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Journal of Functional Analysis 256 (2009) 2123–2156

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Spectral analysis of the semi-relativistic Pauli–Fierz hamiltonian

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Received 2 June 2008; accepted 23 September 2008

Available online 5 October 2008

Communicated by L. Gross

Abstract

We consider a charged particle, spin $\frac{1}{2}$, with relativistic kinetic energy and minimally coupled to the quantized Maxwell field. Since the total momentum is conserved, the Hamiltonian admits a fiber decomposition as $H(P)$, $P \in \mathbb{R}^3$. We study the spectrum of $H(P)$. In particular we prove that, for non-zero photon mass, the ground state is exactly two-fold degenerate and separated by a gap, uniformly in P, from the rest of the spectrum.

Keywords: Semi-relativistic quantum electrodynamics; Ground state of fixed total momentum

1. Introduction and main results

Let us consider a classical point charge, charge e, mass M, position q, velocity $\dot{q}$, coupled to the Maxwell field with electric field E and magnetic field B. The coupling to the field is through a rigid charge distribution $\varphi : \mathbb{R}^3 \to \mathbb{R}_+$ normalized as $\int dx\, \varphi(x) = 1$. Then the equations of motion for the coupled system read, in units where $c = 1$,

$$\frac{\partial}{\partial t} B = -\nabla \wedge E, \qquad \frac{\partial}{\partial t} B = \nabla \wedge E - e\varphi(\cdot - q)\dot{q},$$
$$\nabla \cdot E = e\varphi(\cdot - q), \qquad \nabla \cdot B = 0,$$

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0022-1236/$ – see front matter
doi:10.1016/j.jfa.2008.09.014

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(M\big(1-\dot{q}^2\big)^{-1/2}\dot{q}\big)=e\big(E*\varphi(q)+\dot{q}\wedge(B*\varphi)(q)\big) \tag{1}$$

with $*$ denoting convolution. The uncoupled system, $e=0$, is Lorentz invariant. But the choice of the rigid charge distribution singles out a specific reference frame and hence makes the model semi-relativistic, only.

The canonical quantization of (1) results in a quantum evolution governed by the semi-relativistic Pauli–Fierz hamiltonian. Our goal is to study spectral properties of this operator. While the nonrelativistic counterpart has been investigated in considerable detail, no spectral results seem to be available for the semi-relativistic case.

The quantization procedure for (1) is described, e.g., in [24]. One writes (1) in Lagrangian form and Legendre transforms to Hamiltonian structure in using the Coulomb gauge. Since our prime example will be an electron (charge $-e$), we want to include spin $\frac{1}{2}$. As for the nonrelativistic hamiltonian this amounts to replacing $(p+eA)^2$ by $(\sigma\cdot(p+eA))^2$ with σ the 3-vector of Pauli spin matrices. As a result one obtains the semi-relativistic Pauli–Fierz hamiltonian, which is given by

$$H_{\mathrm{sr}}=\sqrt{\big(\sigma\cdot\big(-\mathrm{i}\nabla_x+eA(x)\big)\big)^2+M^2}+H_{\mathrm{f}}. \tag{2}$$

Without restriction of generality we set $e\geqslant 0$. H_{sr} acts in $L^2(\mathbb{R}^3;\mathbb{C}^2)\otimes\mathfrak{F}$, where $\mathfrak{F}$ is the photon Fock space

$$\mathfrak{F}=\sum_{n\geqslant 0}^{\oplus}L^2\big(\mathbb{R}^3\times\{1,2\}\big)^{\otimes_{\mathrm{s}}n}.$$

$A(x)$ is the quantized vector potential defined through

$$A(x)=\sum_{\lambda=1,2}\int_{|k|\leqslant\Lambda}\frac{\mathrm{d}k}{\sqrt{2(2\pi)^3\omega(k)}}\,\varepsilon(k,\lambda)\big(\mathrm{e}^{\mathrm{i}k\cdot x}a(k,\lambda)+\mathrm{e}^{-\mathrm{i}k\cdot x}a(k,\lambda)^*\big),$$

where $\varepsilon(k,\lambda)$, $\lambda=1,2$, is the pair of polarization vectors. $k/|k|,\varepsilon(k,1),\varepsilon(k,2)$ are a dreibein depending measurably on k. For convenience we use the sharp ultraviolet cutoff Λ which corresponds to setting $\hat{\varphi}(k)=(2\pi)^{-3/2}$ for $|k|\leqslant\Lambda$ and $\hat{\varphi}(k)=0$ otherwise, $\hat{}$ denoting Fourier transform. Our results are equally valid for a smooth cutoff. $a(k,\lambda)$, $a(k,\lambda)^*$ are the annihilation and creation operators which satisfy the standard commutation relations

$$\big[a(k,\lambda),a(k',\lambda')^*\big]=\delta_{\lambda\lambda'}\delta(k-k'),$$
$$\big[a(k,\lambda),a(k',\lambda')\big]=0=\big[a(k,\lambda)^*,a(k',\lambda')^*\big].$$

H_{f} is the field energy,

$$H_{\mathrm{f}}=\sum_{\lambda=1,2}\int_{\mathbb{R}^3}\mathrm{d}k\,\omega(k)a(k,\lambda)^*a(k,\lambda). \tag{3}$$

For the Maxwell field the dispersion relation is

$$\omega(k) = |k|.$$

Mathematically it is convenient to introduce the photon mass m_{ph} through the choice

$$\omega(k) = \sqrt{k^2 + m_{\mathrm{ph}}^2}.$$

Readers will find more precise definitions of $A(x)$ and H_{f} in Appendix A.

Remark 1. For a fixed configuration of the vector potential the classical hamiltonian function is

$$H_{\mathrm{cl}}(p,q) = \sqrt{\big(p - eA(q)\big)^2 + M^2}. \tag{4}$$

We picked here the "naive" quantization $p \rightsquigarrow -\mathrm{i}\nabla_x$, $q \rightsquigarrow x$, which is fairly common in the physics community [3]. Alternatives would be either Weyl or magnetic Weyl quantization [16].

By translation invariance the total momentum, i.e., the sum of the momentum of the charge and the field momentum, is conserved. The generator of translations is the total momentum operator $P_{\mathrm{tot}} = -\mathrm{i}\nabla_x + P_{\mathrm{f}}$ with

$$P_{\mathrm{f}} = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \mathrm{d}k\, k a(k,\lambda)^* a(k,\lambda).$$

It strongly commutes with the hamiltonian H_{sr}, namely, $\exp[-\mathrm{i}a \cdot P_{\mathrm{tot}}]\exp[-\mathrm{i}tH_{\mathrm{sr}}] = \exp[-\mathrm{i}tH_{\mathrm{sr}}]\exp[-\mathrm{i}a \cdot P_{\mathrm{tot}}]$ for all $a \in \mathbb{R}^3$ and $t \in \mathbb{R}$. Therefore H_{sr} admits the direct integral decomposition

$$\mathcal{U} H_{\mathrm{sr}} \mathcal{U}^* = \int_{\mathbb{R}^3}^{\oplus} H_{\mathrm{sr}}(P)\,\mathrm{d}P, \tag{5}$$

$$H_{\mathrm{sr}}(P) = \sqrt{\big(P - P_{\mathrm{f}} + eA(0)\big)^2 + \sigma \cdot B(0) + M^2} + H_{\mathrm{f}} \tag{6}$$

acting in $\mathbb{C}^2 \otimes \mathfrak{F}$, $B(0) = \nabla \wedge A(0)$. The unitary $\mathcal{U}$ is defined by $\mathcal{U} = \mathcal{F}_x \exp[\mathrm{i}x \cdot P_{\mathrm{f}}]$ where $\mathcal{F}_x$ is the Fourier transformation with respect to x. We will provide a mathematically rigorous definition of H_{sr} and $H_{\mathrm{sr}}(P)$ in Section 2.

As the most basic information on $H_{\mathrm{sr}}(P)$ we want to study its spectral gap and the multiplicity of its ground state. To have a guideline, one restricts $H_{\mathrm{sr}}(P)$ with $e = 0$ to one-photon excitations only [6]. Then this restricted operator has a single, doubly degenerate eigenvalue

$$E_0^{\mathrm{r}}(P) = \sqrt{P^2 + M^2} \tag{7}$$

and continuous spectrum with bottom

$$E_{\mathrm{c}}^{\mathrm{r}}(P) = \inf_k \Big(\sqrt{(P-k)^2 + M^2} + \omega(k)\Big). \tag{8}$$

Now in the non-relativistic case, kinetic energy $P^2/2M$, the corresponding version of (7), (8) imply that for small P there is a gap which vanishes at P_c, $P_c \simeq M$. On the other hand for the semi-relativistic case, (7), (8) imply that the restricted hamiltonian has a spectral gap which closes as $1/|P|$ for $|P| \to \infty$.

One of our long term goals is to control the effective dynamics of a charge subject to slowly varying external potentials and coupled to the radiation field as in (2). Very crudely, one considers the subspace of $L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathfrak{F}$ spanned by the ground states of $H_{sr}(P)$ with $P \in \mathbb{R}^3$ and constructs the effective dynamics as an approximate solution to the full dynamics lying close to that subspace. In principle this problem can be handled by space-adiabatic perturbation theory [19,25,26], which as one basic input uses that $H_{sr}(P)$ has a uniform spectral gap, i.e., for all $P \in \mathbb{R}^3$,

$$\inf\bigl\{\operatorname{spec}\bigl(H_{sr}(P)\bigr) \setminus \bigl\{E_{sr}(P)\bigr\}\bigr\} - E_{sr}(P) =: C_g(P) \geqslant C_0 > 0, \tag{9}$$

where $E_{sr}(P) = \inf \operatorname{spec}(H_{sr}(P))$. The heuristic argument above indicates that $H_{sr}(P)$ does not satisfy the gap condition (9). Of course, one could imagine to avoid the uniform gap through a suitable restriction on the allowed initial wave functions. But this seems to be a major technical enterprise. Therefore we propose to modify somewhat $H_{sr}(P)$ such that the small energy behavior is changed only little, while at large P the gap condition (9) holds. The simplest way is to put a factor γ, $0 < \gamma \leqslant 1$, in front of the square root, to say, (2) is modified to

$$H = \gamma \sqrt{\bigl(\sigma \cdot \bigl(-\mathrm{i}\nabla_x + eA(x)\bigr)\bigr)^2 + M^2} + H_f. \tag{10}$$

Then (4) carries a prefactor γ and (5), (6) are modified to

$$\mathcal{U} H \mathcal{U}^* = \int_{\mathbb{R}^3}^{\oplus} H(P)\, \mathrm{d}P, \tag{11}$$

$$H(P) = \gamma \sqrt{\bigl(P - P_f + eA(0)\bigr)^2 + \sigma \cdot B(0) + M^2} + H_f. \tag{12}$$

The heuristics based on (7) and (8) indicates that $H(P)$ should have a uniform gap and this will be one of the main results of our paper.

To get started we have to ensure the self-adjointness of H and of $H(P)$, see Section 2 for details.

Proposition 1.1. *For any $0 < \gamma \leqslant 1$, $\Lambda < \infty$ and $0 \leqslant m_{ph}$, there exists $e_* > 0$ such that, for all $e < e_*$, H is self-adjoint on $\operatorname{dom}(|-\mathrm{i}\nabla_x|) \cap \operatorname{dom}(H_f)$. Moreover H is essentially self-adjoint on any core of the free Hamiltonian $H_0 = \gamma\sqrt{-\Delta_x + M^2} + H_f$.*

Proposition 1.2. *Choose γ, Λ, m_{ph} arbitrarily as Proposition 1.1. Let e_* be given by Proposition 1.1. Then, for all $e < e_*$ and $P \in \mathbb{R}^3$, $H(P)$ is self-adjoint on $\operatorname{dom}(H_f)$. Moreover $H(P)$ is essentially self-adjoint on any core of the operator $H_0(P) = \gamma\sqrt{(P - P_f)^2 + M^2} + H_f$.*

Remark 2. There are further parts of our proof which will require small e_*. Therefore we did not attempt to optimize e_* in every step.

The spectral analysis of the nonrelativistic Pauli–Fierz hamiltonian was initiated by J. Fröhlich in his PhD thesis [6]. Our first main result is the extension of his methods to the semi-relativistic case. While the result could be anticipated from [6,17,23], the actual proof is surprisingly technical, since the minimal coupling is under the square root, see Section 3.

Theorem 1.3. *Set $\Lambda, \gamma, m_{\mathrm{ph}}$ arbitrarily as in Proposition* 1.2. *Choose e as $e < e_*$. Let*

$$\Delta(P) = \inf_{k \in \mathbb{R}^3} \big(E(P-k) + \omega(k) - E(P)\big)$$

where

$$E(P) = \inf \operatorname{spec}\big(H(P)\big), \qquad \Sigma(P) = \inf \operatorname{ess.spec}\big(H(P)\big).$$

Then one has

$$\Sigma(P) - E(P) = \Delta(P)$$

for all $P \in \mathbb{R}^3$.

If $m_{\mathrm{ph}} > 0$ and $\gamma < 1$ it is easily seen that $\Delta(P) \geqslant C_0 > 0$ uniformly. However this does not yet establish a spectral gap in the sense of (9), because beyond the ground state there could be other eigenvalues in the interval $[E(P), \Sigma(P)]$. In the literature there are two methods to count the number of eigenvalues. One is through positive commutator, Mourre type estimates and the other uses a pull through in order to estimate the overlap between the Fock vacuum and the ground state. For sufficiently small P both methods yield the desired result. However, a uniform bound on the spectral gap seems to be difficult to achieve by such techniques. Therefore we introduce a novel method based on operator monotonicity, which we learned from the masterly works of Lieb and Loss [13,14], together with the min-max principle. While the case $\gamma = 1$ is not worked out detail, our method is still available to investigate the spectral gap of $H_{\mathrm{sr}}(P)$.

Progressed so far, one still has to determine the degeneracy of the ground state. For the nonrelativistic Pauli–Fierz model this is discussed in [12]. Later on we learned a very simple and general argument from M. Loss. We reproduce his result and show that it is applicable to the semi-relativistic Pauli–Fierz hamiltonian.

We summarize our main result in

Theorem 1.4. *Fix $0 < \gamma < 1$ and $0 < m_{\mathrm{ph}}$. Then there exists $e_* > 0$ independent of P, such that, for all $e < e_*$ and $P \in \mathbb{R}^3$, the following properties hold.*

(i) *One has*

$$\Sigma(P) - E(P) \geqslant (1-\gamma) m_{\mathrm{ph}} - e c_1 - \mathcal{O}\big(e^2\big) > 0$$

for all $P \in \mathbb{R}^3$, where c_1 and $\mathcal{O}(e^2)$ are independent of P. In particular $E(P)$ is an eigenvalue.

