Abstract

A procedure is set up for obtaining lower eigenvalue bounds for pencils of matrices $A - \nu B$ where $A$ is a Stieltjes matrix and $B$ is positive definite, under assumptions suitable for the estimation of asymptotic convergence rates of locally perturbed factorization iterative schemes. Using these results and a formerly developed approach for estimating upper bounds, we widely confirm Gustafsson's conjecture concerning the nonnecessity of Axelsson's perturbations. In so doing, we however keep local perturbations, thereby enlarging the number of applications where their sufficiency is proven; their necessity remains, on the other hand, an open question.

1. Introduction

To motivate the present work, let us first recall that, when solving the linear system of order $n$

$$Ax = b$$

(1.1)

by an iterative scheme

$$Bx_{m+1} = Cx_m + b$$

(1.2)

where

$$A = B - C$$

(1.3)
and when accelerating the convergence by Chebyshev or conjugate gradient relaxation—assuming that $A$ and $B$ are symmetric and positive definite—the convergence rate is governed by the ratio $q = \lambda_1 / \lambda_n$ of the extreme eigenvalues of the pencil $A - qB$.

Upper eigenvalue bounds, specially dedicated to the analysis of polynomially accelerated factorization schemes, i.e. where $B$ is an incomplete $LU$ factorization of $A$ [which we shall write $B = LP^{-1}U$ with $L = U^{-T}$ and $P = \text{diag}(U)$], have been obtained in [4] and used in [5] to analyse the role of distributed Axelsson's perturbations under appropriate assumptions covering, among others, those of Axelsson's original work. The conclusion of [5] was essentially that, under the same assumptions as in [4], the distributed perturbations of Axelsson are unnecessary, as suggested by Gustafsson in [9] —or more precisely, unnecessary whenever the length of the longest increasing path in the graph of $L$ is not too large.

It should be pointed out here that the main purpose of [4] and [5] was perhaps less to reach a conclusion on this particular topic than to set up an algebraic approach to the convergence analysis of polynomially accelerated factorization methods under practically relevant assumptions. It is clear however in this respect that the assumptions made in [4] and [5] are too strong. Typically, the inequality (3.4) of Theorem 3.2 in [4], i.e.

$$((L - U)x)_i \leq (Ax)_i \quad \text{for} \quad 1 \leq i \leq n,$$  \hspace{1cm} (1.4)

where $x$ denotes some appropriate positive vector [or its weaker form (3.11) appearing in Theorem 3.3 of the same paper], while usually satisfied for most indices $i$, often does not hold for all indices.

The way around this difficulty was suggested by the work of Gustafsson [7–9] and consists in the introduction of "local perturbations" [nonzero for those indices where (1.4) does not hold]. A straightforward adaptation of the approach developed in [4], reported in Section 2, confirms that, as far as upper bounds are concerned, this is indeed a viable solution.

Unfortunately, our previous analysis fails when local perturbations are introduced. The reason is that lower bounds are necessary as well as upper bounds and that a simple extension of the argument used in [4] and [5] would underestimate these lower bounds by an order of magnitude. The purpose of the present work is to elaborate a methodology, detailed in Section 3, for obtaining accurate lower bounds in the presence of local perturbations.

The accuracy of these bounds has been tested on many examples, two of which are discussed in Section 4, the other ones being briefly commented on. It is worthwhile to mention that our lower bounds were always found at least reasonably accurate. The conclusions to be drawn on this basis concerning the scope of the algebraic approach, so completed, are reported in Section 5.
Standard definitions and notation used throughout the paper are the following. The order relation between real matrices and vectors of the same dimensions is the usual componentwise order: if $A = (a_{ij})$ and $B = (b_{ij})$, then $A \leq B$ if $a_{ij} \leq b_{ij}$ for all $i, j$, while $A < B$ if $a_{ij} < b_{ij}$ for all $i, j$; $A$ is said to be nonnegative (positive) if $A \geq 0$ ($A > 0$). If $A = (a_{ij})$ is an $n \times n$ matrix, we denote by $P = \text{diag}(A)$ the diagonal $n \times n$ matrix with entries $p_{ii} = a_{ii}$. By $e$ we denote the vector whose all components are equal to unity.

