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A study of inverse trigonometric integrals associated with three-variable Mahler measures, and some related identities

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Abstract

We prove several identities relating three-variable Mahler measures to integrals of inverse trigonometric functions. After deriving closed forms for most of these integrals, we obtain ten explicit formulas for three-variable Mahler measures. Several of these results generalize formulas due to Condon and Lalín. As a corollary, we also obtain three q-series expansions for the dilogarithm.

Keywords: Mahler measure; Special functions; Polylogarithms; Trilogarithm; Elliptic functions

1. Introduction

In this paper we will undertake a systematic study of each of the inverse trigonometric integrals

$$\mathrm{T}(v,w)=\int_0^1 \frac{\tan^{-1}(vx)\tan^{-1}(wx)}{x}\,\mathrm{d}x,$$

$$\mathrm{S}(v,w)=\int_0^1 \frac{\sin^{-1}(vx)\sin^{-1}(wx)}{x}\,\mathrm{d}x,$$

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$$\mathrm{TS}(v, w) = \int_0^1 \frac{\tan^{-1}(vx)\sin^{-1}(wx)}{x}\,\mathrm{d}x.$$

This class of integrals arises when trying to find closed form expressions for the Mahler measures of certain three-variable polynomials.

Recall that the Mahler measure of an n-dimensional polynomial, $P(x_1, \ldots, x_n)$, can be defined by

$$\mathrm{m}\big(P(x_1, \ldots, x_n)\big) = \int_0^1 \cdots \int_0^1 \log\big|P\big(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n}\big)\big|\,\mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n.$$

In the last few years, numerous papers have established explicit formulas relating multi-variable Mahler measures to special constants. Smyth proved the first result [3] with

$$\mathrm{m}(1 + x + y + z) = \frac{7}{2\pi^2}\zeta(3),$$

where the Riemann zeta function is defined by $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

In this paper, we will prove a number of new formulas relating three-variable Mahler measures to the aforementioned trigonometric integrals. Many of our identities generalize previously known results. We will list a few of our main results in this introductory section.

For our first example, we can use various properties of $\mathrm{T}(v, w)$ to show that

$$\begin{aligned}
&\mathrm{m}\left(1 - v^4\left(\frac{1-x}{1+x}\right)^2 + \left(y + v^2\left(\frac{1-x}{1+x}\right)\right)^2 z\right) \\
&\quad = \frac{4}{\pi}\int_0^v \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u - \frac{8}{\pi^2}\mathrm{T}\left(v, \frac{1}{v}\right) + \frac{1}{2}\mathrm{m}\left(1 - v^4\left(\frac{1-x}{1+x}\right)^2\right).
\end{aligned} \tag{1.1}$$

This reduces to one of Lalin's formulas [7] when $v = 1$:

$$\mathrm{m}\big((1 + y)(1 + z) + (1 - z)(x - y)\big) = \frac{7}{2\pi^2}\zeta(3) + \frac{\log(2)}{2}. \tag{1.2}$$

We can use the double arcsine integral, $\mathrm{S}(v, w)$, to prove that if $v \in [0, 1]$:

$$\begin{aligned}
\mathrm{m}\big(v(1 + x) + y + z\big) &= \frac{2}{\pi}\int_0^v \frac{\sin^{-1}(u)}{u}\,\mathrm{d}u - \frac{4}{\pi^2}\mathrm{S}(v, 1) \\
&= \frac{4}{\pi^2}\left(\frac{\mathrm{Li}_3(v) - \mathrm{Li}_3(-v)}{2}\right).
\end{aligned} \tag{1.3}$$

The second equality has been proved by Vandervelde [12]. Slightly more complicated arguments lead to expressions that include

$$\mathrm{m}\big(1 - x^{1/6} + y + z\big) = \frac{2}{\pi}\int_0^{1/2} \frac{\sin^{-1}(u)}{u}\,\mathrm{d}u - \frac{12}{\pi^2}\,\mathrm{S}\left(\frac{1}{2}, \frac{1}{2}\right). \tag{1.4}$$

This fractional Mahler measure is defined by

$$\mathrm{m}\big(1 - x^{1/6} + y + z\big) = \int_0^1 \mathrm{m}\big(1 - e^{2\pi i u/6} + y + z\big)\,\mathrm{d}u,$$

notice that $\mathrm{m}(1 - x^{1/6} + y + z) \neq \mathrm{m}(1 - x + y + z)$. We can simplify the right-hand side of Eq. (1.4) by either expressing $\mathrm{S}(\frac{1}{2}, \frac{1}{2})$ as a linear combination of L-functions, or in terms of a famous binomial sum:

$$\mathrm{S}\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^3\binom{2n}{n}}.$$

Condon [5] proved an identity that Boyd and Rodriguez Villegas conjectured:

$$\mathrm{m}\big(1 + x + (1 - x)(y + z)\big) = \frac{28}{5\pi^2}\zeta(3). \tag{1.5}$$

His proof also showed (in a slightly disguised form) that

$$\mathrm{TS}(2, 1) = \frac{\pi}{2}\int_0^2 \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u - \frac{7}{5}\zeta(3). \tag{1.6}$$

We have generalized Condon's identity to show that

$$\mathrm{m}\left(1 + x + \frac{v}{2}(1 - x)(y + z)\right) = \frac{2}{\pi}\int_0^v \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u - \frac{4}{\pi^2}\,\mathrm{TS}(v, 1), \tag{1.7}$$

where Eq. (4.18) expresses $\mathrm{TS}(v, 1)$ in terms of polylogarithms. We can use this result to prove a number of new formulas, including:

$$\begin{aligned}
&\mathrm{m}\left(x + \frac{v^2}{4}(1 + x)^2 + \left(y + \frac{v}{2}(1 + x)\right)^2 z\right) \\
&\quad = \frac{2}{\pi}\int_0^v \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u - \frac{4}{\pi^2}\,\mathrm{TS}(v, 1) + \frac{1}{2}\mathrm{m}\left(x + \frac{v^2}{4}(1 + x)^2\right).
\end{aligned} \tag{1.8}$$

When $v = 2$ this reduces to an interesting identity for $\zeta(3)$ and the golden ratio:

$$\mathrm{m}\big(x + (1+x)^2 + (1+x+y)^2 z\big) = \frac{28}{5\pi^2}\zeta(3) + \log\left(\frac{1+\sqrt{5}}{2}\right). \tag{1.9}$$

We will show that all of the integrals $\mathrm{TS}(v, w)$, $\mathrm{T}(v, w)$, and $\mathrm{S}(v, w)$ have closed form expressions in terms of polylogarithms. The special case of $\mathrm{TS}(v, 1)$ will warrant extra attention, as it is related to an interesting family of binomial sums. Our closed forms are all derived through elementary methods.

2. Preliminaries: A description of the method, and some two-dimensional Mahler measures

Although there are many conjectured formulas for multi-variable Mahler measures, most are extremely difficult, if not impossible, to prove. Rather than attempting to prove any of these conjectures, we will take an easier approach. By investigating promising functions, and rewriting them as Mahler measures, we can recover a number of useful formulas.

Our first step was to determine a class of functions that we could relate to Mahler's measure. We chose the three integrals $\mathrm{TS}(v, w)$, $\mathrm{S}(v, w)$, and $\mathrm{T}(v, w)$, based on Condon's evaluation of $\mathrm{TS}(2, 1)$, Eq. (1.6). Condon's formula naturally suggested the existence of a generalized Mahler measure formula involving $\mathrm{TS}(v, 1)$. From there, it was a small step to consider the similar functions $\mathrm{TS}(v, w)$, $\mathrm{T}(v, w)$, and $\mathrm{S}(v, w)$.

We will use the following method to express $\mathrm{TS}(v, 1)$, $\mathrm{S}(v, 1)$, and $\mathrm{T}(v, 1/v)$ as three-variable Mahler measures. First, a simple integration by parts changes each function into a two-dimensional integral, containing either a nested arcsine or arctangent integral. Recall that the following integrals define the arctangent and arcsine integrals, respectively:

$$\int_0^w \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u, \qquad \int_0^v \frac{\sin^{-1}(u)}{u}\,\mathrm{d}u.$$

A typical formula for $\mathrm{TS}(v, 1)$, Eq. (3.8), can be proved with little trouble:

$$\mathrm{TS}(v, 1) = \frac{\pi}{2}\int_0^v \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u - \int_0^{\pi/2}\int_0^{v\sin(\theta)} \frac{\tan^{-1}(z)}{z}\,\mathrm{d}z\,\mathrm{d}\theta.$$

Next, substituting a two-dimensional Mahler measure for the nested arctangent or arcsine integral will allow us to obtain a three-dimensional Mahler measure evaluation. Theorem 3.2, Proposition 5.2, and Theorem 7.3 contain our main results from using this method.

Expressing the arcsine and arctangent integrals in terms of Mahler's measure represents the main difficulty in this approach. In the remainder of this section we will establish four two-variable Mahler measures for the arctangent integral, and one two-variable Mahler measure for the arcsine integral.

Since many of our results involve polylogarithms, this will be a good place to define the polylogarithm.

Definition 2.1. If $|z| < 1$, then the polylogarithm of order k is defined by

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}.$$

We call $\mathrm{Li}_2(z)$ the dilogarithm, and we call $\mathrm{Li}_3(z)$ the trilogarithm.

Theorem 2.2 requires a formula of Cassaigne and Maillot [9]. In particular, Cassaigne and Maillot showed that

$$\pi\,\mathrm{m}(a+bx+cy) = \begin{cases} D\bigl(\frac{|a|}{|b|}e^{i\gamma}\bigr) + \alpha\log|a| + \beta\log|b| + \gamma\log|c|, & \text{if “}\triangle\text{”}, \\ \pi\log(\max\{|a|,|b|,|c|\}), & \text{otherwise.} \end{cases}$$

The “$\triangle$” condition states that $|a|$, $|b|$, and $|c|$ form the sides of a triangle. If “$\triangle$” is true, then α, β, and γ denote the radian measures of the angles opposite to the sides of length $|a|$, $|b|$, and $|c|$, respectively. In this formula, $D(z)$ denotes the Bloch–Wigner dilogarithm. As usual,

$$D(z) = \mathrm{Im}\bigl(\mathrm{Li}_2(z)\bigr) + \log|z|\arg(1-z).$$

Now that we have stated Cassaigne and Maillot’s formula, we will prove Theorem 2.2.

Theorem 2.2. *If* $0 \leqslant v \leqslant 1$ *and* $w \geqslant 0$, *then*

$$\int_0^v \frac{\sin^{-1}(u)}{u}\,\mathrm{d}u = \frac{\pi}{2}\mathrm{m}(2v+y+z), \tag{2.1}$$

$$\int_0^w \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u = \frac{\pi}{2}\mathrm{m}\bigl(1+w^2+(y+w)^2z\bigr) - \frac{\pi}{4}\log\bigl(1+w^2\bigr). \tag{2.2}$$

Proof. To prove Eq. (2.1) first recall the usual formula for this arcsine integral,

$$\int_0^v \frac{\sin^{-1}(u)}{u}\,\mathrm{d}u = \frac{1}{2}\mathrm{Im}\bigl(\mathrm{Li}_2\bigl(e^{2i\sin^{-1}(v)}\bigr)\bigr) + \sin^{-1}(v)\log(2v), \tag{2.3}$$

which is valid whenever $0 \leqslant v \leqslant 1$.

Now apply Cassaigne and Maillot’s formula to $\mathrm{m}(2v+y+z)$; we are in the “$\triangle$” case since $0 \leqslant v \leqslant 1$. It follows from a little trigonometry that

$$\pi\,\mathrm{m}(2v+y+z) = D\bigl(e^{2i\sin^{-1}(v)}\bigr) + 2\sin^{-1}(v)\log(2v).$$

Since $|e^{2i\sin^{-1}(v)}| = 1$, $D(e^{2i\sin^{-1}(v)}) = \mathrm{Im}(\mathrm{Li}_2(e^{2i\sin^{-1}(v)}))$, hence we obtain

$$\pi\,\mathrm{m}(2v+y+z) = \mathrm{Im}\bigl(\mathrm{Li}_2\bigl(e^{2i\sin^{-1}(v)}\bigr)\bigr) + 2\sin^{-1}(v)\log(2v).$$

Comparing this last formula to Eq. (2.3), we have

$$\frac{\pi}{2}\mathrm{m}(2v+y+z)=\int_0^v \frac{\sin^{-1}(u)}{u}\,\mathrm{d}u.$$

To prove Eq. (2.2) first recall that if $0 \leqslant w \leqslant 1$, then

$$\int_0^w \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u=\operatorname{Im}\bigl(\mathrm{Li}_2(iw)\bigr).$$

Next observe that by Cassaigne and Maillot's formula

$$\begin{aligned}\pi\,\mathrm{m}\bigl(\sqrt{1+w^2}+wy+z\bigr)&=D\bigl(e^{\pi i/2}w\bigr)+\tan^{-1}(w)\log(w)+\frac{\pi}{2}\log\bigl(\sqrt{1+w^2}\bigr)\\&=\operatorname{Im}\bigl(\mathrm{Li}_2(iw)\bigr)+\frac{\pi}{4}\log\bigl(1+w^2\bigr).\end{aligned}$$

Making a change of variables in the Mahler measure, it is clear that

$$\begin{aligned}\mathrm{m}\bigl(\sqrt{1+w^2}+wy+z\bigr)&=\frac{1}{2}\bigl\{\mathrm{m}\bigl(\sqrt{1+w^2}+(1+wy)iz\bigr)+\mathrm{m}\bigl(\sqrt{1+w^2}-(1+wy)iz\bigr)\bigr\}\\&=\frac{1}{2}\mathrm{m}\bigl(1+w^2+(1+wy)^2z^2\bigr)\\&=\frac{1}{2}\mathrm{m}\bigl(1+w^2+(y+w)^2z\bigr).\end{aligned}$$

It follows that for $0 \leqslant w \leqslant 1$ we have

$$\int_0^w \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u=\frac{\pi}{2}\mathrm{m}\bigl(1+w^2+(y+w)^2z\bigr)-\frac{\pi}{4}\log\bigl(1+w^2\bigr).$$

We can extend this formula to the entire positive real line. Suppose that $w=1/w'$ where $w' \geqslant 1$, then

$$\begin{aligned}\int_0^{1/w'} \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u&=\frac{\pi}{2}\mathrm{m}\left(1+\frac{1}{w'^2}+\left(y+\frac{1}{w'}\right)^2 z\right)-\frac{\pi}{4}\log\left(1+\frac{1}{w'^2}\right)\\&=\frac{\pi}{2}\mathrm{m}\bigl(1+w'^2+(y+w')^2z\bigr)-\frac{\pi}{4}\log\bigl(1+w'^2\bigr)-\frac{\pi}{2}\log(w').\end{aligned}$$

Since the arctangent integral obeys the functional equation [10]

$$\int_0^{w'} \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u=\frac{\pi}{2}\log(w')+\int_0^{1/w'} \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u, \tag{2.4}$$

it follows that

$$\int_0^{w'} \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u = \frac{\pi}{2}\mathrm{m}\big(1 + w'^2 + (y + w')^2 z\big) - \frac{\pi}{4}\log\big(1 + w'^2\big).$$

Therefore Eq. (2.2) holds for all $w \geqslant 0$. □

The next theorem proves that Eq. (2.2) is not unique. Using results from Theorem 6.5, we can derive three more Mahler measures for the arctangent integral.