(ii) *One has*

$$\inf\big\{\operatorname{spec}\big(H(P)\big) \setminus \big\{E(P)\big\}\big\} - E(P) \geqslant (1 - e c_2 - \gamma) m_{\mathrm{ph}} - e c_3 - \mathcal{O}\big(e^2\big) \tag{13}$$

for all P, where c_2, c_3 and $\mathcal{O}(e^2)$ are independent of P.
(iii) *$E(P)$ is exactly doubly degenerate.*

Remark 3. We remark that the lowest energy $E(P)$ and a possible spectral gap are also of importance, e.g., in scattering theory. We refer to [1,7,8,10,11,22] for the investigation of related models and to [2] for $E(P)$ when the infrared cutoff is removed.

This paper is organized as follows. In Section 2 we define the semi-relativistic Pauli–Fierz Hamiltonians and prove Propositions 1.1 and 1.2. In Section 3 we prove Theorem 1.3. Section 4 is devoted to show the degeneracy of eigenvalues by Kramers' degeneracy theorem. In Section 5 some energy inequalities are established. We prove Theorem 1.4 in Section 6. Some auxiliary results are proven in Appendices A–E.

2. Self-adjointness

2.1. Dirac operators

As a preliminary, we introduce two Dirac operators which will simplify our study.

Let us define a Dirac operator D by

$$D = \alpha \cdot \big(-\mathrm{i}\nabla_x + eA(x)\big) + M\beta$$

living in $L^2(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathfrak{F}$. This is essentially self-adjoint on $C_0^\infty(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathfrak{F}_{\mathrm{fin}}$ by the Nelson's commutator theorem [20] with the test operator $-\Delta_x + H_{\mathrm{f}}$. Here

$$\mathfrak{F}_{\mathrm{fin}} = \mathrm{Lin}\big\{a(f_1)^* \dots a(f_n)^*\Omega,\ \Omega \bigm| f_1(\cdot,\lambda_1), \dots, f_n(\cdot,\lambda_n) \in C_0^\infty\big(\mathbb{R}^3\big) \\ \text{for all } \lambda_1, \dots, \lambda_n \in \{1,2\} \text{ and } n \in \mathbb{N}\big\},$$

where $\mathrm{Lin}\{\dots\}$ means the linear span of the set $\{\dots\}$ and Ω is the Fock vacuum defined by $\Omega = 1 \oplus 0 \oplus 0 \oplus \cdots$. We denote the closure of D by the same symbol. We note that

$$D^2 = T + M^2,$$
$$|D| = \sqrt{T + M^2},$$

where the self-adjoint operator T is expressed as

$$T = \begin{pmatrix} (\sigma \cdot (-\mathrm{i}\nabla_x + eA(x)))^2 & 0 \\ 0 & (\sigma \cdot (-\mathrm{i}\nabla_x + eA(x)))^2 \end{pmatrix}$$

on $C_0^\infty(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathfrak{F}_{\mathrm{fin}}$.

Next let us define the following Dirac operator

$$D(P) = \alpha \cdot \big(P - P_{\mathrm{f}} + eA(0)\big) + M\beta$$

acting in $\mathbb{C}^4 \otimes \mathfrak{F}$. Again this is essentially self-adjoint on $\mathbb{C}^4 \otimes \mathfrak{F}_{\mathrm{fin}}$ by the Nelson's commutator theorem with a test operator H_{f}. We denote its closure by the same symbol. Then one can easily observe that

$$\mathcal{U}D\mathcal{U}^* = \int_{\mathbb{R}^3}^{\oplus} D(P)\,\mathrm{d}P,$$

$$D(P)^2 = T(P) + M^2,$$

$$\left|D(P)\right| = \sqrt{T(P) + M^2},$$

where the action of the self-adjoint operator $T(P)$ is concretely given as

$$T(P) = \begin{pmatrix} (\sigma \cdot (P - P_{\mathrm{f}} + eA(0)))^2 & 0 \\ 0 & (\sigma \cdot (P - P_{\mathrm{f}} + eA(0)))^2 \end{pmatrix}$$

on $\mathbb{C}^4 \otimes \mathfrak{F}_{\mathrm{fin}}$.

2.2. *Definition of the Hamiltonians*

Our definition of H and $H(P)$ are as follow:

$$H = \gamma|D| + H_{\mathrm{f}},$$

$$H(P) = \gamma\left|D(P)\right| + H_{\mathrm{f}}.$$

In this paper we occasionally identify a direct sum operator $A \oplus A$ with A if no confusion occurs. Hence the above definitions mean that $H \oplus H = \gamma|D| + H_{\mathrm{f}} \oplus H_{\mathrm{f}}$ and $H(P) \oplus H(P) = \gamma|D(P)| + H_{\mathrm{f}} \oplus H_{\mathrm{f}}$.

2.3. *Proof of Proposition 1.1*

For each $\varphi \in C_0^\infty(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathfrak{F}_{\mathrm{fin}}$, one has

$$\big\||D|\varphi\big\|^2 = \langle\varphi, D^2\varphi\rangle \leqslant \mathrm{const}\big\|(H_0 + \mathbb{1})\varphi\big\|^2.$$

Since $C_0^\infty(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathfrak{F}_{\mathrm{fin}}$ is a core of H_0, one concludes that $\mathrm{dom}(H_0) \subseteq \mathrm{dom}(|D|)$. Also note that, for $D_0 := \alpha \cdot (-\mathrm{i}\nabla_x) + M\beta$, one has $\mathrm{dom}(H_0) \subseteq \mathrm{dom}(|D_0|)$. Let H_{I} be the interaction term given by

$$H_{\mathrm{I}} = |D| - |D_0|.$$

By the above arguments, $\mathrm{dom}(H_0) \subseteq \mathrm{dom}(H_{\mathrm{I}})$ holds. Using the formula

$$|a| = \frac{1}{\pi}\int_0^\infty \mathrm{d}t\,\frac{1}{\sqrt{t}}\frac{a^2}{t + a^2}, \tag{14}$$

one has

$$|D| - |D_0| = \frac{1}{\pi}\int_0^\infty \mathrm{d}t\,\sqrt{t}\bigl(t+D^2\bigr)^{-1}\bigl\{2eA(x)\cdot\bigl(-\mathrm{i}\nabla_x\bigr) + e^2A(x)^2 + e\sigma\cdot B(x)\bigr\}\bigl(t+D_0^2\bigr)^{-1}, \tag{15}$$

where $B(x) = \nabla_x \wedge A(x)$. Observe that

$$\begin{aligned}
&\bigl\|A_j(x)(-\mathrm{i}\partial_j)\bigl(t+D_0^2\bigr)^{-1}(H_0+\mathbb{1})^{-1}\bigr\| \\
&\quad\leqslant \bigl\|A_j(x)(-\mathrm{i}\partial_j)|-\mathrm{i}\partial_j|^{-1/2}(H_0+\mathbb{1})^{-1}\bigr\|\bigl\||-\mathrm{i}\partial_j|^{1/2}\bigl(t+D_0^2\bigr)^{-1}\bigr\| \\
&\quad\leqslant \mathrm{const}\,t^{-3/4}
\end{aligned}$$

for $j = 1, 2, 3$, and

$$\begin{aligned}
\bigl\|A(x)^2(H_0+\mathbb{1})^{-1}\bigr\| &\leqslant \mathrm{const}, \\
\bigl\|\sigma\cdot B(x)(H_0+\mathbb{1})^{-1}\bigr\| &\leqslant \mathrm{const}.
\end{aligned}$$

Combining these with (15), one obtains

$$\begin{aligned}
\bigl\|H_\mathrm{I}(H_0+\mathbb{1})^{-1}\bigr\| &\leqslant \mathrm{const}\bigl(e+e^2\bigr)\int_0^\infty \mathrm{d}t\,\sqrt{t}\bigl(t+M^2\bigr)^{-1}\bigl\{t^{-3/4} + \bigl(t+M^2\bigr)^{-1}\bigr\} \\
&\leqslant \mathcal{O}(e).
\end{aligned} \tag{16}$$

Hence there exists e_* such that $\|H_\mathrm{I}(H_0+\mathbb{1})^{-1}\| < 1$ for all $e < e_*$. Now we can apply the Kato–Rellich theorem [20] to obtain the assertion in Proposition 1.1.

2.4. Proof of Proposition 1.2

By (16), one has

$$\|H_\mathrm{I}\psi\| \leqslant \mathcal{O}(e)\bigl\|(H_0+\mathbb{1})\psi\bigr\|. \tag{17}$$

For each $k_0 \in \mathbb{R}^3$, choose ψ as $\mathcal{U}\psi = |B_{\varepsilon,k_0}|^{-1/2}\chi_{B_{\varepsilon,k_0}} \otimes \varphi$ where $\varphi \in \mathbb{C}^2 \otimes \mathfrak{F}_\mathrm{fin}$, χ_S is the characteristic function of the set S, $B_{\varepsilon,k_0} = \{k \in \mathbb{R}^3 \mid |k-k_0| < \varepsilon\}$ and $|B_{\varepsilon,k_0}| = 4\pi\varepsilon^3/3$. It follows from (17) that

$$|B_{\varepsilon,k_0}|^{-1}\int_{B_{\varepsilon,k_0}} \mathrm{d}k\,\bigl\|H_\mathrm{I}(k)\varphi\bigr\|^2 \leqslant \mathcal{O}\bigl(e^2\bigr)|B_{\varepsilon,k_0}|^{-1}\int_{B_{\varepsilon,k_0}} \mathrm{d}k\,\bigl\|\bigl(H_0(k)+\mathbb{1}\bigr)\varphi\bigr\|^2, \tag{18}$$

where $H_\mathrm{I}(P) = |D(P)| - |D_0(P)|$. Since $H_\mathrm{I}(P)\varphi$ and $(H_0(P)+\mathbb{1})\varphi$ are strongly continuous in P, we can take the limit as $\varepsilon \downarrow 0$ and obtain that

$$\bigl\|H_\mathrm{I}(k_0)\varphi\bigr\| \leqslant \mathcal{O}(e)\bigl\|\bigl(H_0(k_0)+\mathbb{1}\bigr)\varphi\bigr\|.$$

Since k_0 is arbitrary and $\mathbb{C}^2 \otimes \mathfrak{F}_{\mathrm{fin}}$ is a core of $H_0(P)$, we have that $\|H_{\mathrm{I}}(P)(H_0(P)+\mathbb{1})^{-1}\| \leqslant \mathcal{O}(e)$ for all P. Now we can apply the Kato–Rellich theorem [20] and obtain the assertion in the proposition. □

3. Spectral properties

3.1. Preliminaries

In this section, we will prove Theorem 1.3. To this end, we need some preliminaries.

Let j_1 and j_2 be two localization functions on $\mathbb{R}^3$ so that $j_1^2 + j_2^2 = 1$ and j_1 is supported in a ball of radius R. For each vector $f = f(k,\lambda)$ in $L^2(\mathbb{R}^3 \times \{1,2\})$, we define an operator $\mathscr{J}_i$ $(i = 1,2)$ by

$$(\mathscr{J}_i f)(k,\lambda) = j_i(-\mathrm{i}\nabla_k) f(k,\lambda).$$

Now we define a linear operator $\mathscr{J} : L^2(\mathbb{R}^3 \times \{1,2\}) \to L^2(\mathbb{R}^3 \times \{1,2\}) \oplus L^2(\mathbb{R}^3 \times \{1,2\})$ by

$$\mathscr{J} f = \mathscr{J}_1 f \oplus \mathscr{J}_2 f$$

for each $f \in L^2(\mathbb{R}^3 \times \{1,2\})$.

Let U be the natural isometry from $\mathfrak{F}(L^2(\mathbb{R}^3 \times \{1,2\}) \oplus L^2(\mathbb{R}^3 \times \{1,2\}))$ to $\mathfrak{F} \otimes \mathfrak{F}$ where $\mathfrak{F}(L^2(\mathbb{R}^3 \times \{1,2\}) \oplus L^2(\mathbb{R}^3 \times \{1,2\}))$ is the Fock space over $L^2(\mathbb{R}^3 \times \{1,2\}) \oplus L^2(\mathbb{R}^3 \times \{1,2\})$, see Appendix A. Concrete action of U is given by

$$\begin{aligned} &U a(f_1 \oplus g_1)^* \ldots a(f_n \oplus g_n)^* \Omega^{\oplus} \\ &\quad = \left[a(f_1)^* \otimes \mathbb{1} + \mathbb{1} \otimes a(g_1)^*\right] \ldots \left[a(f_n)^* \otimes \mathbb{1} + \mathbb{1} \otimes a(g_n)^*\right] \Omega \otimes \Omega, \end{aligned}$$

where $\Omega^{\oplus}$ is the Fock vacuum in $\mathfrak{F}(L^2(\mathbb{R}^3 \times \{1,2\}) \oplus L^2(\mathbb{R}^3 \times \{1,2\}))$. The following operator

$$\check{\Gamma}(\mathscr{J}) := U\Gamma(\mathscr{J})$$

plays an important role in our proof. The importance of $\check{\Gamma}(\mathscr{J})$ was discovered by Dereziński and Gérard [4].

In Appendix C we show the following formula.

Lemma 3.1 *(Localization formula). Let*

$$H^{\otimes}(P) = \gamma \sqrt{\left\{\sigma \cdot \left(P - P_{\mathrm{f}} \otimes \mathbb{1} - \mathbb{1} \otimes P_{\mathrm{f}} + eA(0) \otimes \mathbb{1}\right)\right\}^2 + M^2} + H_{\mathrm{f}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}}$$

acting in $\mathbb{C}^2 \otimes \mathfrak{F} \otimes \mathfrak{F}$. Choose e as $e < e_*$, where e_* is given in Proposition 1.2*. Then, for all $\varphi \in \mathbb{C}^2 \otimes \mathfrak{F}_{\mathrm{fin}} \otimes \mathfrak{F}_{\mathrm{fin}}$, one obtains*

$$\left|\left\langle \varphi, \left(H(P) - \check{\Gamma}(\mathscr{J})^* H^{\otimes}(P) \check{\Gamma}(\mathscr{J})\right)\varphi\right\rangle\right| \leqslant o_R(1) \left\|\left(H(P) + \mathbb{1}\right)\varphi\right\|^2,$$

where $o_R(1)$ is a function of R vanishing as $R \to \infty$.

Finally we note the following lemma.