All graph concepts used below refer to ordered undirected graphs.

2. UPPER EIGENVALUE BOUNDS

The results reported here are in the spirit of Reference [4]. The first one shows that, if any failure of the assumption (1.4) is compensated by a (sufficiently large) perturbation in the RHS of the condition $(Rx)_i = (Ax)_i$, then we essentially keep the same upper bound as in Theorem 0.2 of [4]; in particular, no distributed perturbation, i.e. of the type $XDx$ where $\lambda$ is a positive scalar, as considered in [5], need be introduced.

**Theorem 2.1.** Let $A = (a_{ij})$ be a Stieltjes $n \times n$ matrix and $U = (u_{ij})$ be an upper triangular real $n \times n$ matrix with positive diagonal entries and such that $u_{ij} \leq a_{ij}$ for $1 \leq i < j \leq n$; set $D = \text{diag}(A)$, $P = \text{diag}(U)$, $L = U^T$, and $B = LP^{-1}U$. If there exists a positive $n$-vector $x$ and a nonnegative diagonal matrix $\Lambda$ such that

$$Ax \geq 0,$$

$$Bx = Ax + \Lambda Dx,$$

$$(Px)_i \leq ((I + \Lambda)Dx)_i \quad \text{if} \quad i \in I,$$

$$((L - U)x)_i \leq (Ax)_i \quad \text{if} \quad i \in N \setminus I,$$

where $\Lambda = (\lambda_i \delta_i)$ with

$$\lambda_i = \begin{cases} \lambda & \text{if} \quad i \in I, \\ 0 & \text{if} \quad i \in N \setminus I, \end{cases}$$
and where $I$ is a subset of the set of indices $N = \{1, 2, \ldots, n\}$, then we have

$$\nu < \frac{1}{\lambda} + l + 1,$$

where $\nu$ denotes any eigenvalue of the pencil $\Lambda - \nu B$, and $l - 1$ is the length of the longest increasing path in the (ordered) graph $G_\nu - I(L)$ deduced from the graph of $L$ by discarding the last node and its adjacent edges.

N.B. By "increasing path," we mean a path $P_{i_1}P_{i_2}\ldots P_{i_k}$ such that $i_1 < i_2 < \cdots < i_k$. By $G_k(L)$ we denote the subgraph of the graph of $L$ determined by the nodes $P_i$ for $i = 1, 2, \ldots, k$.

**Proof.** We first observe [by induction on the successive row of (2.2)] that $UX \geq 0$. Next, defining

$$\tau_i = \frac{((P - U)x)_i}{(Px)_i},$$

we prove by induction on $l_i$ that

$$1 - \tau_i \geq \frac{1}{\lambda + l_i + 2},$$

where $l_i$ is the length of the longest increasing path of the (ordered) graph of $L$, ending at $i$.

If $i \in I$, we have by (2.2) and (2.3) that

$$(UX)_i = (Ax)_i - ((L - P)P^{-1}UX)_i + \lambda a_{ii}x_i$$

$$\geq \lambda a_{ii}x_i \geq \frac{\lambda}{1 + \lambda} p_{ii}x_i,$$

hence

$$1 - \tau_i \geq \frac{\lambda}{1 + \lambda} = \frac{1}{\frac{\lambda}{\lambda + 1} + l_i + 2} \geq \frac{1}{\lambda + l_i + 2}.$$

Therefore, at each step of the induction proof, we need only consider the case $i \in N \setminus I$. 
If \( l_i = 0 \), there is no entry in the \( i \)th row of \( L - P \); then, if \( i \in N \setminus I \), we have by (2.2) and (2.4)

\[
(Ux)_i = (Ax)_i \geq ((P - U)x)_i
\]

and hence

\[
1 - \tau_i \geq \tau_i;
\]

thus \( \tau_i \leq \frac{1}{2} \), or

\[
1 - \tau_i \geq \frac{1}{2} \geq \frac{1}{\frac{1}{\lambda} + 2} = \frac{1}{\frac{1}{\lambda} + l_i + 2},
\]

since \( l_i = 0 \).

