_i^+ x_{i+1}^- \notin E$ for some $x_i \in N(H) - \{x_P\}$. Set $S = C(x_{i-1}^+, x_i)$, if $x_{i-1}^+ x_i^- \in E(G)$, and $S = C(x_{i-1}, x_i)$, otherwise. Since $|N(S)| < s$ we have $s - 1 \geq \kappa(G)$.

We first deal with the $C$-count. As in Case 1 we have $|C(x_i, x_{i+1})| \geq d_C(x_i^+) + 2$ for all $x_i \in N(H)$. Let $w_i = x_i^+$ for $i = 1, \ldots, k + 1$.

$$\begin{align*}
|C| \geq & \sum_{i=1}^{k+1} d_C(w_i) + (s - k - 1)D + 2s \\
\geq & (k - 1)(d_C(u) + s - 1) + 2(d_C(w) + s - 1) + 2 \\
& + (s - k - 1)D - (s - 1)(k - 1),
\end{align*}$$

for some non-adjacent vertices $u$ and $w$ on $C$.

Similarly we evaluate the degree-sum $d(w_1) + \cdots + d(w_k) + d(v)$ to deduce (ii). This settles Case 3, and completes the proof of Lemma 12. $\Box$

Observe that Lemmas 8 and 12 yield Theorem 11. Note also that in the case, when $s = 2$, we have shown

**Corollary 4.** Let $N(H) = \{x_1, x_2\}$ and $x_1^+ x_2^- \in E(G)$. If $x_1$ has a neighbor in $C(x_1^+, x_2^-) \setminus \{x_2\}$, then $|C \cup H| \geq 2s - 2$.

In the following Lemma 13 we handle the case, when $H$ is not strongly linked in $G$.

**Lemma 13.** Let $H$ be not strongly linked in $G$. There exist non-adjacent vertices $u$ and $w$ in $G$ such that

(i) $|C| \geq (k - 1)d(u) + 2d(w) - k(k - 1)$, if $G$ is $k$-connected;

(ii) $n \geq (k - 1)d(u) + 2d(w) - k(k - 2)$, if $G$ is $(k - 1)$-connected.

**Proof.** Since we could assume that $H$ is normally linked in $G$, necessarily $H$ is not hamilton-connected.

*Case 1: $\kappa(H) \geq 3$.*

Using Theorem 9 we determine non-adjacent vertices $v_1$ and $v_2$ in $H$ such that $D \geq d_H(v_1) + d_H(v_2) - 2$. Let $d_H(v_1) \geq d_H(v_2)$. Note that $D \geq d_H(v_h) + 1$ ($h = 1, 2$).

If $s \geq k$, then

$$|C| \geq (s - 1)D + D + 2s \geq (k - 1)(D + s) - k(k - 1) + D + 2s$$

$$\geq (k - 1)(d(v_1) + 1) + 2d(v_2) - k(k - 1) - 2$$

consequently (i).

If $s = k - 1$, then similarly, $|C \cup H| \geq (s + 1)(D + 2) - 1$. Proceeding as in the $C$-count we deduce (ii). This settles Case 1.

In the rest of this proof let $\kappa(H) = 2$. We determine a cut set $\{a, b\}$ of $H$ and distinct end blocks $B_1$ and $B_2$ of $H - b$. For $h = 1, 2$ let $c_h$ denote the cut vertex of $H - b$ in $B_h$, and let $v_h$ be a vertex in $B_h - c_h$ such that $D(B_h) \geq d_H(v_h) - 1$. If possible, we choose $B_1$ and $B_2$ in $H$ so that $c_1 \neq c_2$. Let $D(B_1) \geq D(B_2)$. 

Consider distinct vertices \(u_1\) and \(u_2\) in \(H\). If \(u_1\) or \(u_2\) is in \(B_1\), then clearly \(L_H(u_1, u_2) \geq D(B_1)\). If \(u_1, u_2\) are outside \(B_1\), then moreover \(L_H(u_1, u_2) \geq D(B_1) + 2\). In particular, \(D \geq D(B_1) \geq D(B_2)\). For \(h = 1, 2\) the vertex \(b\) has a neighbor \(b_h\) in \(B_h - c_h\). Let \(Q\) be a longest \((b_1, b_2)\)-path in \(H - b\). Then \(C^* = Q \cup \{bb_1, bb_2\}\) is a cycle and \(|C^*| \geq D(B_1) + D(B_2) + L_H(c_1, c_2) + 2\). Since \(C\) is a longest cycle, clearly \(2D \geq |C^*| \geq 2D(B_2) + L_H(c_1, c_2) + 2\).