Lemma 3.2. *One has*

$$H^{\otimes}(P) \geqslant E(P) + \Delta(P)(\mathbb{1} - P_{\Omega}),$$

where P_{Ω} is the orthogonal projection onto $\mathbb{C}^2 \otimes \mathfrak{F} \otimes \Omega$.

Proof. Remark the following natural identification,

$$\mathbb{C}^2 \otimes \mathfrak{F} \otimes \mathfrak{F} = \sum_{n \geqslant 0}^{\oplus} \mathbb{C}^2 \otimes \mathfrak{F} \otimes L^2\big(\mathbb{R}^3 \times \{1,2\}\big)^{\otimes_s n}.$$

Set $\mathscr{H}_n = \mathbb{C}^2 \otimes \mathfrak{F} \otimes L^2(\mathbb{R}^3 \times \{1,2\})^{\otimes_s n}$. Each vector $\varphi \in \mathscr{H}_n$ can be expressed as a $\mathbb{C}^2 \otimes \mathfrak{F}$-valued symmetric function on $(\mathbb{R}^3 \times \{1,2\})^{\times n}$:

$$\varphi = \varphi(k_1, \lambda_1, \dots, k_n, \lambda_n).$$

Under this identification, the action of $H^{\otimes}(P)$ is given by

$$\begin{aligned} &\big(H^{\otimes}(P)\varphi\big)(k_1, \lambda_1, \dots, k_n, \lambda_n) \\ &\quad = \left(H\left(P - \sum_{i=1}^n k_i\right) + \sum_{i=1}^n \omega(k_i)\right)\varphi(k_1, \lambda_1, \dots, k_n, \lambda_n) \end{aligned}$$

for a suitable $\varphi \in \mathscr{H}_n$. Thus, using the triangle inequality $\omega(k_1 + k_2) \leqslant \omega(k_1) + \omega(k_2)$, one has

$$\begin{aligned} \big\langle \varphi, H^{\otimes}(P)\varphi\big\rangle = \sum_{\lambda_1, \dots, \lambda_n = 1,2} \int \mathrm{d}k_1 \dots \mathrm{d}k_n \Bigg\langle &\varphi(k_1, \lambda_1, \dots, k_n, \lambda_n), \\ &\left(H\left(P - \sum_{i=1}^n k_i\right) + \sum_{i=1}^n \omega(k_i)\right)\varphi(k_1, \lambda_1, \dots, k_n, \lambda_n)\Bigg\rangle \\ \geqslant \big(\Delta(P) + E(P)\big)\|\varphi\|^2. \end{aligned}$$

For $n = 0$, we have $H^{\otimes}(P) \restriction \mathscr{H}_0 = H(P)$. Combining the results, one reaches the assertion in the lemma. □

3.2. *Proof of Theorem 1.3*

3.2.1. *Lower bound of $\Sigma(P) - E(P)$*

In this sub-subsection, we will show the following lower bound.

Proposition 3.3. *Choose $e < e_*$. Then one has that $\Sigma(P) - E(P) \geqslant \Delta(P)$.*

Proof. For any $\lambda \in \text{ess.spec}(H(P))$, we can find a sequence $\{\varphi_n\}_n$ such that $\|\varphi_n\| = 1$, $\text{w-}\lim_{n\to\infty} \varphi_n = 0$ and $\lim_{n\to\infty} \|(H(P)-\lambda)\varphi_n\| = 0$. For each $n \in \mathbb{N}$, one has

$$\langle \varphi_n, H(P)\varphi_n \rangle \geqslant \langle \varphi_n, \check{\Gamma}(\mathscr{J})^* H^{\otimes}(P)\check{\Gamma}(\mathscr{J})\varphi_n \rangle - o_R(1)\big\|\big(H(P)+\mathbb{1}\big)\varphi\big\|^2$$

by Lemma 3.1. Thus using Lemma 3.2 one gets

$$\langle \varphi_n, H(P)\varphi_n \rangle \geqslant E(P) + \Delta(P) - \Delta(P)\big\|P_\Omega \check{\Gamma}(\mathscr{J})\varphi_n\big\|^2 - o_R(1)\big\|\big(H(P)+\mathbb{1}\big)\varphi_n\big\|^2. \quad (19)$$

First we will show that $\lim_{n\to\infty} \|P_\Omega \check{\Gamma}(\mathscr{J})\varphi_n\| = 0$. Remark that $\|P_\Omega \check{\Gamma}(\mathscr{J})\varphi_n\| = \|\Gamma(\mathscr{J}_1)\varphi_n\|$. With N_{f} the number operator given by

$$N_{\mathrm{f}} = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \mathrm{d}k\, a(k,\lambda)^* a(k,\lambda),$$

we also remark that $\langle \varphi_n, N_{\mathrm{f}}\varphi_n \rangle$ is uniformly bounded in n because

$$\langle \varphi_n, N_{\mathrm{f}}\varphi_n \rangle \leqslant m_{\mathrm{ph}}^{-1} \langle \varphi_n, H_{\mathrm{f}}\varphi_n \rangle \leqslant m_{\mathrm{ph}}^{-1} \langle \varphi_n, H(P)\varphi_n \rangle.$$

Thus $\|(\mathbb{1} - \chi_N(N_{\mathrm{f}}))\Gamma(\mathscr{J}_1)\varphi\| \leqslant \|(\mathbb{1} - \chi_N(N_{\mathrm{f}}))\varphi_n\| = o_N(1)$ holds where $o_N(1)$ is a function of N, independent of n, vanishing as $N \to \infty$. Here $\chi_N(s) = 1$ if $0 \leqslant s \leqslant N$ and $\chi(s) = 0$ otherwise, moreover $\chi_N(N_{\mathrm{f}})$ is defined by the functional calculus. On the other hand, $\chi_N(N_{\mathrm{f}})(H_{\mathrm{f}} + \mathbb{1})^{-1/2}\Gamma(\mathscr{J}_1)$ is compact for all N. Thus one finds that

$$\begin{aligned}
&\big\|P_\Omega \check{\Gamma}(\mathscr{J})\varphi_n\big\|^2 \\
&\quad\leqslant 2\big\|\chi_N(N_{\mathrm{f}})\Gamma(\mathscr{J}_1)\varphi_n\big\|^2 + 2\big\|\big(\mathbb{1} - \chi_N(N_{\mathrm{f}})\big)\varphi_n\big\|^2 \\
&\quad= 2\big\langle \chi_N(N_{\mathrm{f}})(H_{\mathrm{f}}+\mathbb{1})^{-1/2}\Gamma(\mathscr{J}_1)^2\varphi_n, (H_{\mathrm{f}}+\mathbb{1})^{1/2}\varphi_n \big\rangle + o_N(1) \\
&\quad= 2\big\|\chi_N(N_{\mathrm{f}})(H_{\mathrm{f}}+\mathbb{1})^{-1/2}\Gamma(\mathscr{J}_1)^2\varphi_n\big\|\,\big\|(H_{\mathrm{f}}+\mathbb{1})^{1/2}\big(H(P)+\mathbb{1}\big)^{-1/2}\big\| \\
&\qquad\times \big\|\big(H(P)+\mathbb{1}\big)^{1/2}\varphi_n\big\| + o_N(1).
\end{aligned}$$

First we take the limit $n \to \infty$. Then, by the compactness of the linear operator $\chi_N(N_{\mathrm{f}})(H_{\mathrm{f}} + \mathbb{1})^{-1/2}\Gamma(\mathscr{J}_1)$, the vector $\chi_N(N_{\mathrm{f}})(H_{\mathrm{f}} + \mathbb{1})^{-1/2}\Gamma(\mathscr{J}_1)^2\varphi_n$ converges to 0 strongly which implies that $\limsup_{n\to\infty} \|P_\Omega \check{\Gamma}(\mathscr{J})\varphi_n\| \leqslant o_N(1)$. Then taking $N \to \infty$, one concludes that $\lim_{n\to\infty} \|P_\Omega \check{\Gamma}(\mathscr{J})\varphi_n\| = 0$.

Taking the limit $n \to \infty$ in both side of (19), one finds

$$\lambda \geqslant E(P) + \Delta(P) - o_N(1)(\lambda+1)^2.$$

Finally taking $R \to \infty$, one obtains the desired assertion. □

3.2.2. Upper bound of $\Sigma(P) - E(P)$

We will complete our proof of Theorem 1.3 by showing the following upper bound.

Proposition 3.4. *Choose e as $e < e_*$. Then we have that $\Sigma(P) - E(P) \leqslant \Delta(P)$.*

Proof. For notational simplicity we set $\gamma = 1$ in this proof. For each $k_0 \in \mathbb{R}^3$, let us define

$$f_{\varepsilon,k_0} = |B_{\varepsilon,k_0}|^{-1/2}\chi_{B_{\varepsilon,k_0}},$$
$$B_{\varepsilon,k_0} = \{k \in \mathbb{R}^3 \mid |k - k_0| \leqslant \varepsilon\},$$

where χ_A is the characteristic function of the measurable set A and $|A|$ means the Lebesgue measure of A. Choose a normalized vector $\varphi_\varepsilon \in \operatorname{ran} E_\Delta(H(P - k_0))$ with $\Delta = [-\varepsilon + z, z + \varepsilon]$, $z = E(P - k_0) + \omega(k_0)$, $\varepsilon > 0$. Here for a self-adjoint operator A, $E_\Delta(A)$ stands for the spectral measure of A for the interval Δ. Let $a_\lambda(f) = \int_{\mathbb{R}^3} \mathrm{d}k\, f(k)^* a(k, \lambda)$. We will show that $a_\lambda(f_{\varepsilon,k_0})^*\varphi_\varepsilon / \|a_\lambda(f_{\varepsilon,k_0})^*\varphi_\varepsilon\|$ is a Weyl sequence for z as $\varepsilon \downarrow 0$. Applying the pull-through formula, one has

$$\begin{aligned} &\big\langle \big(H(P) - z\big) a_\lambda(f_{\varepsilon,k_0})^* \varphi_\varepsilon, \psi\big\rangle \\ &\quad = \int_{\mathbb{R}^3} \mathrm{d}k\, f_{\varepsilon,k_0}(k) \big\{ \big\langle \big(H(P-k) + \omega(k) - z\big)\varphi_\varepsilon, a(k,\lambda)\psi\big\rangle - \big\langle S_{k,\lambda}(P)^*\varphi_\varepsilon, \psi\big\rangle \big\} \end{aligned} \tag{20}$$

for each normalized $\psi \in \mathbb{C}^2 \otimes \mathfrak{F}_{\mathrm{fin}}$, where

$$S_{k,\lambda}(P) = \big|D(P-k)\big| a(k,\lambda) - a(k,\lambda)\big|D(P)\big|.$$

As to the second term in the right-hand side of (20), observe that

$$\begin{aligned} &\left| \int_{\mathbb{R}^3} \mathrm{d}k\, f_{\varepsilon,k_0}(k) \big\langle S_{k,\lambda}(P)^* \varphi_\varepsilon, \psi\big\rangle \right| \\ &\quad \leqslant \int_{\mathbb{R}^3} \mathrm{d}k\, f_{\varepsilon,k_0}(k) \big\| S_{k,\lambda}(P)^* \varphi_\varepsilon \big\| \|\psi\| \\ &\quad \leqslant \int_{\mathbb{R}^3} \mathrm{d}k\, f_{\varepsilon,k_0}(k) \big\| S_{k,\lambda}(P)^* \big(H(P-k) + \mathbb{1}\big)^{-1} \big\| \big\| \big(H(P-k) + \mathbb{1}\big)\varphi_\varepsilon \big\| \\ &\quad \leqslant C \int_{\mathbb{R}^3} \mathrm{d}k\, f_{\varepsilon,k_0}(k) \big(1 + |k|\big) \big|F_0(k,\lambda)\big| \big(E(P-k_0) + 1 + \omega(k_0) + \mathcal{O}\big(|k - k_0|\big) + \varepsilon\big) \end{aligned}$$

by Lemma D.6 and (29) below, where

$$F_x(k,\lambda) = e \frac{\chi_\Lambda(k)\varepsilon(k,\lambda)}{\sqrt{2(2\pi)^3\omega(k)}} \mathrm{e}^{-\mathrm{i}k\cdot x}. \tag{21}$$

Clearly the right-hand side of the above inequality converges to 0 as $\varepsilon \downarrow 0$ because f_{ε,k_0} weakly converges to 0 in $L^2(\mathbb{R}^3)$. Next we will estimate the first term in the right-hand side of (20). One has

$$\left|\int_{\mathbb{R}^3} \mathrm{d}k\, f_{\varepsilon,k_0}(k)\big\langle\big(H(P-k)+\omega(k)-z\big)\varphi_\varepsilon, a(k,\lambda)\psi\big\rangle\right|$$

$$\leqslant \left[\int_{\mathbb{R}^3} \mathrm{d}k\, f_{\varepsilon,k_0}(k)^2 \big\|(N_\mathrm{f}+2)^{1/2}\big(H(P-k)+\omega(k)-z\big)\varphi_\varepsilon\big\|^2\right]^{1/2} \tag{22}$$

$$\times \left[\int_{\mathbb{R}^3} \mathrm{d}k\, \big\|(N_\mathrm{f}+2)^{-1/2}a(k,\lambda)\psi\big\|^2\right]^{1/2}. \tag{23}$$

The term (23) is less than $\|\psi\|^2(=1)$ because

$$\begin{aligned}\int_{\mathbb{R}^3} \mathrm{d}k\, \big\|(N_\mathrm{f}+2)^{-1/2}a(k,\lambda)\psi\big\|^2 &\leqslant \int_{\mathbb{R}^3} \mathrm{d}k\, \big\langle a(k,\lambda)\psi, a(k,\lambda)(N_\mathrm{f}+\mathbb{1})^{-1}\psi\big\rangle\\ &= \big\langle\psi, N_\mathrm{f}(N_\mathrm{f}+\mathbb{1})^{-1}\psi\big\rangle\\ &\leqslant \|\psi\|^2.\end{aligned}$$

As to the term (22) we need a lengthy calculation below. Note that, since $N_\mathrm{f}+2\leqslant m_\mathrm{ph}^{-1}H(P-k)+2$, one has

$$\begin{aligned}&\big\|(N_\mathrm{f}+2)^{1/2}\big(H(P-k)+\omega(k)-z\big)\varphi_\varepsilon\big\|\\ &\quad\leqslant C\big\|\big(H(P-k)+2\big)^{1/2}\big(H(P-k)+\omega(k)-z\big)\varphi_\varepsilon\big\|\\ &\quad\leqslant C\big\|\big(H(P-k)+\omega(k)-z\big)^{3/2}\varphi_\varepsilon\big\| &(24)\\ &\qquad+C\big|2-z+\omega(k)\big|^{1/2}\big\|\big(H(P-k)+\omega(k)-z\big)\varphi_\varepsilon\big\|. &(25)\end{aligned}$$