If \( l_i > 0 \), if the property is true for all nodes \( j \) such that \( l_j < l_i - 1 \) and if \( i \in N \setminus I \), we have by (2.2)

\[
(Ux)_i - \frac{1}{\frac{1}{\lambda} + l_i + 1} ((P - U)x)_i,
\]

\[
= (Ax)_i - ((L - P)P^{-1}Ux)_i - \frac{1}{\frac{1}{\lambda} + l_i + 1} ((P - U)x)_i,
\]

and by the induction hypothesis

\[
- ((L - P)P^{-1}Ux)_i \geq - \frac{1}{\frac{1}{\lambda} + l_i + 1} ((L - P)x)_i.
\]

Hence, by the condition (2.4)

\[
(Ux)_i - \frac{1}{\frac{1}{\lambda} + l_i + 1} ((P - U)x)_i \geq (Ax)_i - \frac{1}{\frac{1}{\lambda} + l_i + 1} ((L - U)x)_i, \geq 0,
\]
or

\[ 1 - \tau_i - \frac{1}{\frac{1}{\lambda} + l_i + 1} \tau_i \geq 0, \]

which entails

\[ 1 - \tau_i \geq \frac{1}{\frac{1}{\lambda} + l_i + 2}. \]

The conclusion follows by Theorem 3.1 of [4]. \( \Box \)

It will become apparent from the results reported in later sections that the node set \( I \) should be kept as small as possible; in this respect the following generalization of Theorem 2.1 may prove helpful in practical applications. We skip its proof, since it is very much the same as the previous one, showing by induction on \( l_i \) (with the same definitions of \( \tau_i \) and \( l_i \)) that

\[ 1 - \tau_i \geq \frac{1}{\frac{1}{\lambda(1 - c)^{l_i - 1} + s_{l_i + 1}}}, \]

where the sequence \( s_l, l \geq 0, \) is defined by the recursion

\[ s_l = \frac{1}{1 - c} (1 + s_{l-1}) \quad \text{for} \quad l > 0 \]

with \( s_0 = 0. \)

**Theorem 2.2.** Under the same assumptions as in Theorem 2.1 except for the condition (2.4) being replaced by

\[ ((I_U)x)_i \leq (Ax)_i + c(Px)_i, \quad \text{if} \quad i \in N \setminus I, \quad (2.9) \]
where $0 \leq c < 1$ and $\lambda(1 - c) \leq 1$, we have

\[
\nu \leq \frac{1}{\lambda} + 1 + \sum_{k=0}^{l-1} (1 - c)^k \quad (1 - c)^l,
\]

where $\nu$ denotes any eigenvalue of the pencil $A - \nu B$, and $l - 1$ is the length of the longest increasing path of the graph $G_{n-1}(I)$.

3. LOWER EIGENVALUE BOUNDS

As mentioned in the introduction, lower eigenvalue bounds constitute the main objective of the present work. No closed form formula will be presented, because none of sufficient accuracy has been obtained so far. Instead, a procedure will be described for obtaining lower bounds in specific situations. The accuracy that can be reached in this way will be illustrated on two typical examples discussed in the next section.

Our procedure is based on Theorem 3.1 developed in the first subsection below. By this result, the estimation of a lower bound is reduced to the determination of a family $\gamma_i$ of upper eigenvalue bounds for auxiliary pencils $\Delta_i - \gamma A_i$ defined for $i \in I$; here $(A_i), \gamma, I$ is a family of $n \times n$ matrices satisfying appropriate conditions while $\Delta_i$ is the $n \times n$ matrix whose only nonzero entry is the diagonal entry on its $i$th line, equal to unity.

The other subsection is intended for practical applications. A technique is described to determine families $(A_i), \gamma, I$ of $n \times n$ matrices satisfying the required conditions; a few restrictions are suggested so that we may confine ourselves to readily available bounds $\gamma_i$, three of which are listed in Table 1 below.