Theorem 2.3. *Suppose that $w \geqslant 0$, then*

$$\int_0^{w} \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u = \frac{\pi}{4}\mathrm{m}\big(\big(1 + w^2\big)(1 + y) + w(1 - y)\big(z + z^{-1}\big)\big), \tag{2.5}$$

$$\int_0^{w} \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u = \frac{\pi}{2}\mathrm{m}\big(\big(y - y^{-1}\big) + w\big(z + z^{-1}\big)\big), \tag{2.6}$$

$$\int_0^{w} \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u = \frac{\pi}{4}\mathrm{m}\big(\big(4(1 + y)^2 - \big(z + z^{-1}\big)^2\big)\big(1 + w^2\big)^2 + \big(z - z^{-1}\big)^2(1 + y)^2\big(1 - w^2\big)^2\big) - \frac{\pi}{4}\log(2) - \frac{\pi}{2}\log(1 + w). \tag{2.7}$$

Proof. Since all three of these formulas have similar proofs, we will only prove Eqs. (2.5) and (2.7). It is necessary to remark, that while Eq. (2.5) follows from Eq. (6.25), and Eq. (2.7) follows from Eq. (6.19), we must start from Eq. (6.16) to prove Eq. (2.6).

Now we will proceed with the proof of Eq. (2.5). From Eq. (6.25) we have

$$\frac{\pi}{4k}\log\left(\frac{1 + k}{1 - k}\right) - \frac{2}{k}\,\mathrm{Im}\big(\mathrm{Li}_2(ir)\big) = \int_0^1 \frac{\sin^{-1}(u)}{1 - k^2u^2}\,\mathrm{d}u,$$

where $k = \frac{2r}{1+r^2}$, and $0 < k < 1$. After an integration by parts this becomes

$$\frac{\pi}{4k}\log\left(\frac{1 + k}{1 - k}\right) - \frac{2}{k}\,\mathrm{Im}\big(\mathrm{Li}_2(ir)\big) = \frac{\pi}{4k}\log\left(\frac{1 + k}{1 - k}\right) - \frac{1}{2k}\int_0^1 \log\left(\frac{1 + ku}{1 - ku}\right)\frac{\mathrm{d}u}{\sqrt{1 - u^2}}.$$

It follows immediately that

$$\mathrm{Im}\big(\mathrm{Li}_2(ir)\big) = \frac{1}{4}\int_0^{\pi/2} \log\left(\frac{1 + k\sin(t)}{1 - k\sin(t)}\right)\mathrm{d}t = \frac{1}{8}\int_0^{2\pi} \log^+\left|\frac{1 + k\sin(t)}{1 - k\sin(t)}\right|\mathrm{d}t.$$

Changing the "$\log^+|\cdot|$" term into a Mahler measure, which we can do by Jensen's formula, yields

$$\operatorname{Im}\bigl(\operatorname{Li}_2(ir)\bigr)=\frac{\pi}{4}\mathrm{m}\left(y+\frac{1+k(z+z^{-1})/2}{1-k(z+z^{-1})/2}\right).$$

Since $k=\frac{2}{r+r^{-1}}$, we have

$$\begin{aligned}\operatorname{Im}\bigl(\operatorname{Li}_2(ir)\bigr)&=\frac{\pi}{4}\mathrm{m}\left(y+\frac{r+r^{-1}+(z+z^{-1})}{r+r^{-1}-(z+z^{-1})}\right)\\&=\frac{\pi}{4}\mathrm{m}\bigl((1+y)(r+r^{-1})+(1-y)(z+z^{-1})\bigr)-\frac{\pi}{4}\mathrm{m}\bigl(r+r^{-1}-(z+z^{-1})\bigr)\\&=\frac{\pi}{4}\mathrm{m}\bigl((1+y)(r+r^{-1})+(1-y)(z+z^{-1})\bigr)-\frac{\pi}{4}\left(\log^+(r)+\log^+\left(\frac{1}{r}\right)\right).\end{aligned}$$

In order to substitute the arctangent integral for $\operatorname{Im}(\operatorname{Li}_2(ir))$, we will assume that $0<r<1$. With this restriction, the formula becomes

$$\begin{aligned}\int_0^r\frac{\tan^{-1}(u)}{u}\,\mathrm{d}u&=\frac{\pi}{4}\mathrm{m}\bigl((1+y)(r+r^{-1})+(1-y)(z+z^{-1})\bigr)-\frac{\pi}{4}\log\left(\frac{1}{r}\right)\\&=\frac{\pi}{4}\mathrm{m}\bigl((1+y)(1+r^2)+r(1-y)(z+z^{-1})\bigr).\end{aligned}\tag{2.8}$$

We can manually verify that Eq. (2.8) holds when $r=0$ and $r=1$, and using Eq. (2.4) we can extend Eq. (2.8) to all $r>1$. Therefore, Eq. (2.5) follows immediately.

Next we will prove Eq. (2.7). Using Eq. (6.16), we can show that

$$2\operatorname{Im}\bigl(\operatorname{Li}_2(ip)\bigr)=\frac{\pi}{2}\log(p)+\int_0^1\frac{\sin^{-1}(u)}{u\sqrt{(1-u^2)(1-k^2u^2)}}\,\mathrm{d}u,$$

where $k=\frac{1-p^2}{1+p^2}$, and $0<k<1$. To satisfy this restriction on k, we will assume that $0<p<1$. After several elementary simplifications, the right-hand side becomes

$$\begin{aligned}&=\frac{\pi}{2}\log(p)+\frac{\pi}{2}\log\left(1+\frac{1}{\sqrt{1-k^2}}\right)+\int_0^1\log\left(1+\sqrt{\frac{1-u^2}{1-k^2u^2}}\right)\frac{\mathrm{d}u}{\sqrt{(1-u^2)}}\\&=\frac{\pi}{2}\log(p)+\frac{\pi}{2}\log\left(\frac{(1+p)^2}{2p}\right)+\int_0^{\pi/2}\log\left(1+\frac{\cos(\theta)}{\sqrt{1-k^2\sin^2(\theta)}}\right)\mathrm{d}\theta\\&=\frac{\pi}{2}\log\left(\frac{(1+p)^2}{2}\right)+\frac{1}{2}\int_0^{2\pi}\log^+\left|1+\frac{\cos(\theta)}{\sqrt{1-k^2\sin^2(\theta)}}\right|\mathrm{d}\theta.\end{aligned}$$

Since $\cos(\pi-\theta)=-\cos(\theta)$, we have

$$2\,\mathrm{Im}\big(\mathrm{Li}_2(ip)\big)=\frac{\pi}{2}\log\left(\frac{(1+p)^2}{2}\right)+\frac{1}{4}\int_0^{2\pi}\log^+\left|1+\frac{\cos(\theta)}{\sqrt{1-k^2\sin^2(\theta)}}\right|\mathrm{d}\theta$$
$$+\frac{1}{4}\int_0^{2\pi}\log^+\left|1-\frac{\cos(\theta)}{\sqrt{1-k^2\sin^2(\theta)}}\right|\mathrm{d}\theta.$$

Applying Jensen's formula yields

$$\begin{aligned}2\,\mathrm{Im}\big(\mathrm{Li}_2(ip)\big)&=\frac{\pi}{2}\log\left(\frac{(1+p)^2}{2}\right)+\frac{1}{4}\int_0^{2\pi}\mathrm{m}\left((1+y)^2-\frac{\cos^2(\theta)}{1-k^2\sin^2(\theta)}\right)\mathrm{d}\theta\\&=\frac{\pi}{2}\log\left(\frac{(1+p)^2}{2}\right)+\frac{\pi}{2}\mathrm{m}\left((1+y)^2-\frac{(z+z^{-1})^2}{4+k^2(z-z^{-1})^2}\right)\\&=\frac{\pi}{2}\log\left(\frac{(1+p)^2}{2}\right)-\frac{\pi}{2}\mathrm{m}\big(4+k^2(z-z^{-1})^2\big)\\&\quad+\frac{\pi}{2}\mathrm{m}\big(\big(4(1+y)^2-(z+z^{-1})^2\big)+k^2(1+y)^2(z-z^{-1})^2\big).\end{aligned}$$

We can simplify the one-dimensional Mahler measure as follows:

$$\mathrm{m}\big(4+k^2(z-z^{-1})^2\big)=2\mathrm{m}\big(2+ik(z-z^{-1})\big)=2\log\big(1+\sqrt{1-k^2}\big)=2\log\left(\frac{(1+p)^2}{1+p^2}\right).$$

Eliminating k yields

$$\begin{aligned}2\,\mathrm{Im}\big(\mathrm{Li}_2(ip)\big)&=\frac{\pi}{2}\log\left(\frac{(1+p)^2}{2}\right)-\pi\log\left(\frac{(1+p)^2}{1+p^2}\right)\\&\quad+\frac{\pi}{2}\mathrm{m}\left(\big(4(1+y)^2-(z+z^{-1})^2\big)+\left(\frac{1-p^2}{1+p^2}\right)^2(1+y)^2(z-z^{-1})^2\right)\\&=-\frac{\pi}{2}\log(2)-\pi\log(1+p)\\&\quad+\frac{\pi}{2}\mathrm{m}\big(\big(4(1+y)^2-(z+z^{-1})^2\big)(1+p^2)^2+(1+y)^2(z-z^{-1})^2(1-p^2)^2\big).\end{aligned}$$

Since $0<p<1$, it follows that

$$\begin{aligned}2\int_0^p\frac{\tan^{-1}(u)}{u}\,\mathrm{d}u&=\frac{\pi}{2}\mathrm{m}\big(\big(4(1+y)^2-(z+z^{-1})^2\big)(1+p^2)^2+(1+y)^2(z-z^{-1})^2(1-p^2)^2\big)\\&\quad-\frac{\pi}{2}\log(2)-\pi\log(1+p).\end{aligned}\tag{2.9}$$

It is relatively easy to verify that Eq. (2.9) holds when $p=0$ and $p=1$. Using Eq. (2.4), we can also extend Eq. (2.9) to $p>1$, which completes the proof of Eq. (2.7). □

3. Relations between $\mathrm{TS}(v,1)$ and Mahler's measure, and a reduction of $\mathrm{TS}(v,w)$ to multiple polylogarithms

The first goal of this section is to establish five identities relating $\mathrm{TS}(v,1)$ to three-variable Mahler measures. We will prove these formulas in Theorem 3.2, using the methods outlined in Section 2. Corollary 3.3 examines a few special cases of these results.

Theorem 3.5 accomplishes the second goal of this section, which is to express $\mathrm{TS}(v,w)$ in terms of multiple polylogarithms. This result, which appears to be new, is stated in Eq. (3.14). The importance of Eq. (3.14) lies in its easy proof, and more importantly in the fact that it immediately reduces $\mathrm{TS}(v,1)$ to multiple polylogarithms. Finally, Proposition 3.6 will demonstrate that the multiple polylogarithms in Eq. (3.14) always reduce to standard polylogarithms.

We will need the following simple lemma to prove Theorem 3.1.

Lemma 3.1. *Assume that v and w are real numbers with $v>0$ and $w\in(0,1]$, then*

$$\mathrm{TS}(v,w)=\tan^{-1}(v)\int_0^{w}\frac{\sin^{-1}(z)}{z}\,\mathrm{d}z-\int_0^{\tan^{-1}(v)}\int_0^{\frac{w}{v}\tan(\theta)}\frac{\sin^{-1}(z)}{z}\,\mathrm{d}z\,\mathrm{d}\theta, \tag{3.1}$$

$$\mathrm{TS}(v,w)=\sin^{-1}(w)\int_0^{v}\frac{\tan^{-1}(u)}{u}\,\mathrm{d}u-\int_0^{\sin^{-1}(w)}\int_0^{\frac{v}{w}\sin(\theta)}\frac{\tan^{-1}(z)}{z}\,\mathrm{d}z\,\mathrm{d}\theta. \tag{3.2}$$

Proof. To prove Eq. (3.1) first integrate $\mathrm{TS}(v,w)$ by parts to obtain

$$\mathrm{TS}(v,w)=\tan^{-1}(v)\int_0^{w}\frac{\sin^{-1}(z)}{z}\,\mathrm{d}z-\int_0^{1}\frac{\mathrm{d}}{\mathrm{d}u}\bigl(\tan^{-1}(vu)\bigr)\int_0^{wu}\frac{\sin^{-1}(z)}{z}\,\mathrm{d}z\,\mathrm{d}u.$$

Making the u-substitution $\theta=\tan^{-1}(vu)$ we have

$$\mathrm{TS}(v,w)=\tan^{-1}(v)\int_0^{w}\frac{\sin^{-1}(z)}{z}\,\mathrm{d}z-\int_0^{\tan^{-1}(v)}\int_0^{\frac{w}{v}\tan(\theta)}\frac{\sin^{-1}(z)}{z}\,\mathrm{d}z\,\mathrm{d}\theta,$$

which completes the proof of the identity.

The proof of Eq. (3.2) follows in a similar manner. □

The fact that Lemma 3.1 expresses $\mathrm{TS}(v,w)$ as a double integral in two different ways, makes $\mathrm{TS}(v,w)$ more versatile than either $\mathrm{S}(v,w)$ or $\mathrm{T}(v,w)$. These two different expansions will allow us to combine $\mathrm{TS}(v,w)$ with Mahler measures for both arctangent and arcsine integrals.