Note that

$$\begin{aligned}&H(P-k)+\omega(k)-z\\ &\quad=\big(H(P-k_0)+\omega(k_0)-z\big)+\big(H(P-k)-H(P-k_0)+\omega(k)-\omega(k_0)\big).\end{aligned}$$

Thus one has

$$\begin{aligned}&\big\|\big(H(P-k)+\omega(k)-z\big)\varphi_\varepsilon\big\|\\ &\quad\leqslant\big\|\big(H(P-k_0)+\omega(k_0)-z\big)\varphi_\varepsilon\big\|+\big\|\big(H(P-k)-H(P-k_0)\big)\varphi_\varepsilon\big\|+\big|\omega(k)-\omega(k_0)\big|\\ &\quad\leqslant\varepsilon+\big\|\big(\big|D(P-k)\big|-\big|D(P-k_0)\big|\big)\varphi_\varepsilon\big\|+\big|\omega(k)-\omega(k_0)\big|. &(26)\end{aligned}$$

We will show that

$$\big\|\big(\big|D(P-k)\big|-\big|D(P-k_0)\big|\big)\big(H(P-k_0)+\mathbb{1}\big)^{-1}\big\| \leqslant \mathcal{O}\big(|k-k_0|\big). \tag{27}$$

To see this, we just note that, by (14),

$$\begin{aligned}&\big|D(P-k)\big|-\big|D(P-k_0)\big|\\&\quad=\frac{1}{\pi}\int_{M^2}^{\infty}\mathrm{d}t\,\sqrt{t}\big(t+\hat{D}(P-k)^2\big)^{-1}\big\{2(-k+k_0)\cdot P_{\mathrm{f}}+k^2-k_0^2+2e(-k+k_0)\cdot A(0)\big\}\\&\qquad\times\big(t+\hat{D}(P-k_0)^2\big)^{-1}.\end{aligned}$$

Hence, by Lemma D.3, one obtains

$$\begin{aligned}&\big\|\big(\big|D(P-k)\big|-\big|D(P-k_0)\big|\big)\big(H(P-k)+\mathbb{1}\big)^{-1}\big\|\\&\quad\leqslant\frac{1}{\pi}\int_{M^2}^{\infty}\mathrm{d}t\,\sqrt{t}t^{-1}\big\|\big\{2(-k+k_0)\cdot P_{\mathrm{f}}+k^2-k_0^2+2e(-k+k_0)\cdot A(0)\big\}(H_{\mathrm{f}}+\mathbb{1})^{-1}\big\|\\&\qquad\times\underbrace{\big\|(H_{\mathrm{f}}+\mathbb{1})\big(t+\hat{D}(P-k_0)^2\big)^{-1}\big(H(P-k_0)+\mathbb{1}\big)^{-1}\big\|}_{\leqslant C(t^{-1}+t^{-3/2}+t^{-2})\text{ by Lemma D.3}}\\&\quad\leqslant\mathcal{O}\big(|k-k_0|\big)\end{aligned} \tag{28}$$

which implies

$$\big\|\big(H(P-k)+\omega(k)-z\big)\varphi_\varepsilon\big\| \leqslant \mathcal{O}\big(|k-k_0|\big)+\varepsilon \tag{29}$$

by (26). As a consequence, the term (25) is estimated as

$$(25)\leqslant\big(\mathcal{O}\big(|k-k_0|\big)+\varepsilon\big)\big|2-z+\omega(k)\big|^{1/2}. \tag{30}$$

To estimate (24), observe that

$$\begin{aligned}&\big\|\big(H(P-k)+\omega(k)-z\big)^{3/2}\varphi_\varepsilon\big\|^2\\&\quad\leqslant\big\|\big(H(P-k)+\omega(k)-z\big)^2\varphi_\varepsilon\big\|\underbrace{\big\|\big(H(P-k)+\omega(k)-z\big)\varphi_\varepsilon\big\|}_{\leqslant\mathcal{O}(|k-k_0|)+\varepsilon\text{ by (29)}}.\end{aligned} \tag{31}$$

Since $\|(H(P-k)+\omega(k)-z)^2(H(P-k_0)+1)^{-2}\|$ is uniformly bounded for $k\in B_{\varepsilon,k_0}$ by Lemma D.5, we obtain

$$(24)\leqslant\mathcal{O}\big(|k-k_0|\big)+C\varepsilon. \tag{32}$$

Collecting (30) and (32), we arrive at

$$\int_{\mathbb{R}^3} \mathrm{d}k\, f_{\varepsilon,k_0}(k)^2 \big\|(N_\mathrm{f}+\mathbb{1})^{1/2}\big(H(P-k)+\omega(k)-z\big)\varphi_\varepsilon\big\|^2$$
$$\leqslant \int_{\mathbb{R}^3} \mathrm{d}k\, f_{\varepsilon,k_0}(k)^2\big(\mathcal{O}\big(|k-k_0|\big)+\varepsilon\big)$$
$$= o_\varepsilon(1).$$

This means that

$$\big\|\big(H(P)-z\big)a_\lambda(f_{\varepsilon,k_0})^*\varphi_\varepsilon\big\| \leqslant o_\varepsilon(1). \tag{33}$$

Set $\Psi_\varepsilon = a_\lambda(f_{\varepsilon,k_0})^*\varphi_\varepsilon/\|a_\lambda(f_{\varepsilon,k_0})^*\varphi_\varepsilon\|$. (Note that, by the CCRs, $\|a_\lambda(f_{\varepsilon,k_0})^*\varphi_\varepsilon\|^2 = \|f_{\varepsilon,k_0}\|^2\|\varphi_\varepsilon\|^2 + \|a_\lambda(f_{\varepsilon,k_0})\varphi_\varepsilon\|^2 \geqslant 1$.) Then one can easily see that Ψ_ε weakly converges to 0 as $\varepsilon \downarrow 0$ and, by (33), $\lim_{\varepsilon\downarrow 0}\|(H(P)-z)\Psi_\varepsilon\| = 0$. Hence $\{\Psi_\varepsilon\}$ is a Weyl sequence. Thus $z = E(P-k_0)+\omega(k_0) \in \mathrm{ess.spec}(H(P))$. Since k_0 is arbitrary, one has the desired assertion in the proposition. □

4. Degenerate eigenvalues

4.1. Abstract Kramers' degeneracy theorem

The following lemma is well known as the *Kramers' degeneracy theorem* which plays a central role in this section.

Lemma 4.1 *(Abstract Kramers' degeneracy theorem). Let ϑ be an antiunitary operator with $\vartheta^2 = -\mathbb{1}$.* (*In applications ϑ is mostly the time reversal operator.*) *Let H be a self-adjoint operator. Assume that H commutes with ϑ. Then each eigenvalue of H is at least doubly degenerate.*

Proof. Let ψ be an eigenvector of H for the eigenvalue μ. The commutativity between H and ϑ implies

$$H\vartheta\psi = \vartheta H\psi = \mu\vartheta\psi.$$

Hence $\vartheta\psi$ is an eigenvector for the same eigenvalue μ.

The antiunitarity of ϑ means that

$$\langle\vartheta\psi, \vartheta\eta\rangle = \langle\eta, \psi\rangle.$$

Therefore

$$\big\langle\vartheta(\vartheta\psi), \vartheta\psi\big\rangle = \langle\psi, \vartheta\psi\rangle$$

which implies $\langle\psi, \vartheta\psi\rangle = 0$, using $\vartheta^2 = -\mathbb{1}$. □

4.2. *Reality preserving operators and degenerate eigenvalues*

Recall that the Hamiltonian $H(P)$ is living in $\mathbb{C}^2 \otimes \mathfrak{F}$. Each vector $\varphi \in \mathbb{C}^2 \otimes \mathfrak{F}$ has the following expression:

$$\varphi = \varphi_1 \oplus \varphi_2,$$
$$\varphi_i = \sum_{n \geqslant 0}^{\oplus} \varphi_i^{(n)}(k_1, \lambda_1, \dots, k_n, \lambda_n), \quad i = 1, 2,$$

under the identification $\mathbb{C}^2 \otimes \mathfrak{F} = \mathfrak{F} \oplus \mathfrak{F}$. For each $\varphi \in \mathbb{C}^2 \otimes \mathfrak{F}$, set

$$J\varphi = j\varphi_1 \oplus j\varphi_2,$$
$$j\varphi_i = \sum_{n \geqslant 0}^{\oplus} \overline{\varphi}_i^{(n)}(k_1, \lambda_1, \dots, k_n, \lambda_n), \quad i = 1, 2.$$

We say that a linear operator A on $\mathfrak{F}$ *preserves the reality with respect to* j if A commutes with j.

Since $a(k, \lambda)$ acts by

$$a(k, \lambda)\varphi_i = \sum_{n \geqslant 0}^{\oplus} \sqrt{n+1}\varphi_i^{(n+1)}(k, \lambda, k_1, \lambda_1, \dots, k_n, \lambda_n),$$

one has $ja(k, \lambda) = a(k, \lambda)j$ and $ja(k, \lambda)^* = a(k, \lambda)^*j$ which imply

$$jP_{\mathrm{f}} = P_{\mathrm{f}}j, \tag{34}$$

$$jA(0) = A(0)j, \tag{35}$$

$$jB(0) = -B(0)j, \tag{36}$$

$$jH_{\mathrm{f}} = H_{\mathrm{f}}j, \tag{37}$$

that is, P_{f}, $A(0)$, $\mathrm{i}B(0)$ and H_{f} preserve the reality with respect to j. (Here $B(0) = \nabla \wedge A(0)$.)

Proposition 4.2. *Let* ϑ *be given by*

$$\vartheta = \sigma_2 J.$$

Then ϑ *is an antiunitary operator satisfying* $\vartheta^2 = -\mathbb{1}$. *Moreover, for all* P *and* e, *we obtain that*

$$\vartheta H(P) = H(P)\vartheta. \tag{38}$$

Thus, by Lemma 4.1, *each eigenvalue of* $H(P)$ *is* at least *doubly degenerate.*

Proof. Since $\sigma_2^2 = \mathbb{1}$, one easily sees the antiunitarity of ϑ. Furthermore using the anticommutativity $\sigma_2 J = -J\sigma_2$, one has $\vartheta^2 = -\mathbb{1}$.

Next we will show (38). Since H_{f} commutes with ϑ by (37), it suffices to show that $|D(P)|$ commutes with ϑ. Our basic idea is simple. Noting the fact $\vartheta\sigma_i = -\sigma_i\vartheta$ for $i = 1, 2, 3$, we can easily see that ϑ commutes with $|D(P)|^2 (= (P - P_{\mathrm{f}} + eA(0))^2 + e\sigma \cdot B(0) + M^2)$ on $\mathbb{C}^2 \otimes \mathfrak{F}_{\mathrm{fin}}$ by (34)–(36). As a consequence we could expect that $|D(P)|$ $(= \sqrt{(P - P_{\mathrm{f}} + eA(0))^2 + e\sigma \cdot B(0) + M^2})$ also commutes with ϑ.

Unfortunately since we do not know whether the subspace $\mathbb{C}^2 \otimes \mathfrak{F}_{\mathrm{fin}}$ is a core of $D(P)^2$ or not, the above arguments are somehow formal. However we can rigorize the arguments as follow. To clarify the M dependence, we write $D(P)$ as $D_M(P)$ in this proof. Let $\tilde{\vartheta} = \vartheta \oplus \vartheta$ acting in $\mathbb{C}^4 \otimes \mathfrak{F}$. Then we see that $\alpha_i\tilde{\vartheta} = -\tilde{\vartheta}\alpha_i$ and $\beta\tilde{\vartheta} = \tilde{\vartheta}\beta$ which imply $\tilde{\vartheta} D_M(P)\tilde{\vartheta}^{-1} = -D_{-M}(P)$ on $\mathbb{C}^4 \otimes \mathfrak{F}_{\mathrm{fin}}$. Since we have already seen that $\mathbb{C}^4 \otimes \mathfrak{F}_{\mathrm{fin}}$ is a core of $D_M(P)$ in Section 2.1, this equality holds as an operator equality. Hence, by the functional calculus, one has $\tilde{\vartheta} f(D_M(P))\tilde{\vartheta}^{-1} = f(-D_{-M}(P))$, where f is real-valued. In the case where $f(s) = \sqrt{s^2}$, we have $f(-D_{-M}(P)) = f(D_M(P))$ because $D_{-M}(P)^2 = D_M(P)^2$ by the anticommutativity between $M\beta$ and $D_{M=0}(P)$. Now one can conclude that $\tilde{\vartheta}|D_M(P)|\tilde{\vartheta}^{-1} = |D_M(P)|$ holds as an operator equality. □

4.3. Comments on related models

The arguments in this section are applicable to other models, e.g.,

$$H_{\mathrm{NR},V} = \frac{1}{2M}\big(\sigma \cdot \big(-\mathrm{i}\nabla_x + eA(x)\big)\big)^2 + V(x) + H_{\mathrm{f}},$$

$$H_{\mathrm{NR}}(P) = \frac{1}{2M}\big(P - P_{\mathrm{f}} + eA(0)\big)^2 + \frac{e}{2M}\sigma \cdot B(0) + H_{\mathrm{f}},$$

$$H_V = \sqrt{\big(-\mathrm{i}\nabla_x + eA(x)\big)^2 + e\sigma \cdot B(x) + M^2} + V(x) + H_{\mathrm{f}}$$

with $V(x) = V(-x)$. As regards to $H_{\mathrm{NR}}(P)$, most of the arguments of Section 4.2 are valid. However, for $H_{\mathrm{NR},V}$ and H_V, we have to change the definition of j. H_V is acting in the Hilbert space $L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathfrak{F}$. Each vector φ in $L^2(\mathbb{R}^3; \mathbb{C}^2) \otimes \mathfrak{F}$ has the form

$$\varphi = \varphi_1 \oplus \varphi_2,$$

$$\varphi_i = \sum_{n\geqslant 0}^{\oplus} \varphi_i^{(n)}(x; k_1, \lambda_1, \ldots, k_n, \lambda_n), \quad i = 1, 2.$$

In this case, we define j as

$$j\varphi_i^{(n)} = \sum_{n\geqslant 0}^{\oplus} \overline{\varphi}_i^{(n)}(-x; k_1, \lambda_1, \ldots, k_n, \lambda_n), \quad i = 1, 2.$$

Then one can check that

$$j(-\mathrm{i}\nabla_x) = (-\mathrm{i}\nabla_x)j,$$

$$jA(x) = A(x)j,$$

$$j(\mathrm{i}B(x)) = (\mathrm{i}B(x))j,$$

$$jV(x) = V(x)j,$$
$$jH_{\mathrm{f}} = H_{\mathrm{f}}j,$$

namely, all these operators preserve the reality with the respect to this new j. Hence defining the time reversal operator as $\vartheta = \sigma_2 J$, one can see that H_V commutes with ϑ. Thus using the abstract Kramers' degeneracy theorem, one concludes that each eigenvalue of H_V is at least doubly degenerate. A similar modification applies to $H_{\mathrm{NR},V}$.