3.1. Theoretical Framework

**Theorem 3.1.** Let $A = (a_{ij})$ be a Stieltjes $n \times n$ matrix and $U = (u_{ij})$ be an upper triangular real $n \times n$ matrix with positive diagonal entries and such that

\[
a_{ij} - \sum_{s=1}^{i} \frac{u_{si} u_{sj}}{u_{ss}} \leq u_{ij} \leq a_{ij}, \quad \text{for} \quad 1 \leq i < j \leq n;
\]

\[\text{(3.1)}\]
set $D = \text{diag}(\Lambda)$, $P = \text{diag}(U)$, $L = U^T$, and $B = LP^{-1}U$. If there exists a positive $n$-vector $x$ and a nonnegative diagonal matrix $\Lambda$ such that

$$Bx = Ax + \Lambda Dx,$$

(3.2)

where $\Lambda = (\lambda_i \delta_i^t)$ with

$$\lambda_i = \begin{cases} \lambda & \text{if } i \in I, \\ 0 & \text{if } i \in N \setminus I, \end{cases}$$

(3.3)

where $I$ is some subset of $N = \{1, 2, \ldots, n\}$, then we have

$$\nu \geq \frac{1}{1 + \lambda \alpha}$$

(3.4)

with

$$\alpha = \max_{i \in I} a_{ii} \gamma_i,$$

(3.5)

$$\gamma_i = \max_{z_i \neq 0} \frac{|z_i|^2}{(z, A_i z)},$$

(3.6)

where $(A_i)_{i \in I}$ is any family of $n \times n$ symmetric and positive definite or semidefinite matrices such that

$$\forall z \in C^n \quad 0 \leq \sum_{i \in I} (z, A_i z) \leq (z, A z),$$

(3.7)

$$\forall i \in I, \forall z \in C^n \quad z_i \neq 0 \implies (z, A_i z) > 0.$$  

(3.8)

Proof. Define $C = B - A$; it follows from our assumptions that the off-diagonal entries of $C$, hence also of $C - \Lambda D$, are nonnegative. On the other hand

$$(C - \Lambda D)x = Bx - Ax - \Lambda Dx = 0.$$  

Therefore $C - \Lambda D$ is negative semidefinite (according to the terminology of
Bermann and Plemmons [6], $\Lambda D - C$ is a singular $M$-matrix. Therefore

$$(z, Bz) = (z, Az) + (z, Cz)$$

$$= (z, Az) + (z, (C - \Lambda D)z) + (z, \Lambda Dz)$$

$$\leq (z, Az) + (z, \Lambda Dz).$$

Thus, for any $z \neq 0$,

$$(z, Bz) \leq \left(1 + \frac{(z, \Lambda Dz)}{(z, Az)}\right)(z, Az).$$

On the other hand,

$$(z, \Lambda Dz) = \sum_{i \in I} \lambda a_{ii} |z_i|^2$$

$$\leq \sum_{i \in I} \lambda a_{ii} (z, A_i z) \gamma_i$$

$$\leq \lambda \alpha \sum_{i \in I} (z, A_i z) \leq \lambda \alpha (z, Az).$$

Hence, for any $z \neq 0$,

$$\frac{(z, \Lambda Dz)}{(z, Az)} \leq \lambda \alpha$$

and

$$(z, Bz) \leq (1 + \lambda \alpha)(z, Az),$$

which concludes the proof.

3.2. Practical Considerations

Given a Stieltjes matrix $A$, we describe in this subsection a practical procedure for determining families $(A_l)_{l \in I}$ of $n \times n$ matrices $A_l = (a_{ij}^l)$ satisfying the requirements of Theorem 3.1. We first proceed with a step-by-step description of our construction; we next prove that the assumptions of Theorem 3.1 are fulfilled; we end with practical considerations pertinent to the determination of the parameters $\gamma_i$. 
Step 1. Choose \( x > 0 \) such that \( Ax \geq 0 \), and let \( J \) be the set of indices \( j \) such that \( (Ax)_j > 0 \). Because \( A \) is a Stieltjes matrix, \( x \) exists and \( J \) is not empty.

Step 2. For each \( l \in I \), determine the structure of \( A_l \) by choosing its graph (or more precisely the graph of its nonzero principal submatrix); each one of these graphs must be a connected subgraph of \( A \) that includes both \( l \) and some \( j \in J \). For each \( j \in J \), let \( r_j \) denote the number of members \( A_l \) of the family \((A_l)_{l \in I}\) whose graph includes \( j \).