Theorem 3.2. *The following Mahler measures hold whenever $v \geqslant 0$:*

$$\mathrm{m}\left(1+x+\frac{v}{2}(1-x)(y+z)\right)=\frac{2}{\pi}\int_0^v \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u-\frac{4}{\pi^2}\operatorname{TS}(v,1), \tag{3.3}$$

$$\mathrm{m}\left(x+\frac{v^2}{4}(1+x)^2+\left(y+\frac{v}{2}(1+x)\right)^2 z\right)$$
$$=\frac{2}{\pi}\int_0^v \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u-\frac{4}{\pi^2}\operatorname{TS}(v,1)+\frac{1}{2}\mathrm{m}\left(x+\frac{v^2}{4}(1+x)^2\right), \tag{3.4}$$

$$\mathrm{m}\left((1+y)\left(1+\frac{v^2}{4}\left(x+x^{-1}\right)^2\right)+\frac{v}{2}(1-y)\left(x+x^{-1}\right)\left(z+z^{-1}\right)\right)$$
$$=\frac{4}{\pi}\int_0^v \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u-\frac{8}{\pi^2}\operatorname{TS}(v,1), \tag{3.5}$$

$$\mathrm{m}\left(\left(z-z^{-1}\right)+\frac{v}{2}\left(x+x^{-1}\right)\left(y+y^{-1}\right)\right)=\frac{2}{\pi}\int_0^v \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u-\frac{4}{\pi^2}\operatorname{TS}(v,1), \tag{3.6}$$

$$\mathrm{m}\left(\left(4(1+y)^2-\left(z+z^{-1}\right)^2\right)\left(1+\frac{v^2}{4}\left(x+x^{-1}\right)^2\right)^2+\left(z-z^{-1}\right)^2(1+y)^2\left(1-\frac{v^2}{4}\left(x+x^{-1}\right)^2\right)^2\right)$$
$$=\frac{4}{\pi}\int_0^v \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u-\frac{8}{\pi^2}\operatorname{TS}(v,1)+\frac{4}{\pi}\int_0^{\pi/2}\log\bigl(1+v\sin(\theta)\bigr)\,\mathrm{d}\theta+\log(2). \tag{3.7}$$

Proof. We will prove Eq. (3.3) first, since it has the most difficult proof. Letting $w=1$ in Eq. (3.1) yields

$$\operatorname{TS}(v,1)=\frac{\pi}{2}\log(2)\tan^{-1}(v)-\int_0^{\tan^{-1}(v)}\int_0^{\tan(\theta)/v}\frac{\sin^{-1}(z)}{z}\,\mathrm{d}z\,\mathrm{d}\theta.$$

Since $0\leqslant \tan(\theta)/v\leqslant 1$, we may substitute Eq. (2.1) for the nested arcsine integral to obtain

$$\operatorname{TS}(v,1)=\frac{\pi}{2}\log(2)\tan^{-1}(v)-\frac{\pi}{2}\int_0^{\tan^{-1}(v)}\mathrm{m}\left(\frac{2}{v}\tan(\theta)+y+z\right)\mathrm{d}\theta$$
$$=\frac{\pi}{2}\log(2)\tan^{-1}(v)-\frac{\pi}{2}\int_0^{\pi/2}\mathrm{m}\left(\frac{2}{v}\tan(\theta)+y+z\right)\mathrm{d}\theta$$
$$+\frac{\pi}{2}\int_{\tan^{-1}(v)}^{\pi/2}\mathrm{m}\left(\frac{2}{v}\tan(\theta)+y+z\right)\mathrm{d}\theta.$$

In the right-hand side integral $\frac{\tan(\theta)}{v} \geqslant 1$, hence by Cassaigne and Maillot's formula

$$\mathrm{m}\left(\frac{2}{v}\tan(\theta) + y + z\right) = \log\left(\frac{2}{v}\tan(\theta)\right).$$

Substituting this result yields

$$\begin{aligned}
\mathrm{TS}(v,1) &= \frac{\pi}{2}\log(2)\tan^{-1}(v) + \frac{\pi}{2}\int_{\tan^{-1}(v)}^{\pi/2} \log\left(\frac{2}{v}\tan(\theta)\right)\mathrm{d}\theta - \frac{\pi}{2}\int_0^{\pi/2} \mathrm{m}\left(\frac{2}{v}\tan(\theta) + y + z\right)\mathrm{d}\theta \\
&= \frac{\pi}{2}\int_0^v \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u - \frac{\pi}{2}\int_0^{\pi/2} \mathrm{m}\left(\tan(\theta) + \frac{v}{2}(y+z)\right)\mathrm{d}\theta \\
&= \frac{\pi}{2}\int_0^v \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u - \frac{\pi^2}{4}\mathrm{m}\left(1 + x + \frac{v}{2}(1-x)(y+z)\right).
\end{aligned}$$

Equation (3.3) follows immediately from rearranging this final identity.

The proofs of Eqs. (3.4) through (3.7) are virtually identical, hence we will only prove Eq. (3.5). Letting $w = 1$ in Eq. (3.2), we have

$$\mathrm{TS}(v,1) = \frac{\pi}{2}\int_0^v \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u - \int_0^{\pi/2}\int_0^{v\sin(\theta)} \frac{\tan^{-1}(z)}{z}\,\mathrm{d}z\,\mathrm{d}\theta. \tag{3.8}$$

Substituting Eq. (2.5) for the nested arctangent integral yields

$$\begin{aligned}
\mathrm{TS}(v,1) &= \frac{\pi}{2}\int_0^v \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u - \frac{\pi}{4}\int_0^{\pi/2} \mathrm{m}\big((1+y)\big(1+v^2\sin^2(\theta)\big) \\
&\qquad + v\sin(\theta)(1-y)\big(z+z^{-1}\big)\big)\,\mathrm{d}\theta \\
&= \frac{\pi}{2}\int_0^v \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u \\
&\quad - \frac{\pi^2}{8}\mathrm{m}\left((1+y)\left(1 - \frac{v^2}{4}\big(x-x^{-1}\big)^2\right) + \frac{v}{2i}(1-y)\big(x-x^{-1}\big)\big(z+z^{-1}\big)\right).
\end{aligned}$$

Letting $x \to ix$, we obtain

$$\begin{aligned}
\mathrm{TS}(v,1) &= \frac{\pi}{2}\int_0^v \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u \\
&\quad - \frac{\pi^2}{8}\mathrm{m}\left((1+y)\left(1 + \frac{v^2}{4}\big(x+x^{-1}\big)^2\right) + \frac{v}{2}(1-y)\big(x+x^{-1}\big)\big(z+z^{-1}\big)\right).
\end{aligned}$$

Equation (3.5) follows immediately from rearranging this final equality.

Finally, we will remark that the while Eq. (3.5) follows from substituting Eq. (2.5) into Eq. (3.8), we must substitute Eq. (2.2) to prove Eq. (3.4), Eq. (3.6) follows from substituting Eq. (2.6), and Eq. (3.7) follows from substituting Eq. (2.7). □

Corollary 3.3. *The formulas in Theorem* 3.2 *reduce, in order, to the following identities when* $v = 2$:

$$\mathrm{m}\big((1+x)+(1-x)(y+z)\big) = \frac{28}{5\pi^2}\zeta(3), \tag{3.9}$$

$$\mathrm{m}\big(x+(1+x)^2+(1+x+y)^2z\big) = \frac{28}{5\pi^2}\zeta(3)+\log\left(\frac{1+\sqrt{5}}{2}\right), \tag{3.10}$$

$$\mathrm{m}\big((1+x+z)\big(1+x^{-1}+z^{-1}\big)+y(1+x-z)\big(1+x^{-1}-z^{-1}\big)\big) = \frac{56}{5\pi^2}\zeta(3), \tag{3.11}$$

$$\mathrm{m}\big(\big(z-z^{-1}\big)+\big(x+x^{-1}\big)\big(y+y^{-1}\big)\big) = \frac{28}{5\pi^2}\zeta(3), \tag{3.12}$$

$$\mathrm{m}\big(\big(4z(1+y)^2-(1+z)^2\big)\big(1+3x+x^2\big)^2+(1-z)^2(1+y)^2\big(1+x+x^2\big)^2\big)$$
$$= \frac{56}{5\pi^2}\zeta(3)+\frac{16}{3\pi}G+\log(2). \tag{3.13}$$

In Eq. (3.13)*, and throughout the rest of the paper, G denotes Catalan's constant. In particular,* $G = 1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}\cdots$.

Proof. As we have already stated, Condon proved Eq. (3.9) in [5]. His proof also showed that

$$\mathrm{TS}(2,1) = \frac{\pi}{2}\int_0^2 \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u - \frac{7}{5}\zeta(3).$$

Using this formula, Eqs. (3.10) through (3.13) follow immediately from Theorem 3.2. □

Theorem 3.2 shows that we can obtain closed forms for several three-variable Mahler measures by reducing $\mathrm{TS}(v,1)$ to polylogarithms. We have proved a convenient closed form for $\mathrm{TS}(v,1)$ in Eq. (4.18). Corollary 4.6 also shows that this closed form immediately implies Condon's evaluation of $\mathrm{TS}(2,1)$. We will postpone further discussion of Eq. (4.18) until Section 4.

We will devote the remainder of this section to deriving a closed form for $\mathrm{TS}(v,w)$ in terms of multiple polylogarithms. For convenience, we will use a slightly non-standard notation for our multiple polylogarithms.

Definition 3.4. Define $\mathrm{F}_j(x)$ by

$$\mathrm{F}_j(x) = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)^j} = \frac{\mathrm{Li}_j(x)-\mathrm{Li}_j(-x)}{2},$$

and define $\mathrm{F}_{j,k}(x,y)$ by

$$\mathrm{F}_{j,k}(x,y) = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)^j}\sum_{m=0}^{n}\frac{y^{2m+1}}{(2m+1)^k}.$$

We will employ this notation throughout the rest of the paper.

Theorem 3.5. *If* $\frac{v}{w} \notin (-i\infty, -i] \cup [i, i\infty)$ *and* $w \in [-1, 1]$, *then we can express* $\mathrm{TS}(v, w)$ *in terms of multiple polylogarithms. Let* $R = \frac{v/w}{1+\sqrt{1+(v/w)^2}}$, *and let* $S = iw + \sqrt{1-w^2}$, *then*

$$\begin{aligned}\mathrm{TS}(v, w) = {}& 2\mathrm{F}_3(R) - \mathrm{F}_3(RS) - \mathrm{F}_3(R/S) - 4\mathrm{F}_{1,2}(R, 1) + 2\mathrm{F}_{1,2}(R, S) + 2\mathrm{F}_{1,2}(R, 1/S) \\ & + i \sin^{-1}(w)\big\{\mathrm{F}_2(RS) - \mathrm{F}_2(R/S) - 2\mathrm{F}_{1,1}(R, S) + 2\mathrm{F}_{1,1}(R, 1/S)\big\}. \end{aligned} \tag{3.14}$$

Proof. First note that by u-substitution

$$\mathrm{TS}(v, w) = \int_0^{\sin^{-1}(w)} \tan^{-1}\left(\frac{v}{w}\sin(\theta)\right)\cot(\theta)\theta \, \mathrm{d}\theta. \tag{3.15}$$

Since $w \in [-1, 1]$, it follows that our path of integration is along the real axis. Next substitute the Fourier series

$$\tan^{-1}\left(\frac{v}{w}\sin(\theta)\right) = 2\sum_{n=0}^{\infty} \frac{R^{2n+1}}{2n+1}\sin\big((2n+1)\theta\big) \tag{3.16}$$

into Eq. (3.15). Swapping the order of summation and integration, we have

$$\mathrm{TS}(v, w) = 2\sum_{n=0}^{\infty} \frac{R^{2n+1}}{2n+1} \int_0^{\sin^{-1}(w)} \sin\big((2n+1)\theta\big)\cot(\theta)\theta \, \mathrm{d}\theta.$$

Uniform convergence justifies this interchange of summation and integration. In particular, Eq. (3.16) converges uniformly whenever $|R| < 1$ and $\theta \in \mathbb{R}$. It is easy to show that $|R| < 1$ except when $\frac{v}{w} \in (-i\infty, -i] \cup [i, i\infty)$, in which case $|R| = 1$. If $|R| = 1$, then Eq. (3.16) no longer converges uniformly, and hence the following arguments do not apply.

Evaluating the nested integral yields

$$\begin{aligned}\mathrm{TS}(v, w) = 4\sum_{n=0}^{\infty} \frac{R^{2n+1}}{2n+1} \Bigg\{ & \sin^{-1}(w) {\sum_{k=0}^{n}}{}' \frac{\sin((2k+1)\sin^{-1}(w))}{2k+1} \\ & - {\sum_{k=0}^{n}}{}' \frac{1 - \cos((2k+1)\sin^{-1}(w))}{(2k+1)^2} \Bigg\}, \end{aligned} \tag{3.17}$$

where ${\sum_{k=0}^{n}}{}' a_k = a_0 + \cdots + a_{n-1} + \frac{a_n}{2}$. Simplifying Eq. (3.17) completes our proof. □

Equation (3.14) deserves a few remarks, since it is a fairly general result. Firstly, observe that a closer analysis of Eq. (3.16) would probably allow us to relax the restriction that $w \in [-1, 1]$. Secondly, Eq. (3.14) most likely has applications beyond the scope of this paper. For example, we can use Eq. (3.14) to reduce the right-hand side of the following equation:

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)^2}\binom{2n}{n}\left(\frac{w}{2}\right)^{2n+1}\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}$$
$$=\mathrm{TS}(1,w)-\frac{\pi}{4}\int_0^w\frac{\sin^{-1}(t)}{t}\,\mathrm{d}t+\frac{\log(2)}{2}\int_0^w\frac{\sinh^{-1}(t)}{t}\,\mathrm{d}t, \tag{3.18}$$

to multiple polylogarithms.

We can use the final result of this section, Proposition 3.6, to reduce $\mathrm{TS}(v,w)$ to regular polylogarithms. This proposition allows us to equate $\mathrm{TS}(v,w)$ with a formula involving around twenty trilogarithms. While a clever usage of trilogarithmic functional equations might simplify this result, it seems more convenient to simply leave Eq. (3.14) in its current form.

Proposition 3.6. *The functions* $\mathrm{F}_{1,1}(x,y)$ *and* $\mathrm{F}_{1,2}(x,y)$ *can be expressed in terms of polylogarithms, we have*:

$$4\mathrm{F}_{1,1}(x,y)=\mathrm{Li}_2\left(\frac{x(1+y)}{1+x}\right)-\mathrm{Li}_2\left(\frac{x(1-y)}{1+x}\right)$$
$$-\mathrm{Li}_2\left(\frac{-x(1+y)}{1-x}\right)+\mathrm{Li}_2\left(\frac{-x(1-y)}{1-x}\right). \tag{3.19}$$

To reduce $\mathrm{F}_{1,2}(x,y)$ *to polylogarithms, apply Lewin's formula, Eq.* (7.5)*, four times to the following identity*:

$$\mathrm{F}_{1,2}(x,y)=\mathrm{F}_3(xy)-\frac{1}{2}\log\big(1-x^2\big)\mathrm{F}_2(xy)+\frac{1}{4}\int_0^x\frac{\log(1-u^2)\log\big(\frac{1+yu}{1-yu}\big)}{u}\,\mathrm{d}u. \tag{3.20}$$

Proof. To prove Eq. (3.20), first swap the order of summation to obtain

$$\mathrm{F}_{1,2}(x,y)=\mathrm{F}_3(xy)+\mathrm{F}_1(x)\mathrm{F}_2(y)-\sum_{n=0}^{\infty}\frac{y^{2n+1}}{(2n+1)^2}\sum_{k=0}^{n}\frac{x^{2k+1}}{2k+1}.$$

Substituting an integral for the nested sum yields

$$\mathrm{F}_{1,2}(x,y)=\mathrm{F}_3(xy)+\mathrm{F}_1(x)\mathrm{F}_2(y)-\sum_{n=0}^{\infty}\frac{y^{2n+1}}{(2n+1)^2}\int_0^x\frac{1-u^{2n+2}}{1-u^2}\,\mathrm{d}u$$
$$=\mathrm{F}_3(xy)+\int_0^x\frac{u}{1-u^2}\mathrm{F}_2(yu)\,\mathrm{d}u.$$

Integrating by parts, the identity becomes

$$\mathrm{F}_{1,2}(x,y)=\mathrm{F}_3(xy)-\frac{1}{2}\log\big(1-x^2\big)\mathrm{F}_2(xy)+\frac{1}{4}\int_0^x\frac{\log(1-u^2)\log\big(\frac{1+yu}{1-yu}\big)}{u}\,\mathrm{d}u,$$

which completes the proof of Eq. (3.20).

We can verify Eq. (3.19) by differentiating each side of the equation with respect to y. □

Finally, observe that we can obtain simple closed forms for $\mathrm{F}_{1,2}(x,1)$ and $\mathrm{F}_{2,1}(1,x)$ from Eq. (4.9).

4. An evaluation of TS$(v,1)$ using infinite series

This evaluation of $\mathrm{TS}(v,1)$ generalizes a theorem due to Condon. Condon proved a formula that Boyd and Rodriguez Villegas conjectured:

$$m\big(1+x+(1-x)(y+z)\big)=\frac{28}{5\pi^2}\zeta(3).$$

Condon's result is equivalent to evaluating $\mathrm{TS}(2,1)$ in closed form. As Theorem 3.2 has shown, generalizing this Mahler measure depends on finding a closed form for $\mathrm{TS}(v,1)$. Equation (4.18) accomplishes this goal by expressing $\mathrm{TS}(v,1)$ in terms of polylogarithms.