5. Energy inequalities

To make sure that there is no further eigenvalues close to $E(P)$ we will find self-adjoint operators $L_+(P)$ and $L_-(P)$ such that

$$L_-(P) \leqslant H(P) \leqslant L_+(P). \tag{39}$$

$L_-(P)$, $L_+(P)$ are given below. They can be easily diagonalized. The min-max principle allows us to obtain bounds as, e.g.,

$$\Sigma(P) - E(P) \geqslant \Sigma\big(L_-(P)\big) - E\big(L_+(P)\big)$$

and more precise information, since the spectrum of $L_\pm(P)$ is available, see Section 6 for details. (Here, for a self-adjoint operator T, $\Sigma(T) = \inf \mathrm{ess.spec}(T)$ and $E(T) = \inf \mathrm{spec}(T)$.)

Proposition 5.1 *(Lower bound). For any* $0 < \gamma < 1$, $0 \leqslant m_{\mathrm{ph}}$ *and* $P \in \mathbb{R}^3$*, one has*

$$H\big(|P|u\big) \geqslant L_-(P)$$

with

$$L_-(P) = \gamma\sqrt{P^2 + M^2} + (1 - \gamma - eC_1)H_{\mathrm{f}} - eC_2 \tag{40}$$

for suitable constants $C_1, C_2 > 0$ *which are independent of* e *and* P*, where* $u = (1, 0, 0)$.

Proof.

Step 1. Let $H_{\mathrm{SL}}(P)$ be the spinless Hamiltonian. In this step, we will show the following operator inequality by extending the method in [13]:

$$H_{\mathrm{SL}}\big(|P|u\big) \geqslant \gamma\sqrt{P^2 + M^2} + (1 - \gamma - eC)H_{\mathrm{f}} - eC \tag{41}$$

with a strictly positive constant C independent of e and P. Clearly

$$\big(|P|u - P_{\mathrm{f}} + eA(0)\big)^2 \geqslant \big(|P| - P_{\mathrm{f}1} + eA(0)_1\big)^2.$$

Thus by the operator monotonicity of the square root (Lemma E.1), one has

$$H_{\mathrm{SL}}\big(|P|u\big) \geqslant \gamma\sqrt{\big(|P| - P_{\mathrm{f}1} + eA(0)_1\big)^2 + M^2} + H_{\mathrm{f}}.$$

Let $f(s)=\sqrt{s^2+M^2}$, $s\in\mathbb{R}$. By Taylor's theorem, one has

$$f\big(|P|+s\big)=f\big(|P|\big)+f'\big(|P|\big)s+\int_0^1 \mathrm{d}t\,(1-t)f''\big(|P|+ts\big)s^2$$

with $f'(s)=s/\sqrt{s^2+M^2}$ and $f''(s)=M^2/(s^2+M^2)^{3/2}$. Applying the functional calculus, we have the following operator equality

$$\begin{aligned}&\sqrt{\big(|P|-P_{\mathrm{f}1}+eA(0)_1\big)^2+M^2}\\&\quad=\sqrt{P^2+M^2}+\frac{|P|}{\sqrt{P^2+M^2}}\big(-P_{\mathrm{f}1}+eA(0)_1\big)\\&\qquad+\int_0^1 \mathrm{d}t\,(1-t)f''\big(|P|+t\big(-P_{\mathrm{f}1}+eA(0)_1\big)\big)\big(-P_{\mathrm{f}1}+eA(0)_1\big)^2.\end{aligned}\tag{42}$$

Since the last term in (42) is a positive operator, one obtains

$$\begin{aligned}&\sqrt{\big(|P|-P_{\mathrm{f}1}+eA(0)_1\big)^2+M^2}\\&\quad\geqslant\sqrt{P^2+M^2}+\frac{|P|}{\sqrt{P^2+M^2}}\big(-P_{\mathrm{f}1}+eA(0)_1\big)\\&\quad\geqslant\sqrt{P^2+M^2}-H_{\mathrm{f}}-\big\|\omega^{-1/2}F_{01}\big\|(H_{\mathrm{f}}+\mathbb{1})\end{aligned}$$

by the standard bounds $|P_{\mathrm{f}1}|\leqslant H_{\mathrm{f}}$ and $eA(0)_1\geqslant-\|\omega^{-1/2}F_{01}\|(H_{\mathrm{f}}+\mathbb{1})$. This proves (41).

Step 2. We will show that

$$\pm\big(H_{\mathrm{SL}}\big(|P|u\big)-H\big(|P|u\big)\big)\leqslant\frac{3\pi}{M}\big\|\big(1+\omega^{-1/2}\big)|k||F_0|\big\|(H_{\mathrm{f}}+\mathbb{1}).\tag{43}$$

To this end, we simply note that, by (14),

$$\begin{aligned}&H_{\mathrm{SL}}\big(|P|u\big)-H\big(|P|u\big)\\&\quad=-\frac{1}{\pi}\int_{M^2}^{\infty}\mathrm{d}s\,\sqrt{s-M^2}\big(s+\big(|P|u-P_{\mathrm{f}}+eA(0)\big)^2\big)^{-1}e\sigma\cdot B(0)\big(s+\hat{D}\big(|P|u\big)^2\big)^{-1},\end{aligned}$$

where $\hat{D}(P)=D(P)-M\beta$. Noting the facts $\|e\sigma\cdot B(0)(H_{\mathrm{f}}+\mathbb{1})^{-1/2}\|\leqslant 6\|(1+\omega^{-1/2})|k||F_0|\|$ and (D.2) in the proof of Lemma D.3, one can see that $\|(H_{\mathrm{SL}}(P)-H(P))(H_{\mathrm{f}}+\mathbb{1})^{-1/2}\|\leqslant 3\pi\|(1+\omega^{-1/2})|k||F_0|\|/M$. Now (43) is obtained.

Step 3 *(Proof of Proposition 5.1).* From (41) and (43) it follows that

$$\begin{aligned} H\big(|P|u\big) &= H_{\mathrm{SL}}\big(|P|u\big) + \big(H\big(|P|u\big) - H_{\mathrm{SL}}\big(|P|u\big)\big) \\ &\geqslant H_{\mathrm{SL}}\big(|P|u\big) - eC(H_{\mathrm{f}} + \mathbb{1}) \\ &\geqslant \gamma\sqrt{P^2+M^2} + (1-\gamma-eC_1)H_{\mathrm{f}} - eC_2. \end{aligned}$$

This proves the desired assertion in the proposition. □

Before we proceed, we remark the following. Let $SO(3)$ be the rotation group. Then there exists a unitary representation π of $SO(3)$ such that

$$\pi_g H(P)\pi_g^{-1} = H\big(g^{-1}P\big) \tag{44}$$

for all $g \in SO(3)$ and $P \in \mathbb{R}^3$, see, e.g., [24]. Thus $E(P)$ is a radial function in P.

Since $E(P)$ is rotationally symmetric in P, one has an immediate corollary.

Corollary 5.2. *Choose $\gamma < 1$ and e sufficiently small as $e < e_*$. One has*

$$E(P) \geqslant \gamma\sqrt{P^2+M^2} - eC_2$$

for all $P \in \mathbb{R}^3$, where C_2 is independent of e, P.

Proposition 5.3 *(Upper bound). One obtains*

$$H\big(|P|u\big) \leqslant L_+(P)$$

with

$$\begin{aligned} L_+(P) = \gamma\Big[&\big(|P|u - P_{\mathrm{f}}\big)^2 + 2|P|\big(H_{\mathrm{f}} + \big\|\omega^{-1/2}|F_0|\big\|\big) \\ &+ 4(H_{\mathrm{f}}+\mathbb{1})P_{\mathrm{f}}^2 + \big\|\big(1+\omega^{-1/2}\big)|F_0|\big\|^2 + \big\|\big(1+\omega^{-1/2}\big)|F_0|\big\|^2(H_{\mathrm{f}}+\mathbb{1}) \\ &+ H_{\mathrm{f}} + \big\||k|^{1/2}|F_0|\big\|^2 + M^2\Big]^{1/2} + H_{\mathrm{f}} \end{aligned} \tag{45}$$

for all P.

Proof. Observe that

$$D\big(|P|u\big)^2 = \big(|P|u - P_{\mathrm{f}}\big)^2 + 2\big(|P|u - P_{\mathrm{f}}\big)\cdot eA(0) + e^2A(0)^2 + e\sigma\cdot B(0) + M^2.$$

Using the fundamental inequalities in Appendix A, one has

$$\begin{aligned} |P|eA(0)_1 &\leqslant |P|\big(H_{\mathrm{f}} + \big\|\omega^{-1/2}|F_0|\big\|\big), \\ eA(0)\cdot P_{\mathrm{f}} &\leqslant 2(H_{\mathrm{f}}+\mathbb{1})P_{\mathrm{f}}^2 + \frac{1}{2}\big\|\big(1+\omega^{-1/2}\big)|F_0|\big\|^2, \\ e^2A(0)^2 &\leqslant \big\|\big(1+\omega^{-1/2}\big)|F_0|\big\|^2(H_{\mathrm{f}}+\mathbb{1}), \end{aligned}$$

$$e\sigma \cdot B(0) \leqslant H_{\mathrm{f}} + \big\| |k|^{1/2} |F_0| \big\|^2.$$

Applying the operator monotonicity of the square root (Lemma E.1), one concludes (45). □

By the above operator inequality, we immediately obtain

$$\begin{aligned}
&\big\langle \eta_\uparrow \otimes \Omega, H\big(|P|u\big)\eta_\uparrow \otimes \Omega\big\rangle \\
&\quad \leqslant \gamma\sqrt{|P|^2 + 2|P|\big\|\omega^{-1/2}|F_0|\big\| + 2\big\|\big(1+\omega^{-1/2}\big)|F_0|\big\|^2 + \big\||k|^{1/2}|F_0|\big\|^2 + M^2},
\end{aligned}$$

where $\eta_\uparrow = (1,0)$. Thus taking the rotational invariance of $E(P)$ in P into consideration, one has a following corollary.

Corollary 5.4. *One has*

$$E(P) \leqslant \gamma\sqrt{\big(|P| + eC_3\big)^2 + M^2 + e^2C_4} \tag{46}$$

for all P, where C_3 and C_4 are independent of P and e.

6. Proof of Theorem 1.4

6.1. Proof of Theorem 1.4(i)

For $a \in \mathbb{R}^d$, $\|a\|_{\mathbb{R}^d}$ means the standard norm in $\mathbb{R}^d$. Then one has, for example, $\omega(k) = \|(k, m_{\mathrm{ph}})\|_{\mathbb{R}^4} = \|(|k|, m_{\mathrm{ph}})\|_{\mathbb{R}^2}$. Applying the triangle inequality and Corollary 5.2, one gets

$$\begin{aligned}
&E(P-k) + \omega(k) \\
&\quad \geqslant \gamma\big\|\big(|P-k|, M\big)\big\|_{\mathbb{R}^2} + \big\|\big(|k|, m_{\mathrm{ph}}\big)\big\|_{\mathbb{R}^2} - eC \\
&\quad \geqslant \gamma\big\|\big(|P-k|, M\big)\big\|_{\mathbb{R}^2} + \gamma\big\|\big(|k|, m_{\mathrm{ph}}\big)\big\|_{\mathbb{R}^2} + (1-\gamma)\big\|\big(|k|, m_{\mathrm{ph}}\big)\big\|_{\mathbb{R}^2} - eC \\
&\quad = \gamma\big\|(P-k, M)\big\|_{\mathbb{R}^4} + \gamma\big\|(k, m_{\mathrm{ph}})\big\|_{\mathbb{R}^4} + (1-\gamma)\big\|(k, m_{\mathrm{ph}})\big\|_{\mathbb{R}^4} - eC \\
&\quad \geqslant \gamma\big\|(P, M + m_{\mathrm{ph}})\big\|_{\mathbb{R}^4} + (1-\gamma)m_{\mathrm{ph}} - eC.
\end{aligned}$$

On the other hand, since $\|\omega^{-1/2}|F_0|\|^2 = \mathcal{O}(e^2)$ etc., one has, by Corollary 5.4, that

$$\begin{aligned}
E(P) &\leqslant \gamma\Big\|\Big(|P| + eC_3, \sqrt{M^2 + \mathcal{O}\big(e^2\big)}\Big)\Big\|_{\mathbb{R}^2} \\
&\leqslant \gamma\big\|\big(|P| + eC_3, M\big)\big\|_{\mathbb{R}^2} + \gamma\Big\|\Big(0, \sqrt{M^2 + \mathcal{O}\big(e^2\big)} - M\Big)\Big\|_{\mathbb{R}^2} \\
&\leqslant \gamma\big\|(P, M)\big\|_{\mathbb{R}^4} + eC_3 + \mathcal{O}\big(e^2\big).
\end{aligned}$$

Thus the desired assertion in the lemma follows. □

6.2. Proof of Theorem 1.4(ii) and (iii)

We denote the infimum of $\mathrm{spec}(L_-(P))$ and $\mathrm{ess.spec}(L_-(P))$ by $\mathcal{E}_-(P)$ and $\Sigma_-(P)$, respectively. Clearly one has

$$\mathcal{E}_-(P) = \gamma\sqrt{P^2+M^2} - eC,$$
$$\Sigma_-(P) = \gamma\sqrt{P^2+M^2} + (1-\gamma-eC_1)m_{\mathrm{ph}} - eC_2.$$

Let $\mathcal{E}_+(P)$ be the function of P which is appearing on the right-hand side of (46). Then using similar arguments as in the proof of Theorem 1.4(i), one sees that

$$0 \leqslant \mathcal{E}_-(P) - \mathcal{E}_+(P) \leqslant C'e + \mathcal{O}(e^2) \tag{47}$$

with $e < e_*$. (Here it should be noted that e_* is independent of P.) Thus taking the fact $\mathcal{E}_-(P) \leqslant E(P) \leqslant \mathcal{E}_+(P)$ into consideration, one has

$$0 \leqslant E(P) - \mathcal{E}_-(P) \leqslant C'e + \mathcal{O}(e^2) \tag{48}$$

for $e < e_*$, which means $E(P)$ is close to $\mathcal{E}_-(P)$ uniformly in P. Also we note that

$$\Sigma_-(P) - \mathcal{E}_-(P) \geqslant (1 - eC - \gamma)m_{\mathrm{ph}}. \tag{49}$$

Let $E_1(P)$ be the first excited eigenvalue of $H(P)$ (or possibly be $\Sigma(P)$ if there is no such excited state). Then by the operator inequality (40) and the min-max principle [21], one has

$$E_1\big(|P|u\big) \geqslant \Sigma_-(P).$$

(Note that, by Proposition 4.2, $E(P)$ is always degenerate.) With the help of (44), one sees that $E_1(P)$ is radial and

$$E_1(P) \geqslant \Sigma_-(P) \tag{50}$$

for all $P \in \mathbb{R}^3$. Thus, combining this with (48), we arrive at

$$\begin{aligned} E_1(P) - E(P) &\geqslant \Sigma_-(P) - \mathcal{E}_+(P) \\ &\geqslant \Sigma_-(P) - \mathcal{E}_-(P) + \big(\mathcal{E}_-(P) - \mathcal{E}_+(P)\big) \\ &\geqslant (1 - eC_1 - \gamma)m_{\mathrm{ph}} - C'e - \mathcal{O}(e^2) \end{aligned}$$

for $e < e_*$. This proves (ii) in the theorem.