Case 2: \(c_1 \neq c_2\).

We call \(C[x_i, x_{i+1}]\) a better segment, if \(x_i \in N(B_1 - c_1) \cup N(b)\) and \(x_{i+1} \in N(B_2 - c_2)\) vice versa. Since \(H\) is normally linked in \(G\), we have \(|C(x_i, x_{i+1})| \geq D(B_1) + D(B_2) + L_H(c_1, c_2) + 2\), if \(C[x_i, x_{i+1}]\) is a better segment.

Now \(2D \geq 2D(B_2) + 3\), consequently \(D \geq D(B_2) + 2\) and \(|C(x_i, x_{i+1})| \geq D + 2 \geq d_H(v_2) + 3\) for all segments \(C[x_i, x_{i+1}]\). If \(C[x_i, x_{i+1}]\) is better, then \(|C(x_i, x_{i+1})| \geq 2D(B_2) + 3 \geq 2d_H(v_2) + 1\).

Thus it remains the subcase, when there exists no better segment. Note that (ii) follows readily since \(|H| \geq |B_1| + |B_2| + 2 \geq 2d_H(v_2) + 4\).

Therefore it remains to show (i), when \(s = k\) and consequently \(k = \kappa(G)\) by hypothesis.

Consider \(x_j \in N_C(B_1 - c_1)\). Since there exist no better segments, we have \(x_j - 1, x_j + 1 \notin N(B_2 - c_2) \cup N(b)\) and necessarily \(N_C(B_2 - c_2) = N(H) - \{x_j - 1, x_j + 1\}\). This in turn implies \(N_C(B_1 - c_1) \cup N(b) = N(H) - \{x_j - 1, x_j + 1\}\) and \(k = 4 = \kappa(G)\). For \(h = 1, 2\) there exist vertices \(b_h, z_h, c_h, b_h'\) such that \(b_h' \in N(b)\) and \(N_C(z_h) \neq \emptyset\), furthermore \(|B_h| = 2\) or \(b_h' \neq z_h\). Otherwise \(G\) would have a \((k - 1)\)-vertex cut. We now show that in fact \(|C(x_j, x_{j+1})| \geq D(B_1) + D(B_2) + 4\) for some \(x_j\). This estimate readily yields (i).

Pick a path \(Q_0 = Q_0[c_1, c_2]\) in \(H - b\) and a neighbor \(z_0\) of \(x_{j+1}\) in \(H\). Let \(Q_0'\) be a shortest path in \(H\) from \(z_0\) to an element \(z_0'\) in \(V(Q_0 \cup B_1 \cup B_2) \cup \{b\}\). We may assume \(z_0' = c_2\). If \(z_0 \in Q_0\), then there exists a path of length at least \(L_{H - b}(c_1, b_1') + 2 + L_H(b_2', z_2)\) connecting \(z_0\) and \(z_2\). If \(z_0 = b\), then there exists a \((z_0, z_2)\)-path of length at least \(1 + L_{H - b}(b_1, z_2) \geq 1 + D(B_1) + D(B_2) + 2\) in \(H\). This settles Case 2.

Case 3: \(c_1 = c_2\).

By construction \(c_1\) is the only cut vertex of \(H - b\). By symmetry we may assume that \(b\) is the only cut vertex of \(H - c_1\). Now consider a component \(B^*\) of \(H - b - c_1\). If \(|B^*| \geq 2\), then \(|N(c_1) \cap B^*| \geq 2\) and \(|N(b) \cap B^*| \geq 2\). Moreover \(B^*\) is 2-connected, if \(|B^*| \geq 3\).

Let \(B_1^*, \ldots, B_r^*\) be the components of \(H - c_1 - b\), where \(D(B_1^*) \leq \cdots \leq D(B_r^*)\). For \(1 \leq r \leq r\) let \(w_r\) be a vertex of \(B_r^*\) such that \(D(B_r^*) \geq d_{B_r^*}(w_r)\) and hence \(D(B_r^*) \geq d_H(w_r)\).

We call \(C[x_j, x_{j+1}]\) exceptional, if \(N_H(x_j) \cup N_H(x_{j+1}) = \{c_1, b\}\). For such segments we have \(|C[x_j, x_{j+1}]| \geq D(B_1^*) + 2 \geq d_H(w_r)\) (\(1 \leq r \leq r\)). We call \(C[x_j, x_{j+1}]\) normal, if \(N_H(x_j) \cup N_H(x_{j+1}) \subseteq B_r^*\) for some \(r\). For such segments we have \(|C(x_j, x_{j+1})| \geq D(B_r^*) + 6 \geq d_H(w_r) + 4\) whenever \(r \neq r\). We call the remaining segments better.