This calculation of $\mathrm{TS}(v,1)$ is based on several series transformations. The first step is to expand $\mathrm{TS}(v,1)$ in a Taylor series; observe that the following formula holds whenever $|v|<1$:

$$\mathrm{TS}(v,1)=\frac{\pi}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}v^{2k+1}-\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}\frac{(2v)^{2k+1}}{\binom{2k}{k}}. \tag{4.1}$$

We can easily prove Eq. (4.1) by starting from Eq. (3.8). Formula (4.1) shows that $\mathrm{TS}(v,1)$ is analytic in the open unit disk. Unfortunately Eq. (4.1) does not converge when $v=2$, and hence it cannot be used to calculate $\mathrm{TS}(2,1)$. It will be necessary to find an analytic continuation of $\mathrm{TS}(v,1)$ in order to carry out any useful computations.

The following family of functions will play a crucial role in our calculations.

Definition 4.1. Define $h_n(v)$ by the infinite series,

$$h_n(v)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^n}\frac{(2v)^{2k+1}}{\binom{2k}{k}}. \tag{4.2}$$

Using the definition of $h_3(v)$, combined with the identity

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}v^{2k+1}=\int_0^v\frac{\tan^{-1}(u)}{u}\,\mathrm{d}u,$$

it follows that Eq. (4.1) can be rewritten as

$$\mathrm{TS}(v,1)=\frac{\pi}{2}\int_0^v\frac{\tan^{-1}(u)}{u}\,\mathrm{d}u-\frac{1}{2}h_3(v). \tag{4.3}$$

Finding a closed form for $\mathrm{TS}(v,1)$ we will entail finding a closed form for $h_3(v)$. Theorem 4.5 accomplishes this goal, however first we need to prove several auxiliary lemmas. The idea behind our proof is very simple: first we will find a closed form for $h_2(v)$ and then integrate it to find a closed form for $h_3(v)$.

Batir recently used this method in an interesting paper [1] to obtain a formula that is equivalent to Eq. (4.15). Unfortunately Batir seems to have missed Eq. (4.12), so we will provide a full derivation of this important result.

Lemma 4.2. *The function $h_2(v)$ is analytic if $v \notin (-i\infty, -i] \cup [i, i\infty)$. Furthermore, we can express $h_2(v)$ in terms of the dilogarithm,*

$$h_2(v) = 4\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}\left(\frac{v}{1+\sqrt{1+v^2}}\right)^{2k+1}$$
$$= 2\,\mathrm{Li}_2\left(\frac{v}{1+\sqrt{1+v^2}}\right) - 2\,\mathrm{Li}_2\left(\frac{-v}{1+\sqrt{1+v^2}}\right). \tag{4.4}$$

Proof. We use the following elementary identity to prove Eq. (4.4),

$$\frac{2^{4k}}{(2k+1)^2\binom{2k}{k}} = \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1}\frac{(2k)!}{(k+j+1)!(k-j)!}. \tag{4.5}$$

Substituting Eq. (4.5) into the definition of $h_2(v)$, we have

$$h_2(v) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}\frac{(2v)^{2k+1}}{\binom{2k}{k}} = 4\sum_{k=0}^{\infty}(-1)^k\left(\frac{v}{2}\right)^{2k+1}\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}\frac{(2k)!}{(k+j+1)!(k-j)!}.$$

If we assume that $|v| < 1$, then the series converges uniformly, hence we may swap the order of summation to obtain

$$h_2(v) = 4\sum_{j=0}^{\infty}\frac{1}{2j+1}\sum_{k=0}^{\infty}(-1)^{k+2j}\frac{(2k+2j)!}{(k+2j+1)!k!}\left(\frac{v}{2}\right)^{2k+2j+1}$$
$$= 4\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2}\left(\frac{v}{2}\right)^{2j+1}\sum_{k=0}^{\infty}\frac{\left(j+\frac{1}{2}\right)_k (j+1)_k}{(2j+2)_k}\frac{(-v^2)^k}{k!},$$

where $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$. But then we have

$$h_2(v) = 4\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2}\left(\frac{v}{2}\right)^{2j+1} {}_2F_1\left[\left.\begin{matrix} j+\frac{1}{2}, j+1 \\ 2j+2 \end{matrix}\right| -v^2\right],$$

where ${}_2F_1\left[\left.\begin{smallmatrix} a,b \\ c \end{smallmatrix}\right| x\right]$ is the usual hypergeometric function. A standard hypergeometric identity [6] shows that

$${}_2F_1\left[\left.\begin{matrix} j+\frac{1}{2}, j+1 \\ 2j+2 \end{matrix}\right| -v^2\right] = \frac{2^{2j+1}}{\left(1+\sqrt{1+v^2}\right)^{2j+1}},$$

from which we obtain

$$h_2(v) = 4\sum_{j=0}^{\infty} \frac{1}{(2j+1)^2}\left(\frac{v}{1+\sqrt{1+v^2}}\right)^{2j+1},$$

concluding the proof of the identity.

We can use Eq. (4.4) to analytically continue $h_2(v)$ to a larger domain. Recall that $\mathrm{Li}_2(r) - \mathrm{Li}_2(-r)$ is analytic whenever $r \notin (-\infty,-1] \cup [1,\infty)$, and $\frac{v}{1+\sqrt{1+v^2}}$ is analytic whenever $v \notin (-i\infty,-i] \cup [i,i\infty)$. Since we have already assumed that $v \notin (-i\infty,-i] \cup [i,i\infty)$, we simply have to show that the range of $r = \frac{v}{1+\sqrt{1+v^2}}$ does not intersect the set $\{(-\infty,-1] \cup [1,\infty)\}$.

Some elementary calculus shows that

$$|r| = \left|\frac{v}{1+\sqrt{1+v^2}}\right| \leqslant 1$$

for all $v \in \mathbb{C}$, with equality occurring only when $v \in (-i\infty,-i] \cup [i,i\infty)$. It follows that $h_2(v)$ is analytic on $\mathbb{C} - \{(-i\infty,-i] \cup [i,i\infty)\}$. □

Since we have now expressed $h_2(v)$ in terms of dilogarithms, we can find a closed form for $h_1(v)$ by differentiating Eq. (4.4):

$$h_1(v) = \frac{2}{\sqrt{1+v^2}} \log\left(v+\sqrt{1+v^2}\right). \tag{4.6}$$

In Theorem 4.5, we will integrate Eq. (4.4) to find a closed form for $h_3(v)$ involving trilogarithms. To prove this theorem, we first need to establish two lemmas. Lemma 4.3 evaluates a necessary integral, while Lemma 4.4 expresses $\mathrm{F}_{2,1}(1,x)$ in terms of polylogarithms.

Lemma 4.3. *If $j \geqslant 0$ is an integer, and $r = \frac{v}{1+\sqrt{1+v^2}}$, then we have the following identity*:

$$\int_0^v \frac{1}{u}\left(\frac{u}{1+\sqrt{1+u^2}}\right)^{2j+1} \mathrm{d}u = \log\left(\frac{1+r}{1-r}\right) + \frac{r^{2j+1}}{2j+1} - 2\sum_{k=0}^{j} \frac{r^{2k+1}}{2k+1}. \tag{4.7}$$

Proof. To evaluate the integral

$$w_j(v) = \int_0^v \frac{1}{u}\left(\frac{u}{1+\sqrt{1+u^2}}\right)^{2j+1} \mathrm{d}u,$$

first make the substitution $z = \frac{u}{1+\sqrt{1+u^2}}$. In particular, we can show that

$$u = \frac{2z}{1-z^2} \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}z} = 2\frac{(1+z^2)}{(1-z^2)^2}.$$

Therefore we have

$$w_j(v)=\int_0^r z^{2j}\left(\frac{1+z^2}{1-z^2}\right)\mathrm{d}z=\int_0^r \frac{2}{1-z^2}\,\mathrm{d}z-\int_0^r \frac{1-z^{2j}}{1-z^2}\,\mathrm{d}z-\int_0^r \frac{1-z^{2j+2}}{1-z^2}\,\mathrm{d}z.$$

Next substitute the geometric series $\frac{1-z^{2j}}{1-z^2}=\sum_{k=0}^{j-1} z^{2k}$ into each of the right-hand integrals, and swap the order of summation and integration to obtain

$$w_j(v)=\int_0^r \frac{2}{1-z^2}\,\mathrm{d}z-\sum_{k=0}^{j-1}\frac{r^{2k+1}}{2k+1}-\sum_{k=0}^{j}\frac{r^{2k+1}}{2k+1}=\log\left(\frac{1+r}{1-r}\right)+\frac{r^{2j+1}}{2j+1}-2\sum_{k=0}^{j}\frac{r^{2k+1}}{2k+1}.$$

□

Lemma 4.4. *The following double polylogarithm*:

$$\mathrm{F}_{2,1}(1,x)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\sum_{k=0}^{n}\frac{x^{2k+1}}{2k+1} \tag{4.8}$$

can be evaluated in closed form. If $|x|<1$,

$$\begin{aligned}8\mathrm{F}_{2,1}(1,x)&=4\,\mathrm{Li}_3(x)-\mathrm{Li}_3\bigl(x^2\bigr)-4\,\mathrm{Li}_3(1-x)-4\,\mathrm{Li}_3\left(\frac{x}{1+x}\right)+4\zeta(3)\\&\quad+\log\left(\frac{1+x}{1-x}\right)\mathrm{Li}_2\bigl(x^2\bigr)+\frac{\pi^2}{2}\log(1+x)+\frac{\pi^2}{6}\log(1-x)\\&\quad+\frac{2}{3}\log^3(1+x)-2\log(x)\log^2(1-x).\end{aligned} \tag{4.9}$$

Proof. We will verify Eq. (4.9) by differentiating each side of the identity. First observe that the infinite series in Eq. (4.8) converges uniformly whenever $|x|\leqslant 1$, hence term by term differentiation is justified at all points in the open unit disk. It follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{F}_{2,1}(1,x)&=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\left(\frac{1-x^{2n+2}}{1-x^2}\right)\\&=\frac{\pi^2}{8}\left(\frac{1}{1-x^2}\right)-\frac{x}{1-x^2}\left(\mathrm{Li}_2(x)-\frac{1}{4}\mathrm{Li}_2\bigl(x^2\bigr)\right),\end{aligned} \tag{4.10}$$

whenever $|x|<1$.

Let $\varphi(x)$ denote the right-hand side of Eq. (4.9). Taking the derivative of $\varphi(x)$ we obtain:

$$\begin{aligned}\frac{\mathrm{d}\varphi}{\mathrm{d}x}&=\frac{4}{x}\mathrm{Li}_2(x)-\frac{2}{x}\mathrm{Li}_2\bigl(x^2\bigr)+\frac{4}{1-x}\mathrm{Li}_2(1-x)\\&\quad-4\left(\frac{1}{x}-\frac{1}{1+x}\right)\mathrm{Li}_2\left(\frac{x}{1+x}\right)+\frac{2}{1-x^2}\mathrm{Li}_2\bigl(x^2\bigr)\\&\quad-\frac{2}{x}\bigl(\log^2(1+x)-\log^2(1-x)\bigr)+\frac{\pi^2}{2}\left(\frac{1}{1+x}\right)-\frac{\pi^2}{6}\left(\frac{1}{1-x}\right)\\&\quad+\frac{2}{1+x}\log^2(1+x)-\frac{2}{x}\log^2(1-x)+\frac{4}{1-x}\log(x)\log(1-x).\end{aligned} \tag{4.11}$$

We can simplify Eq. (4.11) by eliminating $\mathrm{Li}_2(1-x)$ and $\mathrm{Li}_2(\frac{x}{1+x})$ with the functional equations:

$$\mathrm{Li}_2(1-x)=\frac{\pi^2}{6}-\log(x)\log(1-x)-\mathrm{Li}_2(x),$$
$$\mathrm{Li}_2\left(\frac{x}{1+x}\right)=-\frac{1}{2}\log^2(1+x)+\mathrm{Li}_2(x)-\frac{1}{2}\mathrm{Li}_2\left(x^2\right).$$

Substituting these identities into Eq. (4.11) and simplifying, we are left with

$$\frac{\mathrm{d}\varphi}{\mathrm{d}x}=\pi^2\left(\frac{1}{1-x^2}\right)-\frac{8x}{1-x^2}\left(\mathrm{Li}_2(x)-\frac{1}{4}\mathrm{Li}_2\left(x^2\right)\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left\{8\mathrm{F}_{2,1}(1,x)\right\}.$$

Equation (4.10) justifies this final step. Since the derivatives of $8\mathrm{F}_{2,1}(1,x)$ and $\varphi(x)$ are equal on the open unit disk, and since both functions vanish at zero, we may conclude that $8\mathrm{F}_{2,1}(1,x)=\varphi(x)$. □

The proof of Eq. (4.9) requires a remark. Despite the fact that the right-hand side of Eq. (4.9) is single valued and analytic whenever $|x|<1$, the individual terms involving $\mathrm{Li}_3(1-x)$ and $\log(x)$ are multivalued for $x\in(-1,0)$. To avoid all ambiguity, we can simply use $\mathrm{F}_{2,1}(1,x)=-\mathrm{F}_{2,1}(1,-x)$ to calculate the function at negative real arguments.

Theorem 4.5. *The function $h_3(v)$ is analytic on $\mathbb{C}-\{(-i\infty,-i]\cup[i,i\infty)\}$. If $v\notin(-i\infty,-i]\cup[i,i\infty)$, then $h_3(v)$ can be expressed in terms of polylogarithms. Let $r=\frac{v}{1+\sqrt{1+v^2}}$, then*

$$h_3(v)=\frac{1}{2}\mathrm{Li}_3\left(r^2\right)+4\mathrm{Li}_3(1-r)+4\mathrm{Li}_3\left(\frac{r}{1+r}\right)-4\zeta(3)-\log\left(\frac{1+r}{1-r}\right)\mathrm{Li}_2\left(r^2\right)$$
$$-\frac{2\pi^2}{3}\log(1-r)-\frac{2}{3}\log^3(1+r)+2\log(r)\log^2(1-r). \tag{4.12}$$

We can recover an equivalent form of Condon's identity by letting $v=2$:

$$h_3(2)=\frac{14}{5}\zeta(3). \tag{4.13}$$

Proof. This proof is very simple since we have already completed all of the hard computations. Observe from Eq. (4.2) that if $|v|<1$,

$$h_3(v)=\int_0^v\frac{h_2(u)}{u}\,\mathrm{d}u. \tag{4.14}$$

Lemma 4.2 shows that $h_2(v)$ is analytic provided that $v\notin(-i\infty,-i]\cup[i,i\infty)$. If we assume that the path of integration does not pass through either of these branch cuts, then it is easy to see that Eq. (4.14) provides an analytic continuation of $h_3(v)$ to $\mathbb{C}-\{(-i\infty,-i]\cup[i,i\infty)\}$.