For a self-adjoint operator A, let $E_K(A)$ be its spectral measure for the interval $(-\infty, K)$ and let $P_{\mathrm{pp}}(A)$ be the projection onto the linear space spanned by all eigenstates. Since, by Proposition 5.1, one has the operator inequality $L_-(P) \leqslant H(|P|u)$, the following property holds,

$$\mathrm{tr} P_{\mathrm{pp}}\big(H\big(|P|u\big)\big) E_{\Sigma_-(P)}\big(H\big(|P|u\big)\big) \leqslant \mathrm{tr} P_{\mathrm{pp}}\big(L_-(P)\big) E_{\Sigma_-(P)}\big(L_-(P)\big) = 2$$

by the min-max principle. Applying (44), one has that

$$\mathrm{tr}P_{\mathrm{pp}}\big(H(P)\big)E_{\Sigma_-(P)}\big(H(P)\big) = \mathrm{tr}P_{\mathrm{pp}}\big(H\big(|P|u\big)\big)E_{\Sigma_-(P)}\big(H\big(|P|u\big)\big) \leqslant 2.$$

Thus $H(P)$ has at most two eigenstates with corresponding eigenvalue less than $\Sigma_-(P)$. On the other hand, one already knows that $E(P) < \Sigma_-(P) \leqslant E_1(P)$ for $e < e_*$ by (48), (49) and (50). Therefore $E(P)$ is at most doubly degenerate.

Acknowledgments

We would like to thank M. Loss for explaining to us how to use Kramers degeneracy in our context. This research is supported by the DFG under the grant SP181/24.

Appendix A. Second quantization and basic inequalities

Let $\mathfrak{h}$ be a complex Hilbert space. The Fock space over $\mathfrak{h}$ is defined by

$$\mathfrak{F}(\mathfrak{h}) = \sum_{n\geqslant 0}^{\oplus} \mathfrak{h}^{\otimes_{\mathrm{s}} n},$$

where $\mathfrak{h}^{\otimes_{\mathrm{s}} n}$ means the n-fold symmetric tensor product of $\mathfrak{h}$ with the convention $\mathfrak{h}^{\otimes_{\mathrm{s}} 0} = \mathbb{C}$. The vector $\Omega = 1 \oplus 0 \oplus \cdots \in \mathfrak{F}(\mathfrak{h})$ is called the Fock vacuum.

We denote by $a(f)$ the annihilation operator on $\mathfrak{F}(\mathfrak{h})$ with a test vector $f \in \mathfrak{h}$, its adjoint $a(f)^*$, called the creation operator, is defined by

$$a(f)^*\varphi = \sum_{n\geqslant 0}^{\oplus} \sqrt{n+1}\, f \otimes_{\mathrm{s}} \varphi^{(n)}$$

for a suitable $\varphi = \sum_{n\geqslant 0}^{\oplus} \varphi^{(n)} \in \mathfrak{F}(\mathfrak{h})$. By definition, $a(f)$ is densely defined, closed, and antilinear in f. We frequently write $a(f)^{\#}$ to denote either $a(f)$ or $a(f)^*$. Creation and annihilation operators satisfy the canonical commutation relations

$$\big[a(f), a(g)^*\big] = \langle f, g\rangle_{\mathfrak{h}} \mathbb{1},$$
$$\big[a(f), a(g)\big] = 0 = \big[a(f)^*, a(g)^*\big]$$

on a suitable subspace of $\mathfrak{F}(\mathfrak{h})$, where $\mathbb{1}$ denotes the identity operator. We introduce a particular subspace of $\mathfrak{F}(\mathfrak{h})$ which will be used frequently. Let $\mathfrak{s}$ be a subspace of $\mathfrak{h}$. We define

$$\mathfrak{F}_{\mathrm{fin}}(\mathfrak{s}) = \mathrm{Lin}\big\{a(f_1)^* \ldots a(f_n)^*\Omega,\ \Omega \ \big|\ f_1, \ldots, f_n \in \mathfrak{s},\ n \in \mathbb{N}\big\},$$

where Lin$\{\ldots\}$ means the linear span of the set $\{\ldots\}$. If $\mathfrak{s}$ is dense in $\mathfrak{h}$, so is $\mathfrak{F}_{\mathrm{fin}}(\mathfrak{s})$ in $\mathfrak{F}(\mathfrak{h})$.

For a densely defined closable operator c on $\mathfrak{h}$, $\mathrm{d}\Gamma(c) : \mathfrak{F}(\mathfrak{h}) \to \mathfrak{F}(\mathfrak{h})$ is defined by

$$\mathrm{d}\Gamma(c) \restriction \mathrm{dom}(c)^{\otimes_{\mathrm{s}} n} = \sum_{j=1}^{n} \mathbb{1} \otimes \cdots \otimes \underset{j\mathrm{th}}{c} \otimes \cdots \otimes \mathbb{1} \tag{A.1}$$

and

$$\mathrm{d}\Gamma(c)\Omega = 0$$

where $\mathrm{dom}(c)$ means the domain of the linear operator c. Here in the jth summand c is at the jth entry. Clearly $\mathrm{d}\Gamma(c)$ is closable and we denote its closure by the same symbol. As a typical example, the number operator N_f is given by $N_\mathrm{f} = \mathrm{d}\Gamma(\mathbb{1})$.

In the case where $\mathfrak{h} = L^2(\mathbb{R}^3 \times \{1,2\})$, the annihilation and creation operator can be expressed as the operator-valued distributions $a(k,\lambda)$, $a(k,\lambda)^*$ by

$$a(f) = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \mathrm{d}k\, \bar{f}(k,\lambda) a(k,\lambda), \qquad a(f)^* = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \mathrm{d}k\, f(k,\lambda) a(k,\lambda)^*.$$

Let F be a measurable function on $\mathbb{R}^3$ and let the multiplication operator associated with F be denoted by the same symbol: $(Ff)(k,\lambda) = F(k) f(k,\lambda)$ for $f \in L^2(\mathbb{R}^3 \times \{1,2\})$. Then one can formally express $\mathrm{d}\Gamma(F)$ as

$$\mathrm{d}\Gamma(F) = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \mathrm{d}k\, F(k) a(k,\lambda)^* a(k,\lambda).$$

For $F(k) = \omega(k)$, one has the expression (3) of $H_\mathrm{f} = \mathrm{d}\Gamma(\omega)$.

Lemma A.1. *One has the following.*

(i) $\|a(f)\varphi\| \leqslant \|\omega^{-1/2} f\| \|H_\mathrm{f}^{1/2}\varphi\|$.
(ii) $\|a(f)^*\varphi\| \leqslant \|(1+\omega^{-1/2}) f\| \|(H_\mathrm{f}+\mathbb{1})^{1/2}\varphi\|$.
(iii) $a(f) + a(f)^* \leqslant H_\mathrm{f} + \|\omega^{-1/2} f\|^2$.
(iv) $\|(a(f)+a(f)^*)\varphi\| \leqslant 2\|(1+\omega^{-1/2}) f\| \|(H_\mathrm{f}+\mathbb{1})^{1/2}\varphi\|$.
(v) $|\langle \varphi, a(f)^{\#_1} a(g)^{\#_2}\varphi\rangle| \leqslant \|(1+\omega^{-1/2}) f\| \|(1+\omega^{-1/2}) g\| \langle \varphi, (H_\mathrm{f}+\mathbb{1})\varphi\rangle$.

Appendix B. Invariant domains

Lemma B.1. *Let A be self-adjoint and H be positive and self-adjoint. Assume the following.*

(i) $(H+\mathbb{1})^{-1}\mathrm{dom}(A) \subseteq \mathrm{dom}(A)$.
(ii) $|\langle Hu, Au\rangle - \langle Au, Hu\rangle| \leqslant C\|(H+\mathbb{1})u\|^2$ *for all* $u \in \mathrm{dom}(A) \cap \mathrm{dom}(H)$.
(iii) $[H, A](H+\mathbb{1})^{-1}$ *can be extended to a bounded operator.*

Then one has $\mathrm{e}^{\mathrm{i}tA}\mathrm{dom}(H) = \mathrm{dom}(H)$ *for all* $t \in \mathbb{R}$.

Proof. See [5, Lemma 2]. □

Appendix C. Localization estimate

In this appendix, we will establish Lemma 3.1 which is essential for the proof of Theorem 1.3. Unfortunately the proof is technically complicated because of the square root structure. We repeat the statement which we want to prove.

Lemma 3.1. *Choose e as $e < e_*$. For all $\varphi \in \mathbb{C}^2 \otimes \mathfrak{F}_{\mathrm{fin}} \otimes \mathfrak{F}_{\mathrm{fin}}$, one obtains*

$$\left|\left\langle \varphi, \left(H(P) - \check{\Gamma}(\mathscr{J})^* H^{\otimes}(P)\check{\Gamma}(\mathscr{J})\right)\varphi\right\rangle\right| \leqslant o_R(1)\left\|\left(H(P)+\mathbb{1}\right)\varphi\right\|^2,$$

where $o_R(1)$ is a function of R vanishing as $R \to \infty$.

Proof. Let us define a Dirac operator by

$$D^{\otimes}(P) = \alpha \cdot \left(P - P_{\mathrm{f}} \otimes \mathbb{1} - \mathbb{1} \otimes P_{\mathrm{f}} + eA(0) \otimes \mathbb{1}\right) + M\beta.$$

Then this is essentially self-adjoint on $\mathbb{C}^4 \otimes \mathfrak{F}_{\mathrm{fin}} \otimes \mathfrak{F}_{\mathrm{fin}}$ by Nelson's commutator theorem with a test operator $H_{\mathrm{f}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{f}}$. We denote its closure by the same symbol. Remark that $H^{\otimes}(P)$ is defined by the similar way in Section 2.

We also introduce

$$\hat{D}(P) = D(P) - M\beta, \qquad \hat{D}^{\otimes}(P) = D^{\otimes}(P) - M\beta.$$

Using the formula (14), one has

$$\left|D(P)\right| = \frac{1}{\pi}\int_{M^2}^{\infty} \mathrm{d}t\, \frac{\hat{D}(P)^2 + M^2}{\sqrt{t - M^2}(t + \hat{D}(P)^2)},$$
$$\left|D^{\otimes}(P)\right| = \frac{1}{\pi}\int_{M^2}^{\infty} \mathrm{d}t\, \frac{\hat{D}^{\otimes}(P)^2 + M^2}{\sqrt{t - M^2}(t + \hat{D}^{\otimes}(P)^2)}.$$

Hence

$$\begin{aligned}
&\left|D(P)\right| - \check{\Gamma}(\mathscr{J})^*\left|D^{\otimes}(P)\right|\check{\Gamma}(\mathscr{J}) \\
&\quad = \frac{1}{\pi}\int_{M^2}^{\infty} \mathrm{d}t\, \sqrt{t - M^2}\left(t + \hat{D}(P)^2\right)^{-1}\left\{\hat{D}(P)G(P) + G(P)\hat{D}(P) - G(P)^2\right\} \\
&\qquad \times \left(t + \tilde{D}^{\otimes}(P)^2\right)^{-1},
\end{aligned}$$

where

$$G(P) = \hat{D}(P) - \tilde{D}^{\otimes}(P)$$

with $\tilde{D}^{\otimes}(P) = \check{\Gamma}(\mathscr{J})^*\hat{D}^{\otimes}(P)\check{\Gamma}(\mathscr{J})$. Remark the following fact

$$\left\|G(P)(N_{\mathrm{f}} + \mathbb{1})^{-1}\right\| \leqslant o_R(1),$$

see, e.g., [9,15]. (It should be noted that the positive photon mass is crucial here.) By Lemma C.1 below, we estimate as

$$\begin{aligned}
&\big\|(N_{\mathrm f}+\mathbb{1})^{-1}\big(t+\hat D(P)^2\big)^{-1}\big\{\hat D(P)G(P)+G(P)\hat D(P)-G(P)^2\big\}\\
&\qquad\times\big(t+\tilde D^{\otimes}(P)^2\big)^{-1}(N_{\mathrm f}+\mathbb{1})^{-1}\big\|\\
&\quad\leqslant 2\big\|(N_{\mathrm f}+\mathbb{1})^{-1}\big(t+\hat D(P)^2\big)^{-1}(N_{\mathrm f}+\mathbb{1})\big\|\big\|(N_{\mathrm f}+\mathbb{1})^{-1}G(P)\big\|\big\|(N_{\mathrm f}+\mathbb{1})^{-1}\hat D(P)\big\|\\
&\qquad\times\big\|(N_{\mathrm f}+\mathbb{1})\big(t+\tilde D^{\otimes}(P)^2\big)^{-1}(N_{\mathrm f}+\mathbb{1})^{-1}\big\|\\
&\qquad+\big\|(N_{\mathrm f}+\mathbb{1})^{-1}\big(t+\hat D(P)^2\big)^{-1}(N_{\mathrm f}+\mathbb{1})\big\|\big\|(N_{\mathrm f}+\mathbb{1})^{-1}G(P)\big\|^2\\
&\qquad\times\big\|(N_{\mathrm f}+\mathbb{1})\big(t+\tilde D^{\otimes}(P)^2\big)^{-1}(N_{\mathrm f}+\mathbb{1})^{-1}\big\|\\
&\quad\leqslant o_R(1)\big(t^{-2}+t^{-3/2}+t^{-1}\big)^2
\end{aligned}$$

which implies

$$\big\|(N_{\mathrm f}+\mathbb{1})^{-1}\big(\big|D(P)\big|-\check\Gamma(\mathscr J)^*\big|D^{\otimes}(P)\big|\check\Gamma(\mathscr J)\big)(N_{\mathrm f}+\mathbb{1})^{-1}\big\|\leqslant o_R(1). \tag{C.1}$$

We also note the fact

$$\big\|(N_{\mathrm f}+\mathbb{1})^{-1}\big(H_{\mathrm f}-\check\Gamma(\mathscr J)^*(H_{\mathrm f}\otimes\mathbb{1}+\mathbb{1}\otimes H_{\mathrm f})\check\Gamma(\mathscr J)\big)(N_{\mathrm f}+\mathbb{1})^{-1}\big\|\leqslant o_R(1) \tag{C.2}$$

which is proven in [9]. Collecting (C.1) and (C.2), one sees that $|\langle\varphi,(H(P)-\check\Gamma(\mathscr J)^*H^{\otimes}(P)\check\Gamma(\mathscr J))\varphi\rangle|\leqslant o_R(1)\|(N_{\mathrm f}+\mathbb{1})\varphi\|^2$ holds.