Step 3. For each \( l \in I \), choose arbitrarily the nonzero off-diagonal \( a_{ij}^{(l)} \) of \( A_l \), subject only to the following rules:

\[
a_{ij}^{(l)} = a_{ji}^{(l)} \leq 0, \\
\sum_{l \in I} a_{ij}^{(l)} \geq a_{ii}.
\]

Step 4. For each \( l \in I \), determine the diagonal entries \( a_{ii}^{(l)} \) of \( A_l \) by

\[
a_{ii}^{(l)} = \begin{cases} 
- \sum_{j \neq i} a_{ij}^{(l)} \frac{x_j}{x_i} & \text{if } i \notin J, \\
- \sum_{j \neq i} a_{ij}^{(l)} \frac{x_j}{x_i} + \frac{1}{r_i x_i} (Ax)_i & \text{if } i \in J.
\end{cases}
\]

We now prove that the above procedure meets the requirements of Theorem 3.1.

It is readily seen that the nonzero principal submatrix of each \( A_l \) is a Stieltjes matrix, hence that \( A_l \) is symmetric and positive definite or semidefinite.

Further, letting \( A_0 = (a_{ij}^{(0)}) \) be defined by

\[
A_0 = A - \sum_{l \in I} A_l,
\]

we have that \( A_0 \) is symmetric with

\[
a_{ij}^{(0)} = a_{ij} - \sum_{l \in I} a_{ij}^{(l)} \leq 0.
\]

On the other hand

\[
A_0 x = Ax - \sum_{l \in I} A_l x,
\]
and the formula used in step 4 shows that

$$\sum_{i \in I} (A_i x)_i = \begin{cases} 0 & \text{if } i \notin J, \\ (A x)_i & \text{if } i \in J. \end{cases}$$

Hence it follows that, if the node $i$ belongs to the graph of the nonzero principal matrix of some member $A_i$ of the family $(A_i)_{i \in J}$, then

$$(A_0 x)_i = 0;$$

otherwise

$$(A_0 x)_i = (A x)_i \geq 0$$

and thus $A_0 x \geq 0$, showing that $A_0$ is positive definite or semidefinite.

It follows from the preceding result that the condition (3.7) is satisfied; on the other hand, since the nonzero principal submatrix of $A_i$ is a Stieltjes matrix whose graph includes $i$, the condition (3.8) is also met.

We finally consider the problem of determining the upper eigenvalue bounds $\gamma_i$; some known values are listed in Table 1 for matrices whose graph is a simple path or loop; in order to resort only to those cases, we suggest the following modification of step 2:

**Step 2'.** For each $i \in I$, determine the structure of $A_i$ by choosing its graph among the following:

1. a simple path from $i$ to some $j \in J$,
2. a simple path through $i$, from some $j \in J$ to some $k \in J$ with $j \neq k$,
3. a simple loop including both $i$ and some $j \in J$,

where it is understood that these paths or loops must belong to the graph of $A$. For each $j \in J$, let $r_j$ denote the number of paths and loops so chosen, which include $j$.

In addition, it is apparent from the results listed in Table 1 that one should try

1. to use short paths;
2. to use $(A x)_j > 0$ as large as possible;
3. to avoid sharing the same edges between too many paths.
4. EXAMPLES

By way of illustration, we briefly discuss here two typical examples. To have an easy description of the (Stieltjes) matrices $A = (a_{ij})$ considered in these examples, we shall define each matrix by indicating the values of its entries on a representation of its graph. Specifically, each diagonal entry $a_{ii}$ will be written in a circle representing the $i$th node, and each nonzero off-diagonal entry $a_{ij}$ will be written along the edge $\{i, j\}$; the ordering will be indicated separately.

Our examples are described in this way in Figure 1; in both cases, $A$ is of order $n = N^2$ and ordered according to the lexicographic ordering in the $r-q$
plane of the figure [i.e. that the $i$th node is $i = (r - 1)N + q$]; we thus have $l = 2N - 2$.