Next we will prove Eq. (4.12). Substituting Eq. (4.4) into Eq. (4.14) yields an infinite series for $h_3(v)$ that is valid whenever $v \notin (-i\infty, -i] \cup [i, i\infty)$. We have

$$h_3(v) = 4\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \int_0^v \frac{1}{u}\left(\frac{u}{1+\sqrt{1+u^2}}\right)^{2n+1} \mathrm{d}u.$$

The nested integrals can be evaluated by Lemma 4.3. Letting $r = \frac{v}{1+\sqrt{1+v^2}}$ it is clear that

$$\begin{aligned} h_3(v) &= 4\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}\left(\log\left(\frac{1+r}{1-r}\right) + \frac{r^{2n+1}}{2n+1} - 2\sum_{j=0}^{n} \frac{r^{2j+1}}{2j+1}\right) \\ &= \frac{\pi^2}{2}\log\left(\frac{1+r}{1-r}\right) + 4\,\mathrm{Li}_3(r) - \frac{1}{2}\mathrm{Li}_3\left(r^2\right) - 8\mathrm{F}_{2,1}(1,r), \end{aligned} \tag{4.15}$$

where $\mathrm{F}_{2,1}(1,r)$ has a closed form provided by Eq. (4.9). Since $|r| < 1$ whenever $v \notin (-i\infty, -i] \cup [i, i\infty)$, we may substitute Eq. (4.9) to finish the calculation.

Observe that when $v = 2$, we have $r = \frac{\sqrt{5}-1}{2}$. It is easy to verify that $\frac{3-\sqrt{5}}{2} = r^2 = 1 - r = \frac{r}{1+r}$. Using Eq. (4.12), it follows that

$$\begin{aligned} h_3(2) &= \frac{17}{2}\mathrm{Li}_3\left(\frac{3-\sqrt{5}}{2}\right) - 4\zeta(3) - 3\log\left(\frac{1+\sqrt{5}}{2}\right)\mathrm{Li}_2\left(\frac{3-\sqrt{5}}{2}\right) \\ &\quad + \frac{4\pi^2}{3}\log\left(\frac{1+\sqrt{5}}{2}\right) - \frac{26}{3}\log^3\left(\frac{1+\sqrt{5}}{2}\right). \end{aligned} \tag{4.16}$$

Equation (4.13) follows immediately from substituting the classical formulas for $\mathrm{Li}_3(\frac{3-\sqrt{5}}{2})$ and $\mathrm{Li}_2(\frac{3-\sqrt{5}}{2})$ into Eq. (4.16). □

Notice that Eq. (4.13) is equivalent to a new evaluation of the ${}_4F_3$ hypergeometric function,

$${}_4F_3\left[\begin{matrix} 1, 1, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \end{matrix} \,\middle|\, -4\right] = \frac{7}{10}\zeta(3). \tag{4.17}$$

Corollary 4.6. *Let $r = \frac{v}{1+\sqrt{1+v^2}}$ and suppose that $v \notin (-i\infty, -i] \cup [i, i\infty)$, then*

$$\begin{aligned} \mathrm{TS}(v,1) &= \frac{\pi}{2}\int_0^v \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u - \frac{1}{4}\mathrm{Li}_3\left(r^2\right) - 2\,\mathrm{Li}_3(1-r) - 2\,\mathrm{Li}_3\left(\frac{r}{1+r}\right) \\ &\quad + 2\zeta(3) + \frac{1}{2}\log\left(\frac{1+r}{1-r}\right)\mathrm{Li}_2\left(r^2\right) + \frac{\pi^2}{3}\log(1-r) \\ &\quad + \frac{1}{3}\log^3(1+r) - \log(r)\log^2(1-r), \end{aligned} \tag{4.18}$$

$$\mathrm{TS}(2,1)=\frac{\pi}{2}\int_0^2 \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u-\frac{7}{5}\zeta(3). \tag{4.19}$$

Proof. Equation (4.18) follows immediately from substituting Eq. (4.12) into Eq. (4.3), while Eq. (4.19) follows from combining Eq. (4.13) with Eq. (4.3). □

The fact that we can reduce $h_1(v)$, $h_2(v)$ and $h_3(v)$ to standard polylogarithms is somewhat miraculous. Integrating Eq. (4.15) again, we can show that

$$\begin{aligned} h_4(v)=&\frac{\pi^2}{4}\left(\log\left(1-r^2\right)\log\left(\frac{1-r}{1+r}\right)+2\,\mathrm{Li}_2\left(\frac{1-r}{2}\right)-2\,\mathrm{Li}_2\left(\frac{1+r}{2}\right)\right)\\ &+\pi^2\mathrm{F}_2(r)+4\mathrm{F}_3(r)-8\,\mathrm{F}_{3,1}(1,r)-8\mathrm{F}_{2,2}(1,r)+16\mathrm{F}_{2,1,1}(1,1,r). \end{aligned} \tag{4.20}$$

Considering the complexity of these multiple polylogarithms, it seems unlikely that $h_n(v)$ will reduce to standard polylogarithms for $n \geqslant 4$.

5. Relations between $\mathrm{S}(v,1)$ and Mahler's measure, and a closed form for $\mathrm{S}(v,w)$

In this section we will study the double arcsine integral, $\mathrm{S}(v,w)$. Recall that we defined $\mathrm{S}(v,w)$ with an integral:

$$\mathrm{S}(v,w)=\int_0^1 \frac{\sin^{-1}(vx)\sin^{-1}(wx)}{x}\,\mathrm{d}x.$$

First, we will show that both $\mathrm{S}(v,1)$ and $\mathrm{S}(v,v)$ reduce to standard polylogarithms. Next, we will discuss several interesting results relating $\mathrm{S}(v,1)$ and $\mathrm{S}(v,v)$ to Mahler's measure and binomial sums. Finally, Theorem 5.4 concludes this section by expressing $\mathrm{S}(v,w)$ in terms of polylogarithms.

Theorem 5.1. *Assume that* $0 \leqslant v \leqslant 1$*, then* $\mathrm{S}(v,v)$ *and* $\mathrm{S}(v,1)$ *both have simple closed forms*:

$$\mathrm{S}(v,1)=\frac{\pi}{2}\int_0^v \frac{\sin^{-1}(x)}{x}\,\mathrm{d}x-\left(\frac{\mathrm{Li}_3(v)-\mathrm{Li}_3(-v)}{2}\right), \tag{5.1}$$

$$\begin{aligned} \mathrm{S}(v,v)=&\left(\frac{\mathrm{Li}_3(e^{2i\sin^{-1}(v)})+\mathrm{Li}_3(e^{-2i\sin^{-1}(v)})}{4}\right)-\frac{\zeta(3)}{2}\\ &+\sin^{-1}(v)\left(\frac{\mathrm{Li}_2(e^{2i\sin^{-1}(v)})-\mathrm{Li}_2(e^{-2i\sin^{-1}(v)})}{2i}\right)+\left(\sin^{-1}(v)\right)^2\log(2v). \end{aligned} \tag{5.2}$$

Proof. To prove Eq. (5.1), we will substitute the Taylor series for $\sin^{-1}(vx)$ into the integral $\mathrm{S}(v,1)=\int_0^1 \frac{\sin^{-1}(vx)\sin^{-1}(x)}{x}\,\mathrm{d}x$. After swapping the order of summation and integration, we have

$$\begin{aligned}
\mathrm{S}(v,1) &= 2\sum_{n=0}^{\infty}\frac{1}{2n+1}\binom{2n}{n}\left(\frac{v}{2}\right)^{2n+1}\int_0^1 \sin^{-1}(x)x^{2n}\,\mathrm{d}x \\
&= \pi\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\binom{2n}{n}\left(\frac{v}{2}\right)^{2n+1} - \sum_{n=0}^{\infty}\frac{v^{2n+1}}{(2n+1)^3} \\
&= \frac{\pi}{2}\int_0^v \frac{\sin^{-1}(x)}{x}\,\mathrm{d}x - \left(\frac{\mathrm{Li}_3(v)-\mathrm{Li}_3(-v)}{2}\right).
\end{aligned}$$

To prove (5.2) make the u-substitution $x = \frac{\sin(t)}{v}$, and then integrate by parts as follows:

$$\begin{aligned}
\mathrm{S}(v,v) &= \int_0^1 \frac{(\sin^{-1}(vx))^2}{x}\,\mathrm{d}x = \int_0^{\sin^{-1}(v)} t^2\cot(t)\,\mathrm{d}t \\
&= \left(\sin^{-1}(v)\right)^2\log(v) - 2\int_0^{\sin^{-1}(v)} t\log\bigl(\sin(t)\bigr)\,\mathrm{d}t.
\end{aligned}$$

Next substitute the Fourier series for $\log(\sin(t))$ into the previous equation, recall that

$$\log\bigl(\sin(t)\bigr) = -\log(2) - \sum_{n=1}^{\infty}\frac{\cos(2nt)}{n}$$

is valid for $0 < t < \pi$. Integrating by parts a second time completes the proof. □

The function $\mathrm{S}(v,v)$ provides a connection to a second family of interesting binomial sums. If we recall the formula

$$\left(\sin^{-1}(x)\right)^2 = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(2x)^{2n}}{n^2\binom{2n}{n}},$$

then it is immediately obvious that if $|v| \leqslant 1$ we must have

$$\mathrm{S}(v,v) = \frac{1}{4}\sum_{n=1}^{\infty}\frac{(2v)^{2n}}{n^3\binom{2n}{n}}. \tag{5.3}$$

Comparing Eq. (5.3) with Eq. (5.2) yields a classical formula:

$$\mathrm{S}\left(\frac{1}{2},\frac{1}{2}\right) = \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{n^3\binom{2n}{n}} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{\pi n}{3}\right)}{n^3} - \frac{\zeta(3)}{2} + \frac{\pi}{6}\sum_{n=1}^{\infty}\frac{\sin\left(\frac{\pi n}{3}\right)}{n^2}. \tag{5.4}$$

Proposition 5.2. *If* $v \in [0,1]$ *and* $w \in (0,1]$, *we have*

$$\mathrm{S}(v,w) = \sin^{-1}(w)\int_0^v \frac{\sin^{-1}(u)}{u}\,\mathrm{d}u - \frac{\pi}{2}\int_0^{\sin^{-1}(w)} \mathrm{m}\left(\frac{2v}{w}\sin(\theta)+y+z\right)\mathrm{d}\theta. \tag{5.5}$$

Proof. This proof is similar to the proof of Proposition 3.1. After an integration by parts, and the u-substitution $u = \sin(\theta)/w$, we obtain

$$\mathrm{S}(v,w) = \sin^{-1}(w)\int_0^v \frac{\sin^{-1}(u)}{u}\,\mathrm{d}u - \frac{\pi}{2}\int_0^{\sin^{-1}(w)}\int_0^{\frac{v}{w}\sin(\theta)} \frac{\sin^{-1}(z)}{z}\,\mathrm{d}z\,\mathrm{d}\theta.$$

Since $0 \leqslant v \leqslant 1$ and $0 < w \leqslant 1$, it follows that $0 \leqslant \frac{v}{w}\sin(\theta) \leqslant 1$. Therefore we may complete the proof by substituting Eq. (2.1) for the nested arcsine integral. □

Corollary 5.3. *We can recover Vandervelde's formula by letting* $w = 1$ *in Eq.* (5.5):

$$\mathrm{m}\big(v(1+x)+y+z\big) = \frac{2}{\pi}\int_0^v \frac{\sin^{-1}(u)}{u}\,\mathrm{d}u - \frac{4}{\pi^2}\mathrm{S}(v,1) = \frac{4}{\pi^2}\left(\frac{\mathrm{Li}_3(v)-\mathrm{Li}_3(-v)}{2}\right). \tag{5.6}$$

Notice that if $v = w = \frac{1}{2}$ in Eq. (5.5), we have

$$\begin{aligned}\mathrm{S}\left(\frac{1}{2},\frac{1}{2}\right) &= \frac{\pi}{6}\int_0^{1/2} \frac{\sin^{-1}(u)}{u}\,\mathrm{d}u - \frac{\pi}{2}\int_0^{\pi/6} \mathrm{m}\big(2\sin(\theta)+y+z\big)\,\mathrm{d}\theta \\ &= \frac{\pi}{6}\int_0^{1/2} \frac{\sin^{-1}(u)}{u}\,\mathrm{d}u - \frac{\pi^2}{12}\mathrm{m}\big(1-x^{1/6}+y+z\big). \end{aligned}\tag{5.7}$$

Comparing Eq. (5.7) to Eq. (5.4) allows us to express a famous binomial sum as the Mahler measure of a three-variable algebraic function.

The final result of this section allows us to express $\mathrm{S}(v,w)$ in terms of standard polylogarithms.

Theorem 5.4. *Suppose that* $0 \leqslant v < w \leqslant 1$, *and let* $\theta = \sin^{-1}(w) - \sin^{-1}(v)$. *Then we have*

$$\begin{aligned}2\mathrm{S}(v,w) = {} & \mathrm{S}(v,v) + \mathrm{S}(w,w) - \mathrm{S}\big(\sin(\theta),\sin(\theta)\big) - 2\,\mathrm{Li}_3\left(\frac{v}{w}\right) + \mathrm{Li}_3\left(\frac{v}{w}e^{i\theta}\right) \\ & + \mathrm{Li}_3\left(\frac{v}{w}e^{-i\theta}\right) - i\theta\,\mathrm{Li}_2\left(\frac{v}{w}e^{i\theta}\right) + i\theta\,\mathrm{Li}_2\left(\frac{v}{w}e^{-i\theta}\right) \\ & + \frac{\theta^2}{2}\log\left(1+\frac{v^2}{w^2}-\frac{2v}{w}\cos(\theta)\right). \end{aligned}\tag{5.8}$$

Notice that Eq. (5.2) *reduces* $\mathrm{S}(v,v)$, $\mathrm{S}(w,w)$, *and* $\mathrm{S}(\sin(\theta),\sin(\theta))$ *to standard polylogarithms.*

Proof. The details of this proof are not particularly difficult. First observe the following trivial formula:

$$\mathrm{S}(v,v)-2\mathrm{S}(v,w)+\mathrm{S}(w,w)=\int_0^1\frac{(\sin^{-1}(wu)-\sin^{-1}(vu))^2}{u}\,\mathrm{d}u.$$

Rearranging, and then applying the arcsine addition formula yields

$$2\mathrm{S}(v,w)=\mathrm{S}(v,v)+\mathrm{S}(w,w)-\int_0^1\frac{(\sin^{-1}(wu\sqrt{1-v^2u^2}-vu\sqrt{1-w^2u^2}\,))^2}{u}\,\mathrm{d}u. \tag{5.9}$$

This substitution is justified by the monotonicity of the arcsine function. In particular, $0\leqslant v<w\leqslant 1$ implies that $0\leqslant \sin^{-1}(wu)-\sin^{-1}(vu)\leqslant\frac{\pi}{2}$ for all $u\in[0,1]$.