If \(x_j \in N(B_r^*)\) and \(x_{j+1} \in \{c_1, b\}\), then \(|C(x_j, x_{j+1})| \geq D(B_1^*) + D(B_r^*) + 5 \geq d_H(w_r) + d_H(w_r) + 1\). If finally \(x_j \in N(B_r^*)\) and \(x_{j+1} \in N(B_s^*), \) where \(r \neq s\), then \(|C(x_j, x_{j+1})| \geq D(B_r^*) + D(B_s^*) + 4 \geq d_H(w_r) + d_H(w_s)\).

As in the preceding case one easily obtains (i) and (ii), if there exists a better segment. If there exist no better segments, then there exists at most one \(B_{\rho}\) such that \(N(B_{\rho}) \cap N(H) \neq \emptyset\). In this event \(\kappa(G) = 2\) and (ii) follows from the preceding estimates and the inequality \(|H| \geq 3 + |B_1 \cup B_2|\). The proof of Lemma 13 is now complete. \(\square\)
We are now ready to finish the proof of Theorem 12.

**Proof of Theorem 12.** Consider a component $K$ of $G - C$ which is not strongly linked in $G$. By the previous lemma and Theorem 10 it remains the case when $L(K) < k - 1$. Therefore, (ii) follows by Lemma 8. We have $L(K) < k - 1 \leq 4$. If $K$ is separable, then $d_K(w_1) = 1 \leq d_K(w_2) \leq 2$ for some non-adjacent vertices $w_1$ and $w_2$ in $K$. In this event $D \geq k - 1 > L(K) \geq 2$, consequently $D \geq 3$ and $k \geq 4$. This readily yields (i).

In the rest of this proof we assume that $K$ is not separable. In the case, when $K$ is not normally linked in $G$ (and hence $|K| \geq 2$), we determine distinct vertices $x_p$ and $x_q$ such that $N_K(x_p) \cup N_K(x_q)$ has a unique element $w_0$. Let $w$ be an element of $K - w_0$ such that $D(K) \geq d_K(w)$. Observe that in this event $s > \kappa(G)$ and hence (i) follows readily (cf. (4)).

Finally, let $K$ be normally linked in $G$. Then $K$ is not hamilton-connected. Now $D(K) \geq 2$ and $L(K) \geq 3$, consequently $D(K) = 2 = L(K) - 1 < k - 2$. Therefore, $k = 5$ and $D \geq 4$. Since there exist non-adjacent vertices $w_1$ and $w_2$ in $H$ such that $d_K(w_1) = d_K(w_2) = 2$, we again obtain (i). □

5. **Proof of Theorem 3**

The discrepancy between the bounds in Theorems 11 and 12 on one hand and of Theorem 3 on the other hand (namely $k - 1$) is due to the situation, when $S := N(G - C)$ splits $C$ and all components of $G - C$ are strongly linked in $G$. However, if some component of $G - S$ is not strongly linked in $G$, then the bounds differ by one only.

**Theorem 13.** Let $C$ be a longest cycle in the 2-connected graph $G$ such that $L(G - C) \geq k - 1$ ($k \in \{3, 4, 5\}$). Let furthermore some component of $G - N(G - C)$ be not strongly linked in $G$. There exist non-adjacent vertices $u$ and $w$ in $G$ such that

(i) $|C| \geq (k - 1)d(u) + 2d(w) - k(k - 1) - 1$, if $G$ is $k$-connected;
(ii) $n \geq (k - 1)d(u) + 2d(w) - k(k - 2) - 1$, if $G$ is $(k - 1)$-connected and $|N(G - C)| \geq 3$.

**Proof.** We abbreviate $S = N(G - C)$. Again we determine a component $H$ of $G - C$ such that $L(H) \geq k - 1$ and label $N(H) = \{x_1, \ldots, x_j\}$ as usual. By Lemma 8 and Theorem 11 it remains to consider the case when $S$ splits $C$ and $N(K) = S$ for all components $K$ of $G - S$. Let $L$ be a component of $G - S$ which is not strongly linked in $G$.

In view of Theorem 12 we may assume that all components of $G - C$ are strongly connected in $G$ and consequently $V(L) = V(C(x_j, x_{j+1}))$ for some $j$. Pick $v \in V(H)$ and observe that $D := D(H) = |H| - 1$. In view of Lemma 9 we may assume that $L$ is 2-connected.