The last diagonal entry in example 2 is a parameter, denoted $b$ in Figure 1, and we require $b > 1$, for otherwise $A$ would not be a Stieltjes matrix; it is however quasi-singular.

To describe the factorization method used on these examples, let

$$B = LP^{-1}L^T,$$
where \( P \) is a diagonal matrix with positive diagonal entries and
\[
I = P - F,
\]
where \(-E\) is the strictly lower triangular part of the (Stieltjes) matrix \( A \). To determine \( P \), let \( x > 0 \) be such that
\[
Ax > 0
\]
with
\[
(D \quad E)x > 0,
\]
where \( D = \text{diag}(A) \), and compute the successive entries of \( P \) according to the following rules:

1. if \(((D - 2E^T)x)_i \geq 0\), set
\[
(Px)_i = (Dx)_i - (EP^{-1}E^T x)_i;
\]
2. otherwise, set
\[
(Px)_i = (1 + \lambda)(Dx)_i - (EP^{-1}E^T x)_i,
\]
where the value of \( \lambda \) is still open to discussion (see below).

The choice of \( x \) will not be discussed (its existence follows from the corollary of Theorem 3.2 in [3]); in our examples, we use \( x = e \), where \( e \) is the vector whose all components are equal to unity.

We shall discuss both examples together. Before entering this discussion, let us notice that the factorization algorithm considered here is a locally perturbed version of the one used for the examples worked in [5] and that it does not modify the performance obtained previously in all cases where the results of [4] do apply, since no local perturbations are introduced when \( I \) is empty.

4.1. Local Perturbations

In both examples, local perturbations occur at the nodes \( i \in I \), where
\[
I = \{ i = (r - 1)N + q \text{ with } r = 1 \text{ while } q \neq N \text{ or } q = 1 \text{ while } r \neq N \}.
\]
4.2. Upper Eigenvalue Bounds

In both cases, we have by Theorem 2.1

\[ \nu \leq \frac{1}{\lambda} + 2N - 1, \]

since \( l = 2N - 2 \).

This bound is obviously inaccurate for small values of \( \lambda \), and it can hardly be of any use when \( \lambda \ll 1/N \). For \( \lambda \) of the order of \( 1/N \) or larger, it has the merit of predicting a numerically attractive functional variation with \( N \). Numerical comparisons have shown that this behavior parallels the actual variation of \( \nu_n \) with \( N \), although the error remains very large in all cases.

Theorem 3.1 of [4] gives of course a better bound which, while difficult to handle analytically, can readily be computed. It is not much better however from a qualitative point of view.

To the knowledge of the author, all other bounds published so far are qualitatively worse.

For these reasons, any (qualitatively valid) conclusion that might be drawn from available upper bounds can only display the sufficiency of (sufficiently large) local perturbations, not their necessity.

4.3. Lower Eigenvalue Bounds

We follow the procedure described in Section 3.

Step 1. We use \( x = e \) in both cases; the sets \( J \) are

\[ J = \{ j = (r-1)N + q \text{ with } r = N \text{ or } q = N \} \quad \text{for example 1,} \]

\[ J = \{ n \} \quad \text{for example 2.} \]

Step 2'. We use the subgraphs of the graph of \( A \) represented in Figure 2. We have \( r_j = 1 \) for \( j \in J, j \neq n \), while \( r_n = 0 \) in example 1; all subgraphs include the node \( n \) in example 2, whence \( r_n = 2N - 3 \) in that case.

Step 3. We use, for all \( l \in I \),

\[ a_{ij}^l = a_{ij} \quad \text{in example 1,} \]

\[ a_{ij}^l = \frac{a_{ij}}{r_{ij}} \quad \text{in example 2,} \]
where $r_{ij}$ is the number of paths sharing the edge $\{i, j\}$, i.e. the number of matrices $A_l$ of the family $(A_l)_{l \in I}$ with $a_{ij}^{(l)} \neq 0$.