Next we will make the u-substitution $z=wu\sqrt{1-v^2u^2}-vu\sqrt{1-w^2u^2}$. In particular, we can show that

$$u^2=\frac{z^2}{w^2+v^2-2vw\sqrt{1-z^2}},$$

and we can easily verify that

$$\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}z}=\frac{1}{z}-\frac{vwz}{(v^2+w^2-2vw\sqrt{1-z^2}\,)\sqrt{1-z^2}}.$$

Observe that the new path of integration will run from $z=0$ to $z=\sin(\theta)=w\sqrt{1-v^2}-v\sqrt{1-w^2}$. Therefore, Eq. (5.9) becomes

$$\begin{aligned}2\mathrm{S}(v,w)&=\mathrm{S}(v,v)+\mathrm{S}(w,w)\\&\quad-\int_0^{\sin(\theta)}\big(\sin^{-1}(z)\big)^2\left(\frac{1}{z}-\frac{vwz}{(v^2+w^2-2vw\sqrt{1-z^2}\,)\sqrt{1-z^2}}\right)\mathrm{d}z\\&=\mathrm{S}(v,v)+\mathrm{S}(w,w)-\mathrm{S}\big(\sin(\theta),\sin(\theta)\big)\\&\quad+\int_0^{\sin(\theta)}\big(\sin^{-1}(z)\big)^2\frac{vwz}{(v^2+w^2-2vw\sqrt{1-z^2}\,)\sqrt{1-z^2}}\,\mathrm{d}z.\end{aligned}$$

If we let $t=\sin^{-1}(z)$, then this last integral becomes

$$2\mathrm{S}(v,w)=\mathrm{S}(v,v)+\mathrm{S}(w,w)-\mathrm{S}\big(\sin(\theta),\sin(\theta)\big)+\int_0^{\theta}t^2\frac{vw\sin(t)}{v^2+w^2-2vw\cos(t)}\,\mathrm{d}t. \tag{5.10}$$

Since $0 \leqslant v < w \leqslant 1$, a formula from [6] shows that

$$\frac{vw\sin(t)}{v^2+w^2-2vw\cos(t)} = \sum_{n=1}^{\infty}\left(\frac{v}{w}\right)^n \sin(nt). \tag{5.11}$$

The Fourier series in Eq. (5.11) converges uniformly since $v < w$. It follows that we may substitute Eq. (5.11) into Eq. (5.10), and then swap the order of summation and integration to obtain:

$$2\mathrm{S}(v,w) = \mathrm{S}(v,v) + \mathrm{S}(w,w) - \mathrm{S}\big(\sin(\theta),\sin(\theta)\big) + \sum_{n=1}^{\infty}\left(\frac{v}{w}\right)^n \int_0^{\theta} t^2\sin(nt)\,\mathrm{d}t. \tag{5.12}$$

Simplifying Eq. (5.12) completes the proof of Eq. (5.8). □

6. q-Series for the dilogarithm, and some associated trigonometric integrals

In this section we will prove several double q-series expansions for the dilogarithm. While these formulas are relatively simple, it appears that they are new. The first of these formulas, Eq. (6.8), follows from a few simple manipulations of Eq. (5.1). The remaining formulas follow from integrals that we have evaluated in Theorem 6.5. Recall that Theorem 6.5 figured prominently in the proof of Theorem 2.3.

In this section, the twelve Jacobian elliptic functions will play an important role our calculations. Recall that the Jacobian elliptic functions are doubly periodic and meromorphic on $\mathbb{C}$. The Jacobian sine function, $\mathrm{sn}(u)$, inverts the incomplete elliptical integral of the first kind. If $u \in \mathbb{C}$ is an arbitrary number, then under a suitable path of integration:

$$u = \int_0^{\mathrm{sn}(u)} \frac{\mathrm{d}z}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

The Jacobian amplitude can be defined by the equation $\mathrm{sn}(u) = \sin(\mathrm{am}(u))$, and the Jacobian cosine function is defined by $\mathrm{cn}(u) = \cos(\mathrm{am}(u))$. As usual the complementary sine function is given by $\mathrm{dn}(u) = \sqrt{1-k^2\,\mathrm{sn}^2(u)}$. Notice that every Jacobian elliptic function implicitly depends on k; this parameter k is called the elliptic modulus.

Following standard notation, we will denote the real one-quarter period of $\mathrm{sn}(u)$ by K. Since $\mathrm{sn}(K) = 1$, we may compute K from the usual formula

$$K := K(k) = \int_0^1 \frac{\mathrm{d}z}{\sqrt{(1-z^2)(1-k^2z^2)}} = \frac{\pi}{2}\,{}_2F_1\left[\begin{matrix}\frac{1}{2},\frac{1}{2}\\ 1\end{matrix}\middle|\,k^2\right].$$

Let $K' = K(\sqrt{1-k^2})$, and finally, define the elliptic nome by $q = e^{-\pi K'/K}$.

Proposition 6.1. *If* $k \in (0,1)$*, then we have the following integral*:

$$\int_0^K \operatorname{am}(u)\operatorname{cn}(u)\,\mathrm{d}u = \frac{\pi}{2}\frac{\sin^{-1}(k)}{k} - \left(\frac{\mathrm{Li}_2(k) - \mathrm{Li}_2(-k)}{2k}\right). \tag{6.1}$$

Proof. Taking the derivative of each side of Eq. (5.1), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}k}\mathrm{S}(k,1) = \int_0^1 \frac{\sin^{-1}(x)}{\sqrt{1-k^2x^2}}\,\mathrm{d}x = \frac{\pi}{2}\frac{\sin^{-1}(k)}{k} - \left(\frac{\mathrm{Li}_2(k) - \mathrm{Li}_2(-k)}{2k}\right). \tag{6.2}$$

Making the u-substitution $x = \operatorname{sn}(u)$ completes the proof. □

We will need the following two inversion formulas for the elliptic nome.

Lemma 6.2. *Let* q *be the usual elliptic nome. Suppose that* $q \in (0,1)$*, then* q *is invertible using either of the formulas*:

$$k = \sin\left(4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\frac{q^{n+1/2}}{(1+q^{2n+1})}\right), \tag{6.3}$$

$$k = \tanh\left(4\sum_{n=0}^{\infty}\frac{1}{2n+1}\frac{q^{n+1/2}}{(1-q^{2n+1})}\right). \tag{6.4}$$

Proof. To prove Eq. (6.3) observe that

$$\sin^{-1}(k) = k\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-k^2x^2}} = k\int_0^K \operatorname{cn}(u)\,\mathrm{d}u. \tag{6.5}$$

Recall the Fourier series expansion [6] for $\operatorname{cn}(u)$:

$$\operatorname{cn}(u) = \frac{2\pi}{kK}\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1+q^{2n+1}}\cos\left(\frac{\pi(2n+1)}{2K}u\right). \tag{6.6}$$

Since $0 < q < 1$, this Fourier series converges uniformly. It follows that we may substitute Eq. (6.6) into Eq. (6.5), and then swap the order of summation and integration to obtain:

$$\sin^{-1}(k) = \frac{2\pi}{K}\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1+q^{2n+1}}\int_0^K \cos\left(\frac{\pi(2n+1)}{2K}u\right)\mathrm{d}u = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\frac{q^{n+1/2}}{(1+q^{2n+1})}. \tag{6.7}$$

Equation (6.3) follows immediately from taking the sine of both sides of the equation.

Equation (6.4) can be proved in a similar manner when starting from the integral

$$\tanh^{-1}(k) = k\int_0^1 \frac{dx}{1-k^2x^2}. \qquad \square$$

Next we will utilize the Fourier-series expansions for the Jacobian elliptic functions to prove the following theorem:

Theorem 6.3. *If q is the usual elliptic nome, then the following formula holds for the dilogarithm*:

$$\frac{\mathrm{Li}_2(k)-\mathrm{Li}_2(-k)}{8} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\frac{q^{n+1/2}}{(1+q^{2n+1})} + 4\sum_{\substack{n=0\\m=1}}^{\infty}\frac{1}{(2n+1)^2-(2m)^2}\frac{q^{n+m+1/2}}{(1+q^{2m})(1+q^{2n+1})}. \tag{6.8}$$

Proof. We have already stated the Fourier series expansion for $\mathrm{cn}(u)$ in Eq. (6.6). We will also require the Fourier series [6] for $\mathrm{am}(u)$:

$$\mathrm{am}(u) = \frac{\pi}{2K}u + 2\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1+q^{2n}}\sin\left(\frac{\pi n}{K}u\right). \tag{6.9}$$

Substituting Eqs. (6.6) and (6.9) into the integral in Eq. (6.1), and then simplifying yields:

$$\begin{aligned}\frac{\mathrm{Li}_2(k)-\mathrm{Li}_2(-k)}{8} - \frac{\pi}{8}\sin^{-1}(k) &= -\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\frac{q^{n+1/2}}{(1+q^{2n+1})} + \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\frac{q^{n+1/2}}{(1+q^{2n+1})}\\ &\quad + 4\sum_{n=0,\,m=1}^{\infty}\frac{1}{(2n+1)^2-(2m)^2}\frac{q^{n+m+1/2}}{(1+q^{2m})(1+q^{2n+1})}.\end{aligned} \tag{6.10}$$

This proof is nearly complete, the final step is to substitute the identity

$$\sin^{-1}(k) = 4\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\frac{q^{n+1/2}}{(1+q^{2n+1})}$$

into Eq. (6.10). This formula for $\sin^{-1}(k)$ follows immediately from Lemma 6.2. $\square$

The fact that Eq. (6.8) follow easily from an integral of the form

$$\int_0^K \mathrm{am}(u)\varphi(u)\,du,$$

suggests that we should try to generalize Eq. (6.8) by allowing $\varphi(u)$ to equal one of the other eleven Jacobian elliptic functions. Theorem 6.5 proves that ten of these eleven integrals reduce to dilogarithms and elementary functions. First, Theorem 6.4 will prove that the one exceptional integral can be expressed as the Mahler measure of an elliptic curve.

Theorem 6.4. *The following formulas hold whenever* $k \in (0, 1]$:

$$\mathrm{m}\left(\frac{4}{k}+x+\frac{1}{x}+y+\frac{1}{y}\right) = -\log\left(\frac{k}{1+\sqrt{1-k^2}}\right)+\frac{2}{\pi}\int_0^1 \frac{\sin^{-1}(x)}{x\sqrt{1-k^2x^2}}\,\mathrm{d}x \tag{6.11}$$

$$= -\log\left(\frac{k}{1+\sqrt{1-k^2}}\right)+\frac{2}{\pi}\int_0^K \mathrm{am}(u)\frac{\mathrm{cn}(u)}{\mathrm{sn}(u)}\,\mathrm{d}u. \tag{6.12}$$

Proof. First observe that if $k \in \mathbb{R}$ and $0 < k \leqslant 1$, then

$$\mathrm{m}\left(\frac{4}{k}+x+\frac{1}{x}+y+\frac{1}{y}\right) = -\log\left(\frac{k}{4}\right)+\mathrm{m}\left(1+\frac{k}{4}\left(x+\frac{1}{x}+y+\frac{1}{y}\right)\right).$$

For brevity let $\varphi(k) = \mathrm{m}(1+\frac{k}{4}(x+\frac{1}{x}+y+\frac{1}{y}))$. Making the change of variables $(x, y) \to (\frac{x}{y}, yx)$, we have

$$\begin{aligned}
\varphi(k) &= \mathrm{m}\left(1+\frac{k}{4}\left(x+x^{-1}\right)\left(y+y^{-1}\right)\right)\\
&= \mathrm{m}\left(\frac{k}{4}\left(y+y^{-1}\right)\right)+\mathrm{m}\left(x^2+\frac{4}{k}\left(\frac{1}{y+y^{-1}}\right)x+1\right)\\
&= \log\left(\frac{k}{4}\right)+\mathrm{m}\left(x^2+\frac{4}{k}\left(\frac{1}{y+y^{-1}}\right)x+1\right).
\end{aligned}$$

Applying Jensen's formula with respect to x reduces $\varphi(k)$ to a pair of one-dimensional integrals:

$$\begin{aligned}
\varphi(k) = \log\left(\frac{k}{4}\right) &+ \frac{1}{2\pi}\int_0^{2\pi} \log^{+}\left|\frac{1+\sqrt{1-k^2\cos^2(\theta)}}{k\cos(\theta)}\right|\mathrm{d}\theta\\
&+ \frac{1}{2\pi}\int_0^{2\pi} \log^{+}\left|\frac{1-\sqrt{1-k^2\cos^2(\theta)}}{k\cos(\theta)}\right|\mathrm{d}\theta.
\end{aligned} \tag{6.13}$$

The right-hand integral vanishes under the assumption that $0 < k \leqslant 1$. Therefore, it follows that Eq. (6.13) reduces to

$$\varphi(k) = \log\left(\frac{k}{4}\right)+\frac{2}{\pi}\int_0^{\pi/2} \log\left(\frac{1+\sqrt{1-k^2\cos^2(\theta)}}{k\cos(\theta)}\right)\mathrm{d}\theta.$$

With the observation that $\int_0^{\pi/2}\log(\cos(\theta))\,\mathrm{d}\theta=-\frac{\pi}{2}\log(2)$, this formula becomes:

$$\varphi(k)=\frac{2}{\pi}\int_0^{\pi/2}\log\left(\frac{1+\sqrt{1-k^2\cos^2(\theta)}}{2}\right)\mathrm{d}\theta. \tag{6.14}$$

Making the u-substitution of $x=\cos(\theta)$, we obtain

$$\varphi(k)=\frac{2}{\pi}\int_0^{1}\log\left(\frac{1+\sqrt{1-k^2x^2}}{2}\right)\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x.$$

Integrating by parts to eliminate the logarithmic term yields

$$\begin{aligned}\varphi(k)&=\log\left(\frac{1+\sqrt{1-k^2}}{2}\right)+\frac{2}{\pi}\int_0^1\frac{\sin^{-1}(x)}{x}\left(\frac{1-\sqrt{1-k^2x^2}}{\sqrt{1-k^2x^2}}\right)\mathrm{d}x\\&=\log\left(\frac{1+\sqrt{1-k^2}}{2}\right)+\frac{2}{\pi}\int_0^1\frac{\sin^{-1}(x)}{x\sqrt{1-k^2x^2}}\,\mathrm{d}x-\frac{2}{\pi}\int_0^1\frac{\sin^{-1}(x)}{x}\,\mathrm{d}x.\end{aligned}$$

Since $\int_0^1\frac{\sin^{-1}(x)}{x}\,\mathrm{d}x=\frac{\pi}{2}\log(2)$, it follows that

$$\varphi(k)=\log\left(\frac{1+\sqrt{1-k^2}}{4}\right)+\frac{2}{\pi}\int_0^1\frac{\sin^{-1}(x)}{x\sqrt{1-k^2x^2}}\,\mathrm{d}x,$$

from which we obtain

$$\mathrm{m}\left(\frac{4}{k}+x+\frac{1}{x}+y+\frac{1}{y}\right)=-\log\left(\frac{k}{1+\sqrt{1-k^2}}\right)+\frac{2}{\pi}\int_0^1\frac{\sin^{-1}(x)}{x\sqrt{1-k^2x^2}}\,\mathrm{d}x.$$

To prove Eq. (6.12) simply make the u-substitution $x=\mathrm{sn}(u)$. □

The elliptic curve defined by the equation $4/k+x+1/x+y+1/y=0$ was one of the simplest curves that Boyd studied in [4]. Rodriguez Villegas derived q-series expansions for a wide class of functions defined by the Mahler measures of elliptic curves in [11]. We can recover one of his results by substituting the Fourier series expansions for $\mathrm{am}(u)$ and $\mathrm{cn}(u)/\mathrm{sn}(u)$ into Eq. (6.12).

If we let $k=\sin(\theta)$, and then integrate Eq. (6.11) from $\theta=0$ to $\theta=\frac{\pi}{2}$, we can prove that

$$\mathrm{m}\left(8+\left(z+\frac{1}{z}\right)\left(x+\frac{1}{x}+y+\frac{1}{y}\right)\right)=\frac{4}{\pi}G+\frac{4}{\pi^2}\int_0^1\frac{\sin^{-1}(x)}{x}K(x)\,\mathrm{d}x. \tag{6.15}$$

Using Mathematica, we can reduce the right-hand integral to a rather complicated expression involving balanced hypergeometric functions evaluated at one.