Finally one has to show $\|(N_{\mathrm f}+\mathbb{1})\varphi\|\leqslant C\|(H(P)+\mathbb{1})\varphi\|$. The positive photon mass implies $\|(N_{\mathrm f}+\mathbb{1})\varphi\|\leqslant C\|(H_{\mathrm f}+\mathbb{1})\varphi\|$. Applying Lemma D.2 yields the desired results $\|(N_{\mathrm f}+\mathbb{1})\varphi\|\leqslant C\|(H(P)+\mathbb{1})\varphi\|$. □

Lemma C.1. *For all $t>0$, one has the following*:

(i) $\|(N_{\mathrm f}+\mathbb{1})(t+\hat D(P)^2)^{-1}(N_{\mathrm f}+\mathbb{1})^{-1}\|\leqslant C(t^{-1}+t^{-3/2}+t^{-2})$.
(ii) $\|(N_{\mathrm f}+\mathbb{1})(t+\tilde D^{\otimes}(P)^2)^{-1}(N_{\mathrm f}+\mathbb{1})^{-1}\|\leqslant C(t^{-1}+t^{-3/2}+t^{-2})$.

Proof. (i) The essential idea is taken from [14]. First we will show that $\mathrm e^{\mathrm it\hat D(P)}\mathrm{dom}(N_{\mathrm f})=\mathrm{dom}(N_{\mathrm f})$. It suffices to check the conditions (i), (ii) and (iii) in Lemma B.1. Noting $[N_{\mathrm f},\hat D(P)]=-\alpha\cdot(a(F_0)-a(F_0)^*)$ on $\mathbb C^4\otimes\mathfrak F_{\mathrm{fin}}$, we can check all conditions in Lemma B.1 by Lemma A.1.

Using the formula

$$\big(\hat D(P)^2+t\big)^{-1}=\int_{\mathbb R}\mathrm ds\,g_t(s)\,\mathrm e^{-\mathrm is\hat D(P)}$$

with $g_t(s)=\sqrt{\pi/2t}\,\mathrm e^{-\sqrt t|s|}$, we have

$$\big\|(N_{\mathrm f}+\mathbb{1})\big(\hat D(P)^2+t\big)^{-1}\varphi\big\|\leqslant\int_{\mathbb R}\mathrm ds\,g_t(s)\big\|(N_{\mathrm f}+\mathbb{1})\mathrm e^{-\mathrm is\hat D(P)}\varphi\big\| \tag{C.3}$$

for each normalized $\varphi\in\mathrm{dom}(N_{\mathrm f})$. (We already know that $\mathrm e^{\mathrm is\hat D(P)}\varphi\in\mathrm{dom}(N_{\mathrm f})$.) Set $I_1(s)=\|(N_{\mathrm f}+\mathbb{1})\mathrm e^{-\mathrm is\hat D(P)}\varphi\|$ and $I_{1/2}(s)=\|(N_{\mathrm f}+\mathbb{1})^{1/2}\mathrm e^{-\mathrm is\hat D(P)}\varphi\|$. Then one has

$$\frac{d}{ds} I_1(s)^2 = \langle e^{-is\hat{D}(P)}\varphi, i[\hat{D}(P), (N_f + \mathbb{1})^2] e^{-is\hat{D}(P)}\varphi \rangle$$
$$= \langle e^{-is\hat{D}(P)}\varphi, (e\alpha \cdot E(N_f + \mathbb{1}) + (N_f + \mathbb{1})e\alpha \cdot E) e^{is\hat{D}(P)}\varphi \rangle,$$

where $E = ia(F_0) - ia(F_0)^*$. Accordingly using the standard estimate $\| |E|\varphi\| \leqslant C\|(N_f + \mathbb{1})^{1/2}\varphi\|$, one has

$$\frac{d}{ds} I_1(s)^2 \leqslant C I_{1/2}(s) I_1(s). \tag{C.4}$$

Next we will estimate $I_{1/2}(s)$. Observe that

$$\begin{aligned}\frac{d}{ds} I_{1/2}(s)^2 &= \langle e^{-is\hat{D}(P)}\varphi, i[\hat{D}(P), N_f] e^{-is\hat{D}(P)}\varphi \rangle \\ &\leqslant e \| |E| e^{-is\hat{D}(P)}\varphi \| \\ &\leqslant C \| (N_f + \mathbb{1})^{1/2} e^{-is\hat{D}(P)}\varphi \| \\ &= C I_{1/2}(s).\end{aligned}$$

Solving this inequality, we get $I_{1/2}(s) \leqslant C|s| + I_{1/2}(0)$. Inserting this result into (C.4), one has

$$I_1(s) \leqslant I_1(0) + Cs^2 + C|s| I_{1/2}(0).$$

Combining this with (C.3), we finally obtain the assertion (i) in the lemma.

Noting the fact $\check{\Gamma}(\mathscr{J}) N_f \check{\Gamma}(\mathscr{J})^* = N_f \otimes \mathbb{1} + \mathbb{1} \otimes N_f$, one can apply the similar arguments in the proof of (i) to show (ii). □

Appendix D. Auxiliary estimates

In this appendix, we always choose e as $e < e_*$.

Lemma D.1. *For all $P \in \mathbb{R}^3$ and $e \geqslant 0$, one has*

$$\| |D(P)| (H_f + \mathbb{1})^{-1} \| \leqslant |P| + 3 + 3e \| \omega^{-1/2} F_0 \|.$$

Proof. Noting Lemma A.1 and the fundamental fact $\| |P_{f,i}| (H_f + \mathbb{1})^{-1} \| \leqslant 1$ for $i = 1, 2, 3$, one observes that

$$\begin{aligned}\| |D(P)| (H_f + \mathbb{1})^{-1} \| &= \| D(P) (H_f + \mathbb{1})^{-1} \| \\ &\leqslant |P| + \sum_{i=1,2,3} \| |P_{f,i}| (H_f + \mathbb{1})^{-1} \| + e \sum_{i=1,2,3} \| A(0)_i (H_f + \mathbb{1})^{-1} \| \\ &\leqslant |P| + 3 + 3e \| \omega^{-1/2} F_0 \|.\end{aligned}$$

This proves the assertion. □

Lemma D.2. *For each $n \in \mathbb{N}$, one obtains*

$$\left\| H_{\mathrm{f}}^{n/2}\big(H(P)+\mathbb{1}\big)^{-n/2}\right\| \leqslant \mathrm{const},$$
$$\left\| \big|D(P)\big|^{n/2}\big(H(P)+\mathbb{1}\big)^{-n/2}\right\| \leqslant \mathrm{const},$$

where const *is independent of P.*

Proof. In the similar way in the proof of [7, Lemma 8], one can show that both

$$H_{\mathrm{f}}^{n/2}(H+\mathbb{1})^{-n/2}, \qquad |D|^{n/2}(H+\mathbb{1})^{-n/2} \tag{D.1}$$

are bounded. Thus we conclude the assertion by the fact that $\|(H(P)+\mathbb{1})^{n/2}\varphi\|$ is continuous in P for $\varphi \in \mathbb{C}^4 \otimes \mathfrak{F}_{\mathrm{fin}}$. □

Lemma D.3. *For each $n \in \mathbb{N}$, we obtain*

$$\left\| (H_{\mathrm{f}}+\mathbb{1})^n\big(t+\hat{D}(P)^2\big)^{-1}\big(H(Q)+\mathbb{1}\big)^{-n}\right\| \leqslant \mathrm{const}\big(t^{-1}+t^{-3/2}+\cdots+t^{-n-1}\big)$$

for every $P, Q \in \mathbb{R}^3$, where const *is independent of P and Q.*

Sketch of proof. By Lemma A.1, one can see that $\mathrm{e}^{\mathrm{i}t\hat{D}(P)}\mathrm{dom}(H_{\mathrm{f}}^n) = \mathrm{dom}(H_{\mathrm{f}}^n)$. Let us write

$$K_{m/2}(s) = \left\| (H_{\mathrm{f}}+\mathbb{1})^{m/2}\mathrm{e}^{-\mathrm{i}s\hat{D}(P)}\varphi\right\|$$

for a normalized $\varphi \in \mathrm{dom}(H_{\mathrm{f}}^n)$ with $m \leqslant 2n$. In the case where $m=1$, one has

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}K_{1/2}(s)^2 &= \big\langle \mathrm{e}^{-\mathrm{i}s\hat{D}(P)}\varphi, \mathrm{i}\big[\hat{D}(P), H_{\mathrm{f}}\big]\mathrm{e}^{-\mathrm{i}s\hat{D}(P)}\varphi\big\rangle \\
&= \big\langle \mathrm{e}^{-\mathrm{i}s\hat{D}(P)}\varphi, \mathrm{i}\alpha\cdot\big(a(\omega F)-a(\omega F)^*\big)\mathrm{e}^{-\mathrm{i}s\hat{D}(P)}\varphi\big\rangle \\
&\leqslant CK_{1/2}(s)
\end{aligned}$$

by the Schwarz inequality. Thus $K_{1/2}(s) \leqslant K_{1/2}(0) + C|s|$ holds. In the case where $m=2$, one has, by the similar arguments in the above,

$$\frac{\mathrm{d}}{\mathrm{d}s}K_1(s)^2 \leqslant CK_{1/2}(s)K_1(s) \leqslant \big(K_{1/2}(0)+C|s|\big)K_1(s)$$

which implies $K_1(s) \leqslant K_1(0) + C(K_{1/2}(0)|s| + s^2)$. Repeating this procedure, one can arrive at

$$K_{m/2}(s) \leqslant K_{m/2}(0) + C\big(K_{(m-1)/2}(0)|s| + \cdots + K_{1/2}(0)|s|^{m-1} + |s|^m\big).$$

Therefore using the formula

$$\left\| (H_{\mathrm{f}}+\mathbb{1})^{m/2}\big(t+\hat{D}(P)^2\big)^{-1}\varphi\right\| \leqslant \int_{\mathbb{R}} \mathrm{d}s\, g_t(s)K_{m/2}(s)$$

with $g_t(s) = \sqrt{\pi/2t}\,\mathrm{e}^{-\sqrt{t}|s|}$, one has, by putting $m = 2n$,

$$\big\|(H_{\mathrm{f}} + \mathbb{1})^n\big(t + \hat{D}(P)^2\big)^{-1}(H_{\mathrm{f}} + \mathbb{1})^{-n}\big\| \leqslant C\big(t^{-1} + t^{-3/2} + \cdots + t^{-n-1}\big). \tag{D.2}$$

Finally using Lemma D.2, one concludes the desired assertion in the lemma. □

Lemma D.4. *One has*

$$\big\|\big[H_{\mathrm{f}}, \big|D(P)\big|\big]\big(H(Q) + \mathbb{1}\big)^{-2}\big\| \leqslant \mathrm{const}\big(1 + |P|\big)$$

for all $P, Q \in \mathbb{R}^3$.

Proof. By Lemma D.3 and the following standard formula

$$\big[H_{\mathrm{f}}, \big|D(P)\big|\big] = \frac{1}{\pi}\int_{M^2}^{\infty} \mathrm{d}s\,\sqrt{s - M^2}\big(s + \hat{D}(P)^2\big)^{-1}\big[H_{\mathrm{f}}, \hat{D}(P)^2\big]\big(s + \hat{D}(P)^2\big)^{-1},$$

one computes

$$\begin{aligned}
&\big\|\big[H_{\mathrm{f}}, \big|D(P)\big|\big](H_{\mathrm{f}} + \mathbb{1})^{-2}\big\| \\
&\quad\leqslant \frac{1}{\pi}\int_{M^2}^{\infty} \mathrm{d}s\,\sqrt{s - M^2}\big\|\big(s + \hat{D}(P)^2\big)^{-1}\big\|\big\|\big[H_{\mathrm{f}}, \hat{D}(P)^2\big](H_{\mathrm{f}} + \mathbb{1})^{-2}\big\| \\
&\qquad\times\big\|(H_{\mathrm{f}} + \mathbb{1})^2\big(s + \hat{D}(P)^2\big)^{-1}\big(H(Q) + \mathbb{1}\big)^{-2}\big\| \\
&\quad\leqslant \frac{C}{\pi}\int_{M^2}^{\infty} \mathrm{d}s\,\sqrt{s - M^2}s^{-1}\big(1 + |P|\big)\big(s^{-1} + s^{-3/2} + \cdots + s^{-3}\big) \\
&\quad= \mathrm{const}\big(1 + |P|\big).
\end{aligned}$$

This completes the proof. □

Lemma D.5. *For all* $P, Q \in \mathbb{R}^3$, *one has*

$$\begin{aligned}
&\big\|H(P)^2\big(H(Q) + \mathbb{1}\big)^{-2}\big\| \\
&\quad\leqslant \big(C + |P - Q|\big)^2 + C\big(1 + |P| + |Q| + |P||Q|\big)\big(C + |P - Q|^2\big).
\end{aligned}$$