**Step 4.** We apply the stated formula.

The procedure is then readily concluded with the help of the formulas reported in Table 1, yielding

$$\alpha = \max_{i \in I} a_{ii} \gamma_i = 2N,$$

thus, by Theorem 3.1,

$$\nu \geq \frac{1}{1 + 2N \lambda}$$

for example 1, and

$$\alpha = \max_{i \in I} a_{ii} \gamma_i = 2N(N - 1) + \frac{2N - 3}{b - 1},$$

hence

$$\nu \geq \frac{1}{1 + 2 \left( N(N - 1) + \frac{2N - 3}{b - 1} \right) \lambda}$$

for example 2.
Numerical comparisons covering these and many other examples have shown that the bounds obtained by following our procedure are (at least) reasonably accurate in all cases. The problem of estimating lower bounds seems therefore completely solved.

### 4.4. Convergence Properties

As recalled in the introduction, the convergence properties of the (polynomially accelerated) associated iterative scheme are governed by the ratio $q = \nu_1/\nu_n$ of the pencil $A - vB$, which can now be bounded.

For example 1, we have

$$q = \frac{\nu_1}{\nu_n} \geq \frac{1}{(1 + 2N\lambda)(1/\lambda + 2N - 1)};$$

in particular, when $\lambda = 1/N$,

$$q \geq \frac{1}{3(3N - 1)} = O\left(\frac{1}{N}\right).$$

As far as we know, this result is not covered by existing theories.

For example 2, we have

$$q = \frac{\nu_1}{\nu_n} \geq \frac{1}{1 + 2\left(\frac{N(N - 1) + \frac{2N - 3}{b - 1}}{\lambda}\right)\left(\frac{1}{\lambda} + 2N - 1\right)};$$

and when $\lambda = 1/N$

$$q \geq \frac{1}{1 + 2(N - 1) + \frac{4N - 6}{(b - 1)N}(3N - 1)}.$$

What happens in this case is that the lower bound decreases much faster with increasing $\lambda$ than in the previous case. In order to keep it sufficiently large, one should require $\lambda \ll 1/N$. Unfortunately, available upper bounds are useless in this range.
4.5. Other Examples

Many other examples may be considered, displaying a continuous range of behavior between those reported above. They may be characterized by Card(J) as well as by the values of (Ax)_j for j ∈ J: the decrease of the lower bound with increasing λ is indeed accelerated when anyone of these parameters decreases.

Other families of examples may also be considered, ranging between those requiring no local perturbations and those considered above; Card(I) is then a critical parameter, since the decrease of the lower bound with increasing λ may only be accelerated when Card(I) is increased.

4.6. Another Approach

Though inspired by the work of Gustafsson [7], our technique does not generalize his approach. Such a generalization has been initiated by Axelsson and Barker [2], and it also issued in a procedure to apply in specific cases, but of a more limited scope. Indeed, it makes use of a graph concept, called a “set of path (P_i)_{i ∈ I} based on a node set I” [2], which severely limits the size of the node set I where local perturbations may be introduced. In particular, the two examples discussed here could not be covered in this way.

It is true however that simple generalizations (obtained for example by splitting the set I into two or more subsets) could be used and would lead to the correct order of magnitude (at least in the case of example 1), but a fair comparison would require a deeper refinement of the Axelsson-Barker approach and lies outside the scope of the present work.

5. CONCLUSION

In the author’s opinion, the algebraic analysis of (polynomially accelerated) factorization iterative methods applied to Stieltjes matrices, developed in [4] and [5], was more elegant than the geometrical approach developed by Axelsson’s school. But it was more limited in its scope as a consequence of having disregarded the problem of estimating lower eigenvalue bounds. Since the latter question has been answered here in a (hopefully) complete way, the algebraic approach at least matches now the scope of the geometrical approach.

Further, these results have been obtained without introducing distributed perturbations, showing that the latter are (most probably) never necessary when the length of the longest increasing path in the graph of the matrix L (or U), i.e. the lower (or upper) approximate triangular factor, is not too large. It may be noticed that the latter condition accounts in some sense for the sparsity of the matrix L.
The necessity of local perturbations remains an open question, because no qualitatively valid upper bound is available for small local perturbations. Although this problem has not been addressed here, we have presented some indication of its interest for quasisingular problems.

REFERENCES


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