Theorem 6.5. *We will assume that* $0 < k < 1$ *and that each Jacobian elliptic function has modulus* k. *Let* $p = \sqrt{\frac{1-k}{1+k}}$, $r = \frac{k}{1+\sqrt{1-k^2}}$, *and* $s = \frac{k}{\sqrt{1-k^2}}$, *then*

$$\int_0^K \operatorname{am}(u)\operatorname{sn}(u)\,\mathrm{d}u = \int_0^1 \frac{u\sin^{-1}(u)}{\sqrt{(1-u^2)(1-k^2u^2)}}\,\mathrm{d}u = \frac{\mathrm{Li}_2(is)-\mathrm{Li}_2(-is)}{2ki}, \tag{6.16}$$

$$\int_0^K \operatorname{am}(u)\operatorname{cn}(u)\,\mathrm{d}u = \int_0^1 \frac{\sin^{-1}(u)}{\sqrt{1-k^2u^2}}\,\mathrm{d}u = \frac{\pi}{2}\frac{\sin^{-1}(k)}{k} - \frac{\mathrm{Li}_2(k)-\mathrm{Li}_2(-k)}{2k}, \tag{6.17}$$

$$\int_0^K \operatorname{am}(u)\operatorname{dn}(u)\,\mathrm{d}u = \frac{\pi^2}{8}, \tag{6.18}$$

$$\int_0^K \operatorname{am}(u)\frac{1}{\operatorname{sn}(u)}\,\mathrm{d}u = \int_0^1 \frac{\sin^{-1}(u)}{u\sqrt{(1-u^2)(1-k^2u^2)}}\,\mathrm{d}u = -\frac{\pi}{2}\log(p) + \frac{\mathrm{Li}_2(ip)-\mathrm{Li}_2(-ip)}{i}, \tag{6.19}$$

$$\int_0^K \operatorname{am}(u)\frac{1}{\operatorname{cn}(u)}\,\mathrm{d}u = \infty, \tag{6.20}$$

$$\int_0^K \operatorname{am}(u)\frac{1}{\operatorname{dn}(u)}\,\mathrm{d}u = \int_0^1 \frac{\sin^{-1}(u)}{(1-k^2u^2)\sqrt{1-u^2}}\,\mathrm{d}u = \frac{1}{\sqrt{1-k^2}}\left(\frac{\pi^2}{8} + \frac{\mathrm{Li}_2(r^2)-\mathrm{Li}_2(-r^2)}{2}\right), \tag{6.21}$$

$$\int_0^K \operatorname{am}(u)\frac{\operatorname{sn}(u)}{\operatorname{cn}(u)}\,\mathrm{d}u = \infty, \tag{6.22}$$

$$\int_0^K \operatorname{am}(u)\frac{\operatorname{sn}(u)}{\operatorname{dn}(u)}\,\mathrm{d}u = \int_0^1 \frac{u\sin^{-1}(u)}{(1-k^2u^2)\sqrt{1-u^2}}\,\mathrm{d}u = \frac{\mathrm{Li}_2(r)-\mathrm{Li}_2(-r)}{k\sqrt{1-k^2}}, \tag{6.23}$$

$$\int_0^K \operatorname{am}(u)\frac{\operatorname{cn}(u)}{\operatorname{sn}(u)}\,\mathrm{d}u = \int_0^1 \frac{\sin^{-1}(u)}{u\sqrt{1-k^2u^2}}\mathrm{d}u = \frac{\pi}{2}\log(r) + \frac{\pi}{2}\mathrm{m}\left(\frac{4}{k} + x + \frac{1}{x} + y + \frac{1}{y}\right), \tag{6.24}$$

$$\int_0^K \operatorname{am}(u)\frac{\operatorname{cn}(u)}{\operatorname{dn}(u)}\,\mathrm{d}u = \int_0^1 \frac{\sin^{-1}(u)}{1-k^2u^2}\,\mathrm{d}u = -\frac{\pi}{2k}\log(p) - \frac{\mathrm{Li}_2(ir)-\mathrm{Li}_2(-ir)}{ki}, \tag{6.25}$$

$$\int_0^K \mathrm{am}(u)\frac{\mathrm{dn}(u)}{\mathrm{sn}(u)}\,\mathrm{d}u = 2G, \tag{6.26}$$

$$\int_0^K \mathrm{am}(u)\frac{\mathrm{dn}(u)}{\mathrm{cn}(u)}\,\mathrm{d}u = \infty. \tag{6.27}$$

Proof. First, observe that Eqs. (6.20), (6.22), and (6.27) all follow from the fact that $\mathrm{cn}(K) = 0$. Similarly, Eqs. (6.18) and (6.26) both follow from the formula $\frac{\mathrm{d}}{\mathrm{d}u}\,\mathrm{am}(u) = \mathrm{dn}(u)$.

We already proved Eq. (6.17) in Proposition 6.1, and Eq. (6.24) was proved in Theorem 6.4. This leaves a total of five formulas to prove.

To prove Eq. (6.16), observe that after letting $u = \sqrt{1-z^2}$, we have

$$\int_0^1 \frac{u\sin^{-1}(u)}{\sqrt{(1-u^2)(1-k^2u^2)}}\,\mathrm{d}u = \frac{is}{ki}\int_0^1 \frac{\sin^{-1}(\sqrt{1-z^2})}{\sqrt{1-(is)^2z^2}}\,\mathrm{d}z.$$

If $0 < k \leqslant 1/\sqrt{2}$, then $|s| \leqslant 1$. With this restriction on k, we may expand the square root in a Taylor series to obtain:

$$\begin{aligned} &= \frac{1}{ki}\sum_{m=0}^{\infty}(-1)^m\binom{-\frac{1}{2}}{m}(is)^{2m+1}\int_0^1 \sin^{-1}\bigl(\sqrt{1-z^2}\bigr)z^{2m}\,\mathrm{d}z \\ &= \frac{1}{ki}\sum_{m=0}^{\infty}\frac{(is)^{2m+1}}{(2m+1)^2} = \frac{\mathrm{Li}_2(is) - \mathrm{Li}_2(-is)}{2ki}. \end{aligned} \tag{6.28}$$

Notice that Eq. (6.28) extends to $0 < k < 1$, since both sides of the equation are analytic in this interval. Therefore, Eq. (6.16) follows immediately.

To prove Eq. (6.19) make the u-substitution $u = \frac{z}{\sqrt{1-k^2+z^2}}$. Recalling that

$$\sin^{-1}\left(\frac{z}{\sqrt{1-k^2+z^2}}\right) = \tan^{-1}\left(\frac{z}{\sqrt{1-k^2}}\right),$$

we obtain

$$\int_0^1 \frac{\sin^{-1}(u)}{u\sqrt{(1-u^2)(1-k^2u^2)}}\,\mathrm{d}u = \int_0^\infty \frac{\tan^{-1}\left(\frac{z}{\sqrt{1-k^2}}\right)}{z\sqrt{1+z^2}}\,\mathrm{d}z.$$

Using Mathematica to evaluate this last integral yields:

$$= -\frac{\pi}{2}\log(p) + \sqrt{1-k^2}\,{}_3F_2\left[\begin{matrix}\frac{1}{2},1,1\\ \frac{3}{2},\frac{3}{2}\end{matrix}\,\middle|\,1-k^2\right]$$

$$= -\frac{\pi}{2}\log(p) + \frac{1}{2}\sum_{n=0}^{\infty}\frac{(2\sqrt{1-k^2})^{2n+1}}{(2n+1)^2\binom{2n}{n}}$$
$$= -\frac{\pi}{2}\log(p) + 2\left(\frac{\mathrm{Li}_2(ip)-\mathrm{Li}_2(-ip)}{2i}\right),$$

where Eq. (4.4) justifies the final step.

To prove Eq. (6.21) observe that after the u-substitution $u = \sin(\theta)$ we have

$$\int_0^1 \frac{\sin^{-1}(u)}{(1-k^2u^2)\sqrt{1-u^2}}\,\mathrm{d}u = \int_0^{\pi/2}\frac{\theta}{1-k^2\sin^2(\theta)}\,\mathrm{d}\theta.$$

Now substitute the Fourier series

$$\frac{\sqrt{1-k^2}}{1-k^2\sin^2(\theta)} = 1 + 2\sum_{m=1}^{\infty}(-1)^m\left(\frac{k}{1+\sqrt{1-k^2}}\right)^{2m}\cos(2m\theta) \tag{6.29}$$

into the integral, and simplify to complete the proof.

The proof of Eq. (6.23) follows the same lines as the derivation of Eq. (6.21). Observe that

$$\int_0^1 \frac{u\sin^{-1}(u)}{(1-k^2u^2)\sqrt{1-u^2}}\,\mathrm{d}u = \int_0^{\pi/2}\frac{\theta\sin(\theta)}{1-k^2\sin^2(\theta)}\,\mathrm{d}\theta.$$

Now substitute the Fourier series

$$\frac{k\sqrt{1-k^2}\sin(\theta)}{1-k^2\sin^2(\theta)} = 2\sum_{m=0}^{\infty}(-1)^m\left(\frac{k}{1+\sqrt{1-k^2}}\right)^{2m+1}\sin\bigl((2m+1)\theta\bigr) \tag{6.30}$$

into the integral, and simplify to complete the proof.

Finally, we are left with Eq. (6.25). Expanding $1/(1-k^2u^2)$ in a geometric series yields

$$\int_0^1 \frac{\sin^{-1}(u)}{1-k^2u^2}\,\mathrm{d}u = \sum_{n=0}^{\infty}k^{2n}\int_0^1 \sin^{-1}(u)u^{2n}\,\mathrm{d}u = \sum_{n=0}^{\infty}k^{2n}\left(\frac{\pi/2}{2n+1} - \frac{2^{2n}}{(2n+1)^2\binom{2n}{n}}\right)$$
$$= -\frac{\pi}{4k}\log\left(\frac{1-k}{1+k}\right) - \frac{h_2(ik)}{2ik}.$$

Substituting the closed form for $h_2(ik)$ provided by Eq. (4.4) completes the proof. □

We can obtain each of the following q-series by applying the method from Theorem 6.3 to the formulas in Theorem 6.5.

Corollary 6.6. *Let* $p = \sqrt{\frac{1-k}{1+k}}$, *and let* $r = \frac{k}{1+\sqrt{1-k^2}}$. *The following formulas hold for the dilogarithm*:

$$\frac{\mathrm{Li}_2(k)-\mathrm{Li}_2(-k)}{8}=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\frac{q^{n+1/2}}{(1+q^{2n+1})}+4\sum_{\substack{n=0\\m=1}}^{\infty}\frac{1}{(2n+1)^2-(2m)^2}\frac{q^{m+n+1/2}}{(1+q^{2m})(1+q^{2n+1})},\tag{6.31}$$

$$\frac{\mathrm{Li}_2(r)-\mathrm{Li}_2(-r)}{4}=\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\frac{q^{n+1/2}}{(1+q^{2n+1})}+4\sum_{\substack{n=0\\m=1}}^{\infty}\frac{(-1)^m}{(2n+1)^2-(2m)^2}\frac{q^{m+n+1/2}}{(1+q^{2m})(1+q^{2n+1})},\tag{6.32}$$

$$\frac{\mathrm{Li}_2(ip)-\mathrm{Li}_2(-ip)}{8i}=\frac{G}{4}+\frac{\pi}{16}\log(p)+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\frac{q^{2n+1}}{(1-q^{4n+2})}+4\sum_{\substack{n=0\\m=1}}^{\infty}\frac{(-1)^{n+m}}{(2n+1)^2-(2m)^2}\frac{q^{m+2n+1}}{(1+q^{2m})(1-q^{4n+2})}.\tag{6.33}$$

Proof. As we have already stated, each of these formulas can be proved by substituting Fourier series expansions for the Jacobian elliptic functions into Theorem 6.5.

Using the method described, we have already proved Eq. (6.31) in Theorem 6.3. Equation (6.32) follows in a similar manner from Eq. (6.23).

Equation (6.33) is a little trickier to prove. Expanding Eq. (6.26) in a q-series yields the identity

$$\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1+q^{2n}}\sum_{j=0}^{n-1}\frac{(-1)^j}{2j+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\frac{q^{2n+1}}{1+q^{2n+1}}+4\sum_{\substack{n=0\\m=1}}^{\infty}\frac{(-1)^{n+m}}{(2n+1)^2-(2m)^2}\frac{q^{m+2n+1}}{(1+q^{2m})(1+q^{2n+1})}.\tag{6.34}$$

Next expand Eq. (6.19) in the q-series

$$\frac{\mathrm{Li}_2(ip)-\mathrm{Li}_2(-ip)}{4i}=\frac{G}{2}+\frac{\pi}{8}\log(p)+\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1+q^{2n}}\sum_{j=0}^{n-1}\frac{(-1)^j}{2j+1}+\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\frac{q^{2n+1}}{1-q^{2n+1}}+4\sum_{\substack{n=0\\m=1}}^{\infty}\frac{(-1)^{n+m}}{(2n+1)^2-(2m)^2}\frac{q^{m+2n+1}}{(1+q^{2m})(1-q^{2n+1})},$$

and then combine it with Eq. (6.34) to complete the proof of Eq. (6.33). □

It is important to notice that the nine convergent integrals in Theorem 6.5 only produce three interesting q-series for the dilogarithm. The other q-series we may obtain from Theorem 6.5

really just restate known facts about the elliptic nome. For example, if we expand Eq. (6.21) in a q-series, we will obtain Eq. (6.31) with q replaced by q^2 and k replaced by r^2. This is equivalent to the fact that

$$q\left(\left(\frac{k}{1+\sqrt{1-k^2}}\right)^2\right)=q^2(k).$$

If we let $\ell=(\frac{k}{1+\sqrt{1-k^2}})^2$, then clearly k and ℓ satisfy a second degree modular equation [2].

7. A closed form for $\mathrm{T}(v,w)$, and Mahler measures for $\mathrm{T}(v,1/v)$

Recall that we defined $\mathrm{T}(v,w)$ using the following integral:

$$\mathrm{T}(v,w)=\int_0^1 \frac{\tan^{-1}(vx)\tan^{-1}(wx)}{x}\,\mathrm{d}x. \tag{7.1}$$

Since this integral involves two arctangents, rather than one or two arcsines, $\mathrm{T}(v,w)$ possesses a number of useful properties that $\mathrm{S}(v,w)$ and $\mathrm{TS}(v,w)$ appear to lack.

First observe that $\mathrm{T}(v,w)$ obeys an eight term functional equation. If we let $\mathrm{T}(v)=\int_0^v \frac{\tan^{-1}(x)}{x}\,\mathrm{d}x$, then we can use properties of the arctangent function to prove the following formula:

$$\mathrm{T}(v,w)+\mathrm{T}\left(\frac{1}{v},\frac{1}{w}\right)-\mathrm{T}\left(\frac{w}{v},1\right)-\mathrm{T}\left(\frac{v}{w},1\right)=\frac{\pi}{2}\left(\mathrm{T}(v)+\mathrm{T}\left(\frac{1}{w}\right)-\mathrm{T}\left(\frac{v}{w}\right)-\mathrm{T}(1)\right). \tag{7.2}$$

If $|v|<1$ and $|w|<1$, we can substitute arctangent Taylor series expansions into Eq. (7.1) to obtain:

$$\mathrm{T}(v,w)=\sum_{n=0}^{\infty}\frac{(-1)^n w^{2n+2}}{(2n+2)^2}\sum_{m=0}^{n}\frac{(v/w)^{2m+1}}{2m+1}+\sum_{n=0}^{\infty}\frac{(-1)^n v^{2n+2}}{(2n+2)^2}\sum_{m=0}^{n}\frac{(w/v)^{2m+1}}{2m+1}. \tag{7.3}$$

Equation (7.3) immediately reduces $\mathrm{T}(v,w)$ to multiple polylogarithms. Theorem 7.1 improves upon this result by expressing $\mathrm{T}(v,w)$ in terms of standard polylogarithms.