Proof. First we will show that

$$\big\|H(P)\big(H(Q) + \mathbb{1}\big)^{-1}\big\| \leqslant C + |P - Q|.$$

To this end, observe that

$$
\begin{aligned}
&\big\| H(P)\big(H(Q)+\mathbb{1}\big)^{-1}\big\| \\
&\quad\leqslant \big\|\big|D(P)\big|\big(H(Q)+\mathbb{1}\big)^{-1}\big\| + \big\|H_{\mathrm{f}}\big(H(Q)+\mathbb{1}\big)^{-1}\big\| \\
&\quad\leqslant \big\|D(Q)\big(H(Q)+\mathbb{1}\big)^{-1}\big\| + \big\|\alpha\cdot(P-Q)\big(H(Q)+\mathbb{1}\big)^{-1}\big\| + \big\|H_{\mathrm{f}}\big(H(Q)+\mathbb{1}\big)^{-1}\big\| \\
&\quad\leqslant C+|P-Q|
\end{aligned}
$$

by Lemma D.2. Write

$$
\begin{aligned}
&H(P)^2\big(H(Q)+\mathbb{1}\big)^{-2} \\
&\quad= \big\{H(P)\big(H(Q)+\mathbb{1}\big)^{-1}\big\}^2 + H(P)\big[H(P),\big(H(Q)+\mathbb{1}\big)^{-1}\big]\big(H(Q)+\mathbb{1}\big)^{-1} \\
&\quad= \big\{H(P)\big(H(Q)+\mathbb{1}\big)^{-1}\big\}^2 + H(P)\big(H(Q)+\mathbb{1}\big)^{-1}\big[H(Q),H(P)\big]\big(H(Q)+\mathbb{1}\big)^{-2}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\big\|H(P)^2\big(H(Q)+\mathbb{1}\big)^{-2}\big\| \\
&\quad\leqslant \big(C+|P-Q|\big)^2 + \big(C+|P-Q|\big)\big\|\big[H(Q),H(P)\big]\big(H(Q)+\mathbb{1}\big)^{-2}\big\|.
\end{aligned}
$$

Accordingly what we have to show next is to estimate the operator norm

$$
\big\|\big[H(Q),H(P)\big]\big(H(P)+\mathbb{1}\big)^{-2}\big\|.
$$

Observe that

$$
\begin{aligned}
&\big[H(Q),H(P)\big]\big(H(Q)+\mathbb{1}\big)^{-2} \\
&\quad= \big[\big|D(P)\big|,\big|D(Q)\big|\big]\big(H(Q)+\mathbb{1}\big)^{-2} \qquad \text{(D.3)}\\
&\qquad+ \big[H_{\mathrm{f}},\big|D(P)\big|\big]\big(H(Q)+\mathbb{1}\big)^{-2} \qquad \text{(D.4)}\\
&\qquad+ \big[\big|D(P)\big|,H_{\mathrm{f}}\big]\big(H(Q)+\mathbb{1}\big)^{-2}. \qquad \text{(D.5)}
\end{aligned}
$$

Norm of (D.4) and (D.5) can be estimated by Lemma D.4. As to (D.3), note that

$$
\begin{aligned}
&\big\|\big|D(P)\big|\big|D(Q)\big|\varphi\big\| \\
&\quad\leqslant \big\|\big|D(0)\big|\big|D(Q)\big|\varphi\big\| + |P|\big\|\big|D(Q)\big|\varphi\big\| \\
&\quad\leqslant C\big(\big\|(H_{\mathrm{f}}+\mathbb{1})\big|D(Q)\big|\varphi\big\| + |P|\big\|\big|D(Q)\big|\varphi\big\|\big) \\
&\quad\leqslant C\big(\big\|\big|D(Q)\big|(H_{\mathrm{f}}+\mathbb{1})\varphi\big\| + \big\|\big[H_{\mathrm{f}},\big|D(Q)\big|\big]\varphi\big\| + \big(1+|Q|\big)|P|\big\|(H_{\mathrm{f}}+\mathbb{1})\varphi\big\|\big) \\
&\quad\leqslant C\big(\big(1+|Q|\big)\big\|(H_{\mathrm{f}}+\mathbb{1})^2\varphi\big\| + \big\|\big(H(Q)+\mathbb{1}\big)^2\varphi\big\| + \big(1+|Q|\big)|P|\big\|(H_{\mathrm{f}}+\mathbb{1})\varphi\big\|\big) \\
&\quad\leqslant C\big(1+|P|+|Q|+|P||Q|\big)\big\|\big(H(Q)+\mathbb{1}\big)^2\varphi\big\|.
\end{aligned}
$$

In the above we have used Lemma D.4 from the line four to the next, and from the line five to the final line, we have used Lemma D.1. Collecting the results, one obtains the assertion in the lemma. □

Lemma D.6. *Let*

$$S_{k,\lambda}(P) = \big|D(P-k)\big|a(k,\lambda) - a(k,\lambda)\big|D(P)\big|.$$

Then one has

$$\big\| S_{k,\lambda}(P)^*\big(H(P-k)+\mathbb{1}\big)^{-1}\big\| \leqslant C\big(1+|k|\big)\big|F_0(k,\lambda)\big|,$$

where C is independent of k and P.

Proof. We will show that $\|(H(P-k)+\mathbb{1})^{-1}S_{k,\lambda}\| \leqslant C(1+|k|)|F_0(k,\lambda)|$. Let

$$S_{k,\lambda} = \mathrm{e}^{\mathrm{i}k\cdot x}\mathrm{e}^{\mathrm{i}x\cdot P_{\mathrm{f}}}\big[|D|, a(k,\lambda)\big]\mathrm{e}^{-\mathrm{i}x\cdot P_{\mathrm{f}}}.$$

Then one has

$$\mathcal{F}_x S_{k,\lambda}\mathcal{F}_x^* = \int_{\mathbb{R}^3}^{\oplus} S_{k,\lambda}(P)\,\mathrm{d}P$$

for all $(k,\lambda)\in\mathbb{R}^3\times\{1,2\}$. Hence it suffices to show that

$$\big\|(H_k+\mathbb{1})^{-1}\big[|D-\alpha\cdot k|, a(k,\lambda)\big]\big\| \leqslant C\big(1+|k|\big)\big|F_0(k,\lambda)\big| \tag{D.6}$$

where $H_k = |D-\alpha\cdot k| + H_{\mathrm{f}}$, because

$$\begin{aligned}
&\mathcal{F}_x^*\int_{\mathbb{R}^3}^{\oplus}\big(H(P-k)+\mathbb{1}\big)^{-1}S_{k,\lambda}(P)\,\mathrm{d}P\,\mathcal{F}_x\\
&= \mathrm{e}^{\mathrm{i}x\cdot P_{\mathrm{f}}}(H_k+\mathbb{1})^{-1}\mathrm{e}^{-\mathrm{i}x\cdot P_{\mathrm{f}}}S_{k,\lambda}\\
&= \mathrm{e}^{\mathrm{i}x\cdot P_{\mathrm{f}}}(H_k+\mathbb{1})^{-1}\mathrm{e}^{-\mathrm{i}x\cdot P_{\mathrm{f}}}\mathrm{e}^{\mathrm{i}k\cdot x}\mathrm{e}^{\mathrm{i}x\cdot P_{\mathrm{f}}}\big[|D|, a(k,\lambda)\big]\mathrm{e}^{-\mathrm{i}x\cdot P_{\mathrm{f}}}\\
&= \mathrm{e}^{\mathrm{i}x\cdot P_{\mathrm{f}}}(H_k+\mathbb{1})^{-1}\big[|D-\alpha\cdot k|, a(k,\lambda)\big]\mathrm{e}^{\mathrm{i}k\cdot x}\mathrm{e}^{-\mathrm{i}x\cdot P_{\mathrm{f}}}.
\end{aligned}$$

To this end, we remark that, with $\hat{D} = D - M\beta$,

$$\begin{aligned}
&\big[|D-\alpha\cdot k|, a(k,\lambda)\big]\\
&\quad= \frac{1}{\pi}\int_{M^2}^{\infty}\mathrm{d}s\,\sqrt{s-M^2}\big(s+(\hat{D}-\alpha\cdot k)^2\big)^{-1}\big[(\hat{D}-\alpha\cdot k)^2, a(k,\lambda)\big]\\
&\qquad\times\big(s+(\hat{D}-\alpha\cdot k)^2\big)^{-1}\\
&\quad= -\frac{1}{\pi}\int_{M^2}^{\infty}\mathrm{d}s\,\sqrt{s-M^2}\big(s+(\hat{D}-\alpha\cdot k)^2\big)^{-1}\big\{(\hat{D}-\alpha\cdot k)e\alpha\cdot F_x(k,\lambda)\\
&\qquad+ e\alpha\cdot F_x(k,\lambda)(\hat{D}-\alpha\cdot k)\big\}\big(s+(\hat{D}-\alpha\cdot k)^2\big)^{-1}.
\end{aligned} \tag{D.7}$$

Assume, for a while, that

$$\big\|(H_k+\mathbb{1})\big(s+(\hat{D}-\alpha\cdot k)^2\big)^{-1}(H_k+\mathbb{1})^{-1}\big\| \leqslant C\big(s^{-1}+s^{-3/2}+s^{-2}\big). \tag{D.8}$$

We note the following two estimates:

$$\begin{aligned}&\big\|(H_k+\mathbb{1})^{-1}(\hat{D}-\alpha\cdot k)\alpha\cdot F_x(k,\lambda)\big\|\\ &\quad=\big\|(H+\mathbb{1})^{-1}\hat{D}\alpha\cdot F_0(k,\lambda)\big\|\\ &\quad\leqslant C\big|F_0(k,\lambda)\big|\end{aligned} \tag{D.9}$$

and

$$\begin{aligned}&\big\|(H_k+\mathbb{1})^{-1}\alpha\cdot F_x(k,\lambda)(\hat{D}-\alpha\cdot k)\big\|\\ &\quad=\big\|(H+\mathbb{1})^{-1}\alpha\cdot F_0(k,\lambda)(\hat{D}+\alpha\cdot k)\big\|\\ &\quad\leqslant\big|F_0(k,\lambda)\big|\big\|(H+\mathbb{1})^{-1}\hat{D}\big\|+|k|\big|F_0(k,\lambda)\big|\big\|(H+\mathbb{1})^{-1}\big\|\\ &\quad\leqslant C\big(1+|k|\big)\big|F_0(k,\lambda)\big|,\end{aligned} \tag{D.10}$$

because $\hat{D}(H+\mathbb{1})^{-1}$ is bounded and $\mathrm{e}^{-\mathrm{i}k\cdot x}H_k = H\mathrm{e}^{-\mathrm{i}k\cdot x}$, $\mathrm{e}^{-\mathrm{i}k\cdot x}(\hat{D}-\alpha\cdot k)=\hat{D}\mathrm{e}^{-\mathrm{i}k\cdot x}$. Collecting (D.7), (D.8), (D.9) and (D.10), one has

$$\begin{aligned}&\big\|\big[|D-\alpha\cdot k|, a(k,\lambda)\big](H_k+\mathbb{1})^{-1}\big\|\\ &\quad\leqslant\frac{1}{\pi}\int_{M^2}^{\infty}\mathrm{d}s\,\sqrt{s-M^2}s^{-1}C\big(1+|k|\big)\big|F_0(k,\lambda)\big|\big(s^{-1}+s^{-3/2}+s^{-2}\big)\\ &\quad=C\big(1+|k|\big)\big|F_0(k,\lambda)\big|.\end{aligned}$$

This is what we want to show.

Finally we will prove (D.8). Basic strategy is similar to the proof of Lemma D.3. Let $J_{n/2}(s)=\|(H_k+\mathbb{1})^{n/2}\mathrm{e}^{-\mathrm{i}s(\hat{D}-\alpha\cdot k)}\varphi\|$, $n=1,2$ for $\varphi\in\mathrm{dom}(H_k)$ with $\|\varphi\|=1$. Then, since $[\hat{D}-\alpha\cdot k, H_k]=[\hat{D},H_{\mathrm{f}}]=\alpha\cdot(a(\omega F_x)-a(\omega F_x)^*)$, one can easily modify the proof of Lemma D.3 to conclude that

$$J_1(s)\leqslant J_1(0)+C\big(J_{1/2}(0)|s|+s^2\big),\qquad J_{1/2}(s)\leqslant J_{1/2}(0)+C|s|.$$

Thus, using the formula

$$\big(t+\hat{D}^2\big)^{-1}=\int_{\mathbb{R}}\mathrm{d}s\,g_t(s)\mathrm{e}^{-\mathrm{i}s\hat{D}}$$

with $g_t(s)=\sqrt{\pi/2t}\,\mathrm{e}^{-\sqrt{t}|s|}$ and modifying the proof of Lemma D.3, one can arrive at (D.8). □

Appendix E. Operator monotonicity of the square root

Lemma E.1 *(Operator monotonicity of the square root: unbounded version). Let S and T be two positive self-adjoint operators (not necessarily bounded) with* $\mathrm{dom}(S^{1/2}) \supseteq \mathrm{dom}(T^{1/2})$. *Assume that $S \leqslant T$. Then one has* $\mathrm{dom}(S^{1/4}) \supseteq \mathrm{dom}(T^{1/4})$ *and* $\sqrt{S} \leqslant \sqrt{T}$.

Proof. Set $E_n = E_S([0,n])$ where $E_S(\cdot)$ is the spectral measure of S. Define $S_n = E_n^{1/2} S E_n^{1/2}$. Then one has $S \geqslant S_n \geqslant 0$ for all $n \in \mathbb{N}$. Thus $S_n \leqslant T$ holds for all $n \in \mathbb{N}$. Now one has

$$(\varepsilon T + \mathbb{1})^{-1} S_n (\varepsilon T + \mathbb{1})^{-1} \leqslant (\varepsilon T + \mathbb{1})^{-1} T (\varepsilon T + \mathbb{1})^{-1} \tag{E.1}$$

for every $\varepsilon > 0$. Since both sides of (E.1) are positive and bounded, one can apply the operator monotonicity of the square root for *bounded* positive operators [18] and obtain

$$\sqrt{(\varepsilon T + \mathbb{1})^{-1} S_n (\varepsilon T + \mathbb{1})^{-1}} \leqslant \sqrt{(\varepsilon T + \mathbb{1})^{-1} T (\varepsilon T + \mathbb{1})^{-1}}$$

for all $\varepsilon > 0$. Taking $\varepsilon \downarrow 0$ first, we have $\sqrt{S_n} \leqslant \sqrt{T}$ for each $n \in \mathbb{N}$. It follows that, for $f \in \mathrm{dom}(T^{1/4})$, one has

$$\int_0^n \lambda^{1/2} \, \mathrm{d}\big\| E_S(\lambda) f \big\|^2 \leqslant \big\langle f, \sqrt{T} f \big\rangle$$

by the spectral theorem. Now taking $n \to \infty$, we conclude that $f \in \mathrm{dom}(S^{1/4})$ and $\langle f, \sqrt{S} f \rangle \leqslant \langle f, \sqrt{T} f \rangle$ by the monotone convergence theorem. □

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