First, assume that $L$ is not normally linked in $G$. By definition there exist distinct vertices $x_p$, $x_q$ and $w_0$ in $G$ such that $N_L(x_p) \cup N_L(x_q) = \{w_0\}$. In particular $|N(L - w_0)| \leq s - 1$, consequently $s > \kappa(G)$. Using Lemma 1 we determine distinct vertices $w_1$ and $w_2$ in $L - w_0$ such that $D(L) \geq d_L(w_1)$ and $D(L) \geq d_L(w_2) - 1$. Since $d(w_1) \leq D(L) + s - 2$ and $d(w_2) \leq D(L) + s - 1$, it remains the subcase when $k = 5$ and $L = K_4^-$. But then
\(|C(x_j, x_{j+1})| \geq D + 1 \geq 5\) and \(d_L(w_2) \leq 3\). Using the degree-sum \(d(w_1) + d(w_2) + (k - 1)d(v)\) one readily obtains (i) and (ii).

Finally, let \(L\) be normally linked in \(G\) and hence not hamilton-connected. Therefore, there exist non-adjacent vertices \(w_1\) and \(w_2\) in \(L\) such that \(|C(x_j, x_{j+1})| \geq d_L(w_1) + d_L(w_2)\). Now \(|C| \geq (s - 1)D + d(w_1) + d(w_2) - 1\), and hence the claims (cf. Remark 1). \(\square\)

Now we are ready to give the proof of Theorem 3.

**Proof of Theorem 3.** Let \(H\) be a component of \(G - C\) such that \(L(H) \geq k - 1\). Abbreviate \(S := N(H)\) and \(s = |S|\).

**Case 1:** Some \(C\)-chord joins distinct components of \(C - S\).

By Theorem 11 it remains the subcase when \(N(H) = \{x_1, x_2\}\). In view of Lemmas 8, 10, 10 and 11 we may in addition assume that the \(C\)-chords between \(C(x_1, x_2)\) and \(C(x_2, x_1)\) are edges of the form \(x_j^-x_j^+\) (\(j = 1\) or 2), say \(x_1^-x_1^+ \in E(G)\). Set \(S_1 = \{x_1^-, x_1, x_1^+\}\). Similarly let \(S_2 = \{x_2^-, x_2, x_2^+\}\), if \(x_2^-x_2^+ \in E(G)\), and \(S_2 = \{x_2\}\) otherwise.

By Corollary 4 it remains the subcase when \(|N(L)| = 2\) for all components \(L = (S_1 \cup S_2)\). If \(L\) is such a component and not strongly linked in \(G\), then \(L\) is not hamilton-connected. In this event \(|L| \geq 2d_L(w)\) for some vertex \(w\) of \(L\). (This follows by a standard variation of Dirac’s Theorem.) In this event clearly \(n \geq 2\sigma_2 - 4\). We obtain the same estimate, if \(H\) is not the only component of \(G - C\). This settles Case 1.

In the rest of the proof we assume that \(S\) splits \(C\).

**Case 2:** Some component of \(G - S\) is not strongly linked in \(G\).

By Theorem 13 we again may assume \(s = 2\). As in the previous case we easily obtain (ii).

**Case 3:** All components of \(G - S\) are strongly linked in \(G\).

In view of Corollary 3 we may assume \(s = \kappa(G)\). Consider a longest cycle \(C'\) of \(G\). Let \(H_1, \ldots, H_t\) be all components of \(G - C\) with \(|H_1| \geq |H_2| \geq \cdots \geq |H_t|\). Since each \(H_j\) is strongly linked in \(G\), we have \(|C(x_i, x_{i+1})| \geq |H_1| + 1 \geq |H_j| + 1\) for \(1 \leq i \leq s\). For each \(H_j\) we have \(|C'| = |C| \geq s(|H_j| + 1)\). Hence \(C'\) intersects at least \(s - 1\) components of \(G - S\) and \(S \subseteq V(C')\). Since \(G\) is \(s\)-connected, \(S\) also splits \(C'\).

The proof of Theorem 3 is now complete. \(\square\)

A set of analogous results on longest paths in a graph \(G\) can be obtained by applying our results to the graph \(G + K_1\).

6. Concluding remarks

One of the referees addressed the algorithmic aspect. The constructions in the lemmas and the description of the exceptional classes involve invariants which in general are NP-hard to check. In particular, at several places we need to check \(L(H) = |H| - 1\) for certain subgraphs \(H\) of \(G\) (e.g. Case 1 in the proof of Theorem 3). However, in these instances it suffices to check a corresponding Dirac-type condition. An analogous comment applies in instances where we could construct longer cycles, if one of the claims in a lemma fails to hold. Thus, for our purposes there should be good algorithms available.
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References


Further Reading