Theorem 7.1. *If v and w are real numbers such that $|w/v|\leqslant 1$, then*

$$\begin{aligned}
-4\mathrm{T}(v,w)={}&2\,\mathrm{Li}_3\left(\frac{w}{v}\right)-2\,\mathrm{Li}_3\left(-\frac{w}{v}\right)+\mathrm{Li}_3\left(\frac{1-vi}{1-wi}\right)+\mathrm{Li}_3\left(\frac{1+vi}{1+wi}\right)\\
&-\mathrm{Li}_3\left(\frac{1+vi}{1-wi}\right)-\mathrm{Li}_3\left(\frac{1-vi}{1+wi}\right)-\mathrm{Li}_3\left(\frac{w(1-vi)}{v(1-wi)}\right)\\
&-\mathrm{Li}_3\left(\frac{w(1+vi)}{v(1+wi)}\right)+\mathrm{Li}_3\left(-\frac{w(1+vi)}{v(1-wi)}\right)+\mathrm{Li}_3\left(-\frac{w(1-vi)}{v(1+wi)}\right)\\
&+\log\left(\frac{1+v^2}{1+w^2}\right)\left(\mathrm{Li}_2\left(\frac{w}{v}\right)-\mathrm{Li}_2\left(-\frac{w}{v}\right)\right)
\end{aligned}$$

$$- 4\tan^{-1}(v)\left(\frac{\mathrm{Li}_2(wi)-\mathrm{Li}_2(-wi)}{2i}\right) - 4\tan^{-1}(w)\left(\frac{\mathrm{Li}_2(vi)-\mathrm{Li}_2(-vi)}{2i}\right)$$
$$- \pi\log\left(\frac{1+v^2}{1+w^2}\right)\tan^{-1}(w) + 4\log(v)\tan^{-1}(v)\tan^{-1}(w). \tag{7.4}$$

Proof. Substituting logarithms for the inverse tangents, we obtain

$$-4\mathrm{T}(v,w) = \int_0^1 \log\left(\frac{1+ivu}{1-ivu}\right)\log\left(\frac{1+iwu}{1-iwu}\right)\frac{\mathrm{d}u}{u} = \int_0^{iw} \log\left(\frac{1+\frac{v}{w}u}{1-\frac{v}{w}u}\right)\log\left(\frac{1+u}{1-u}\right)\frac{\mathrm{d}u}{u}.$$

The identity then follows (more or less) immediately from four applications of Lewin's formula

$$\int_0^x \log(1-z)\log(1-cz)\frac{\mathrm{d}z}{z}$$
$$= \mathrm{Li}_3\left(\frac{1-cx}{1-x}\right) + \mathrm{Li}_3\left(\frac{1}{c}\right) + \mathrm{Li}_3(1) - \mathrm{Li}_3(1-cx) - \mathrm{Li}_3(1-x)$$
$$- \mathrm{Li}_3\left(\frac{1-cx}{c(1-x)}\right) + \log(1-cx)\left[\mathrm{Li}_2\left(\frac{1}{c}\right) - \mathrm{Li}_2(x)\right]$$
$$+ \log(1-x)\left[\mathrm{Li}_2(1-cx) - \mathrm{Li}_2\left(\frac{1}{c}\right) + \frac{\pi^2}{6}\right] + \frac{1}{2}\log(c)\log^2(1-x), \tag{7.5}$$

which was proved in [8]. Condon has discussed the intricacies of applying this equation in [5]. □

This closed form for $\mathrm{T}(v,w)$ is quite complicated. Notice that a slight change in the integrand in Eq. (7.1) produces a remarkably simplified formula:

$$\int_0^1 \frac{\tan^{-1}(vx)\tan^{-1}(wx)}{\sqrt{1-x^2}}\,\mathrm{d}x = \pi\sum_{n=0}^{\infty}\frac{\left(\frac{v}{1+\sqrt{1+v^2}}\frac{w}{1+\sqrt{1+w^2}}\right)^{2n+1}}{(2n+1)^2}. \tag{7.6}$$

To prove Eq. (7.6), make the u-substitution $x=\sin(\theta)$, and then apply Eq. (3.16) twice.

There are two special cases of Eq. (7.4) worth mentioning. First observe that $\mathrm{T}(v,\frac{1}{v})$ reduces to a very simple expression. If we let $w\to\frac{1}{v}$ in Eq. (7.4), and perform a few torturous manipulations, we can show that

$$\mathrm{T}\left(v,\frac{1}{v}\right) = \frac{\pi}{2}\,\mathrm{Im}\big[\mathrm{Li}_2(iv)\big] - \frac{1}{2}\big(\mathrm{Li}_3(v^2) - \mathrm{Li}_3(-v^2)\big) + \frac{\log(v)}{2}\big(\mathrm{Li}_2(v^2) - \mathrm{Li}_2(-v^2)\big). \tag{7.7}$$

Lalín obtained an equivalent form of Eq. (7.7) using a different method. (See [7, Appendix 2]. Lalín's formula for $\mathrm{T}(v,1)+\mathrm{T}(\frac{1}{v},1)$ reduces to Eq. (7.7) after applying Eq. (7.2) with $w=1/v$.) Observe that when $w=v$ in Eq. (7.4), we have

$$\mathrm{T}(v,v)=\frac{1}{2}\operatorname{Re}\left[\mathrm{Li}_3\left(\frac{1+vi}{1-vi}\right)-\mathrm{Li}_3\left(-\frac{1+vi}{1-vi}\right)\right]-\frac{7}{8}\zeta(3)$$
$$+2\tan^{-1}(v)\operatorname{Im}\left[\mathrm{Li}_2(iv)\right]-\log(v)\left(\tan^{-1}(v)\right)^2. \tag{7.8}$$

Finally, it appears that $\mathrm{T}(v,1)$ does not reduce to any particularly simple expression. Letting $w\to 1$ fails to simplify Eq. (7.4) in any appreciable way. Expanding $\mathrm{T}(v,1)$ in a Taylor series results in an equally complicated expression:

$$\mathrm{T}(v,1)=\frac{1}{2}\sum_{n=0}^{\infty}\frac{v^{2n+1}}{(2n+1)^2}\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}+\frac{\pi}{4}\int_0^v\frac{\tan^{-1}(x)}{x}\,\mathrm{d}x-\frac{\log(2)}{4}\left(\mathrm{Li}_2(v)-\mathrm{Li}_2(-v)\right). \tag{7.9}$$

Theorem 7.3 relates $\mathrm{T}(v,w)$ to three-variable Mahler measures, and generalizes one of Lalín's formulas. Once again, we will need a simple lemma before we prove our theorem.

Lemma 7.2. *Suppose that v and w are positive real numbers, then*

$$\mathrm{T}(v,w)=\tan^{-1}(v)\int_0^w\frac{\tan^{-1}(u)}{u}\,\mathrm{d}u-\int_0^{\tan^{-1}(v)}\int_0^{(w/v)\tan(\theta)}\frac{\tan^{-1}(z)}{z}\,\mathrm{d}z\,\mathrm{d}\theta, \tag{7.10}$$

$$\mathrm{T}\left(v,\frac{1}{v}\right)=\frac{\pi}{2}\int_0^v\frac{\tan^{-1}(u)}{u}\,\mathrm{d}u-\frac{1}{2}\int_0^{\pi/2}\int_0^{v^2\tan(\theta)}\frac{\tan^{-1}(z)}{z}\,\mathrm{d}z\,\mathrm{d}\theta. \tag{7.11}$$

Proof. While we can verify Eq. (7.10) with a trivial integration by parts, the proof of Eq. (7.11) is slightly more involved.

To prove Eq. (7.11), first let $w=\frac{1}{v}$ in Eq. (7.10). This produces

$$\mathrm{T}\left(v,\frac{1}{v}\right)=\tan^{-1}(v)\int_0^{1/v}\frac{\tan^{-1}(u)}{u}\,\mathrm{d}u-\int_0^{\tan^{-1}(v)}\int_0^{\tan(\theta)/v^2}\frac{\tan^{-1}(z)}{z}\,\mathrm{d}z\,\mathrm{d}\theta. \tag{7.12}$$

Letting $v\to 1/v$ in Eq. (7.12) gives

$$\mathrm{T}\left(\frac{1}{v},v\right)=\tan^{-1}\left(\frac{1}{v}\right)\int_0^v\frac{\tan^{-1}(u)}{u}\,\mathrm{d}u-\int_0^{\tan^{-1}(1/v)}\int_0^{v^2\tan(\theta)}\frac{\tan^{-1}(z)}{z}\,\mathrm{d}z\,\mathrm{d}\theta$$
$$=\left(\frac{\pi}{2}-\tan^{-1}(v)\right)\int_0^v\frac{\tan^{-1}(u)}{u}\,\mathrm{d}u-\int_{\tan^{-1}(v)}^{\pi/2}\int_0^{v^2/\tan(\theta)}\frac{\tan^{-1}(z)}{z}\,\mathrm{d}z\,\mathrm{d}\theta.$$

Now apply Eq. (2.4) twice, which transforms this last identity to

$$\mathrm{T}\left(\frac{1}{v}, v\right) = \left(\frac{\pi}{2} - \tan^{-1}(v)\right)\left(\int_0^{1/v} \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u + \frac{\pi}{2}\log(v)\right)$$
$$- \int_{\tan^{-1}(v)}^{\pi/2} \left(\int_0^{\tan(\theta)/v^2} \frac{\tan^{-1}(z)}{z}\,\mathrm{d}z - \frac{\pi}{2}\log\left(\frac{1}{v^2}\tan(\theta)\right)\right)\mathrm{d}\theta. \tag{7.13}$$

To complete the proof, simply add Eqs. (7.12) and (7.13) together, and then simplify the resulting sum. □

Theorem 7.3. *Suppose that* $v > 0$*, then the following Mahler measures hold*:

$$\mathrm{m}\left(1 - v^4\left(\frac{1-x}{1+x}\right)^2 + \left(y + v^2\left(\frac{1-x}{1+x}\right)\right)^2 z\right)$$
$$= \frac{4}{\pi}\int_0^v \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u - \frac{8}{\pi^2}\mathrm{T}\left(v, \frac{1}{v}\right) + \frac{1}{2}\mathrm{m}\left(1 - v^4\left(\frac{1-x}{1+x}\right)^2\right), \tag{7.14}$$

$$\mathrm{m}\left(1 - v^4\left(\frac{1-x}{1+x}\right)^2 + v^2\left(\frac{1-x}{1+x}\right)\left(\frac{1-y}{1+y}\right)\left(z - z^{-1}\right)\right)$$
$$= \frac{8}{\pi}\int_0^v \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u - \frac{16}{\pi^2}\mathrm{T}\left(v, \frac{1}{v}\right), \tag{7.15}$$

$$\mathrm{m}\left(\left(y - y^{-1}\right) + v^2\left(\frac{1-x}{1+x}\right)\left(z - z^{-1}\right)\right) = \frac{4}{\pi}\int_0^v \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u - \frac{8}{\pi^2}\mathrm{T}\left(v, \frac{1}{v}\right), \tag{7.16}$$

$$\mathrm{m}\left(\left(4(1+y)^2 - \left(z + z^{-1}\right)^2\right)\left(1 - v^4\left(\frac{1-x}{1+x}\right)^2\right)^2 + \left(z - z^{-1}\right)^2(1+y)^2\left(1 + v^4\left(\frac{1-x}{1+x}\right)^2\right)^2\right)$$
$$= \frac{8}{\pi}\int_0^v \frac{\tan^{-1}(u)}{u}\,\mathrm{d}u - \frac{16}{\pi^2}\mathrm{T}\left(v, \frac{1}{v}\right) + \frac{4}{\pi}\int_0^{\pi/2} \log\left(1 + v^2\tan(\theta)\right)\mathrm{d}\theta + \log(2). \tag{7.17}$$

Proof. Each of these results follows, in order, from substituting Eqs. (2.2), (2.5)–(2.7), into Eq. (7.11). □

Corollary 7.4. *If we let* $v = 1$ *in Eq.* (7.14)*, we can recover one of Lalín's formulas* [7]:

$$\mathrm{m}\big((1+z)(1+y) + (1-z)(x-y)\big) = \frac{7}{2\pi^2}\zeta(3) + \frac{\log(2)}{2}. \tag{7.18}$$

Letting $v = 1$ *in Eqs.* (7.15)–(7.17)*, yields in order*:

$$\mathrm{m}\big(4(1+y)+(1-y)\big(x-x^{-1}\big)\big(z-z^{-1}\big)\big)=\frac{14}{\pi^2}\zeta(3), \tag{7.19}$$

$$\mathrm{m}\big((1+x)\big(y-y^{-1}\big)+(1-x)\big(z-z^{-1}\big)\big)=\frac{7}{\pi^2}\zeta(3), \tag{7.20}$$

$$\mathrm{m}\big(16(1+y)^2-4\big(z+z^{-1}\big)^2+(1+y)^2\big(z-z^{-1}\big)^2\big(x+x^{-1}\big)^2\big)=\frac{14}{\pi^2}\zeta(3)+\frac{4}{\pi}G. \tag{7.21}$$

Proof. To prove Eq. (7.18), let $v=1$ in Eq. (7.14). From Eq. (7.7) we know that $\mathrm{T}(1,1)=\frac{\pi}{2}G-\frac{7}{8}\zeta(3)$, hence

$$\begin{aligned}\frac{7}{\pi^2}\zeta(3)+\log(2)&=\mathrm{m}\left(1-\left(\frac{1-x}{1+x}\right)^2+\left(y+\frac{1-x}{1+x}\right)^2 z\right)\\&=\mathrm{m}\big(4x+\big((1+x)y+(1-x)\big)^2 z\big).\end{aligned}$$

Now let $(x,y,z)\to(x,\frac{y}{z},-xz^2)$ to obtain

$$\begin{aligned}\frac{7}{\pi^2}\zeta(3)+\log(2)&=\mathrm{m}\big(4x-\big((1+x)y+(1-x)z\big)^2 x\big)=\mathrm{m}\big(4-\big((1+x)y+(1-x)z\big)^2\big)\\&=2\mathrm{m}\big(2+(1+x)y+(1-x)z\big).\end{aligned}$$

With the final change of variables $(x,y,z)\to(z,\frac{1}{yz},\frac{x}{yz})$, we have

$$\frac{7}{\pi^2}\zeta(3)+\log(2)=2\mathrm{m}\left(2+\frac{(1+z)}{yz}+\frac{(1-z)x}{yz}\right)=2\mathrm{m}\big((1+z)(1+y)+(1-z)(x-y)\big),$$

completing the proof of Eq. (7.18).

The proofs of Eqs. (7.19) through Eq. (7.21) follow almost immediately from our evaluation of $\mathrm{T}(1,1)$. The proof Eq. (7.21) also requires the fairly easy fact that $\int_0^{\pi/2}\log(1+\tan(\theta))\,\mathrm{d}\theta=G+\frac{\pi}{4}\log(2)$. □

8. Conclusion

In principle, we should be able to apply the techniques in this paper to prove formulas for infinitely many three-variable Mahler measures. The main difficulty, which is significant, lies in the challenge of finding infinitely many Mahler measures for the arctangent and arcsine integrals. In Section 2 we proved one such formula for the arcsine integral, and four formulas for the arctangent integral.

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